Prog. Theor. Exp. Phys. **2023** 053B02(27 pages)
DOI: 10.1093/ptep/ptad054

Simple bulk reconstruction in anti-de Sitter/conformal field theory correspondence

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Received February 15, 2023; Accepted April 24, 2023; Published April 25, 2023

In this paper, we show that bulk reconstruction in the anti-de Sitter/conformal field theory (AdS/CFT) correspondence is rather simple and has an intuitive picture, by showing that the HKLL (Hamilton-Kabat-Lifschytz-Lowe) bulk reconstruction formula can be simplified. We also reconstruct the wave packets in the bulk theory from the CFT primary operators. With these wave packets, we discuss the causality and duality constraints and find our picture is the only consistent one. Our picture of the bulk reconstruction can be applied to the asymptotic AdS spacetime.

Subject Index B21

1. Introduction and summary

The holographic principle [1,2] is one of the most important concepts in quantum gravity. The asymptotic anti-de Sitter (AdS) spacetime is the ideal setting for realizing the holographic principle and the AdS/conformal field theory (CFT) correspondence [3] explicitly realizes it. If the holographic principle is true, the bulk gravity theory is equivalent to the lower dimensional theory without gravity.

To understand this, the most important question will be how the states/operators in the bulk gravity theory are reconstructed from the states/operators in the lower dimensional theory without gravity. Indeed, if we can understand this bulk reconstruction, we can also explain how the bulk gravity theory emerges from the lower dimensional theory.

In the AdS/CFT correspondence, this bulk reconstruction has been given for the free bulk theory approximation around the AdS spacetime, which is a large N limit of the corresponding CFT [4–26]. Here this CFT is realized as a gauge theory with a rank N gauge group around the vacuum. In particular, assuming the BDHM (Banks-Douglas-Horowitz-Martinec) relation [4], which is an AdS/CFT dictionary like the GKPW (Gubser-Klebanov-Polyakov-Witten) relation [27,28], the explicit formula of the reconstruction of the bulk local operator by the CFT primary field was given in Ref. [13] and is called HKLL bulk reconstruction. This formula is given as an integration of the CFT primary field over a region in the spacetime of CFT with a weight called

Funded by SCOAP³. SCOAP3 and OUP support the goals of the International Year of Basic Sciences for Sustainable Development.

the smearing function. Unfortunately, this formula is not simple and rather counterintuitive with its causality as we will explain later.

In this paper, we show that the bulk reconstruction in the AdS/CFT correspondence is rather simple and has an intuitive picture, by showing that the HKLL bulk reconstruction formula can be simplified. Indeed, a bulk local operator at a point p is reconstructed from the CFT primary fields integrated over a submanifold M_p in the spacetime for CFT, where M_p is the intersection of the light-cone of the point p and the boundary of the AdS spacetime. Note that this is for the free bulk theory approximation around the AdS spacetime. Note also that this can be regarded as a direct consequence of the time evolution using the equations of motion of the free (light-like propagating) theory and the BDHM relation which identifies the bulk local operators on the boundary as the CFT primary fields. Although this picture was already obtained in Ref. [29] using the identification of the bulk and boundary operators in the energy eigen state basis [26], we mainly use the HKLL reconstruction formula in this paper partly because this formula is well known.

To be more precise, the above picture of the bulk reconstruction is exact for $\Delta = d - 1$ (or when Δ is an integer), where Δ is the conformal dimension of the CFT primary field. We expect that this picture is true for the energy momentum tensor and the conserved currents. For other Δ, the above picture is still valid for bulk local states/operators. If we would like to reconstruct bulk nonlocal operators, which is given by an integration of bulk local operators over a bulk region, we need the CFT primary fields at space-like separated boundary points, not just the ones at light-like separated points, for a generic Δ. In this paper, we concentrate on the bulk local states/operators.

We also reconstruct the wave packets in the bulk theory from the CFT primary operators. The bulk local operator can be represented by a linear combination of these wave packets at the same spacetime point, but moving in different directions. With these wave packets, we discuss the causality and duality constraints and find our picture is the only consistent one.

The identification of bulk operators that correspond to CFT operators in a subregion is also important, in particular for the understanding of the quantum entanglement in the AdS/CFT correspondence. We identify these explicitly and show that the causal wedge naturally appears for the ball-shaped region. However, our results show that a (version of) subregion duality does not hold. This subregion duality is partly based on the HKLL AdS-Rindler reconstruction; moreover, there are some misunderstandings of it. Indeed, we show that bulk correlation functions in the HKLL AdS-Rindler reconstruction and the corresponding ones in the HKLL global AdS reconstruction are different due to the null geodesics discussed in Ref. [30]. This means that the bulk local operators reconstructed from these two reconstructions are different even in the low energy modes.

Our picture of the bulk reconstruction can be applied to the asymptotic AdS spacetime, assuming the BDHM relation. In the CFT, this background is a nontrivial state which has the bulk semiclassical description. In particular, we argue that the bulk reconstruction for a (single-sided) black hole is only possible for the spacetime outside the (stretched) horizon.

This paper is organized as follows. In the next section we give the simple reconstruction of the bulk local operators and bulk wave packets. We also study the bulk operators for CFT operators in a subregion and discuss the subregion duality and the HKLL AdS-Rindler reconstruction. In Sect. 3, we generalize our reconstruction picture to bulk theories on asymptotic AdS spacetimes.

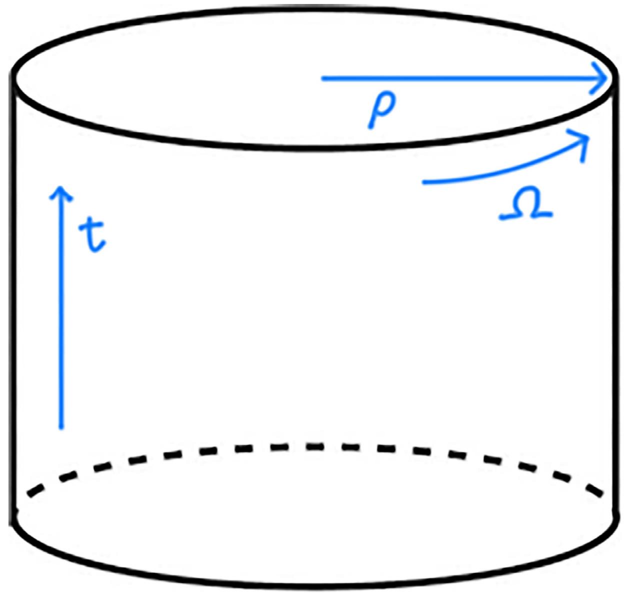

Fig. 1. AdS space.

2. Simple reconstruction of bulk local operators

2.1. *Relation between free scalar field on AdS_{d+1} and large N CFT_d*

In this subsection, we will review the relation between a free scalar field on AdS_{d+1} and CFT_d in the large N limit around the vacuum, where the bulk theory is free.

Let us consider the free scalar field with the following action:

$$S_{\text{scalar}} = \int d^{d+1}x\sqrt{-\det(g)}\left(\frac{1}{2}g^{MN}\nabla_M\phi\nabla_N\phi + \frac{m^2}{2}\phi^2\right), \tag{1}$$

where $M, N = 1, \cdots, d+1$, on global AdS_{d+1}.[1] The AdS_{d+1} metric is

$$ds^2_{\text{AdS}} = -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2 + r^2 d\Omega^2_{d-1}, \tag{2}$$

where $0 \le r < \infty$, $-\infty < t < \infty$, and $d\Omega^2_{d-1}$ is the metric for the $d-1$-dimensional round unit sphere S^{d-1}. We set the AdS scale $l_{AdS} = 1$ in this paper. By the coordinate change $r = \tan\rho$, the metric is also written as

$$ds^2_{\text{AdS}} = \frac{1}{\cos^2(\rho)}\left(-dt^2 + d\rho^2 + \sin^2(\rho)d\Omega^2_{d-1}\right), \tag{3}$$

where $0 \le \rho < \pi/2$. With this coordinate, the AdS spacetime is like a solid cylinder as shown in Fig. 1. The CFT primary field which is dual to the bulk scalar is $\mathcal{O}_\Delta(t, \Omega)$ with the conformal dimension $\Delta = d/2 + \sqrt{m^2 + d^2/4}$ on the cylinder $ds^2_{\text{cyl}} = -dt^2 + d\rho^2 + \sin^2(\rho)d\Omega^2_{d-1}$.[2]

The BDHM relation [4],

$$\lim_{z\to 0}\frac{\phi(t, z, \Omega)}{z^\Delta} \sim \mathcal{O}_\Delta(t, \Omega), \quad \text{where } z = \pi/2 - \rho \tag{4}$$

is the relation between the bulk field and the CFT scalar primary field. Using this relation and the bulk equations of motion, the bulk local field is reconstructed from the CFT primary field, as in the HKLL bulk reconstruction [13], in the large N limit where the bulk theory is free. Thus,

[1] In this paper, we consider the bulk scalar field only although the generalizations of our picture to the gauge fields and the gravitons may be straightforward.

[2] In this paper, we consider this choice of Δ, which implies $\Delta \ge d/2$, for simplicity.

in this limit, we can explicitly study the bulk local operators from the CFT viewpoint. For this study, it is more convenient to express the operators or the states on the energy eigen state basis. This was done explicitly in Ref. [26] and the relation between the bulk local operators and the CFT primary fields is clear, as shown in Appendix A.

2.2. *Reconstruction of bulk local operators*

Below, using the HKLL bulk reconstruction formula [13], we will see how the bulk local operator is reconstructed. The results are rather simple. For example, the bulk local state at the center is given by the time evolution of uniformly distributed CFT states which are localized at the fixed time. Thus, we conclude that through this kind of analysis, we understand how the bulk local operator emerges.

Let us consider the bulk local operator at the center of AdS space, i.e. $\phi(\rho = 0)$, at $t = 0$ and the corresponding bulk local state at the center, $\phi(\rho = 0)|0\rangle$. It was shown in Ref. [29] that this bulk state is essentially equivalent to the following CFT state: $\int_{S^{d-1}} d\Omega\, e^{i\frac{\pi}{2}H}\mathcal{O}_\Delta(\Omega)|0\rangle$, i.e.

$$\phi(\rho = 0)|0\rangle = \int_{S^{d-1}} d\Omega\, e^{i\frac{\pi}{2}H}\mathcal{O}_\Delta(\Omega)|0\rangle, \tag{5}$$

up to an overall numerical constant (Fig. 2). More precisely, this expression is valid only for $\Delta = d - 1$ and the general expression, which includes the time-derivatives, is given in Appendix A. This CFT state $\int_{S^{d-1}} d\Omega\, e^{i\frac{\pi}{2}H}\mathcal{O}_\Delta(\Omega)|0\rangle$ at $t = 0$ is obtained from the CFT state $\int_{S^{d-1}} d\Omega\, \mathcal{O}_\Delta(\Omega)|0\rangle$ at $t = -\pi/2$ by the time evolution to $t = 0$.[3] (The operator $\int_{S^{d-1}} d\Omega\, \mathcal{O}_\Delta(\Omega)$ is a CFT local state averaged over the whole space and then it is invariant under the rotation.)

This equivalence, or the bulk reconstruction, has a simple picture if we regard the CFT local operators as the bulk operators on the boundary using the BDHM relation. The bulk local operator at the center is formed by the boundary operators located on S^{d-1} at $t = -\pi/2$, which are connected to the bulk local operator by the (spherical symmetric) light-like rays.

We note that the bulk local operator contains arbitrary high-momentum modes because it is supported at a point, thus the mass of the bulk scalar field is negligible, which is responsible for the light-like behavior.[4]

One might think this is different from the well-known HKLL bulk reconstruction [13], in which the bulk local operator is constructed from the CFT operators on the time-interval $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, not just on $t = -\pi/2$. Below, we will show that the HKLL bulk reconstruction

[3] Here, we considered the state. The corresponding operator is $\mathcal{O}_\Delta(\Omega, t = -\pi/2) = (e^{-\frac{\pi}{2}\frac{\partial}{\partial t}}\mathcal{O}_\Delta(\Omega, t))|_{t=0}$.

[4] The local state is a kind of a sum of very small wave packets, which have very high momentum, and then it behaves like a massless field. We will explain this more precisely. The local operator itself is not a well-defined operator and we need a smearing of it with a length scale $1/M$, e.g. by the Gaussian factor. Because this M is like an energy cutoff, the number of modes effectively contained in this smeared operator is proportional to M^d. The mass is negligible for energy modes with high energy compared with the mass. If M is comparable with the mass of the scalar field, then mass cannot be neglected. However, for the local operator, M should be much larger than the mass, then the mass is negligible for almost all modes. If we consider the corresponding state, the low energy modes are suppressed for large M because of the normalization of the state which contains a huge number of the modes. Therefore, the effect of the mass is suppressed by the cutoff $1/M$. One might think that the two-point function of the local operator depends on the mass, which contradicts with the above statement. However, it is well known that the two-point function (retarded or Feynman) is divergent if the two operator insertion points are connected by a null-geodesics, even for the massive scalar field. This divergence is regularized by M and the mass is negligible in the large M limit. Thus, there is no contradiction. Of course, if M is not taken to be large compared with the mass, the result depends on the mass.

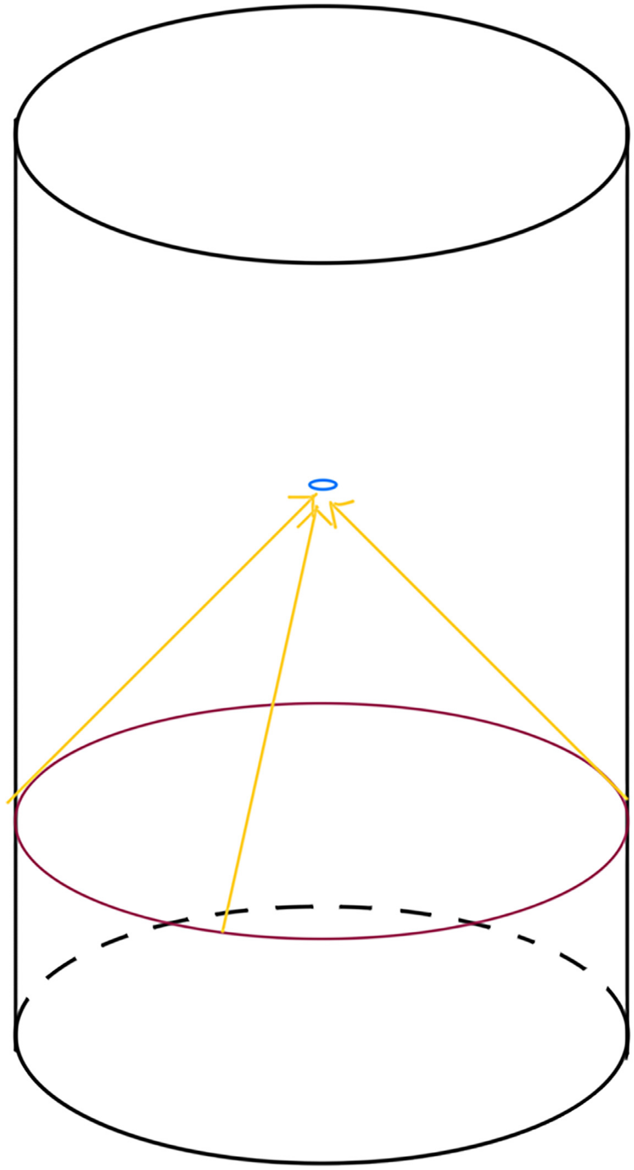

Fig. 2. The blue dot represents the center ($\rho = 0$) at $t = 0$ where we consider the bulk local state $\phi(\rho = 0)|0\rangle$. The red circle represents the $t = -\pi/2$ slice on the boundary where we consider $\int_{S^{d-1}} d\Omega\, \mathcal{O}_\Delta(\Omega)|0\rangle$, which is equivalent to the bulk local state by the time evolution. The yellow lines represent the light-like trajectories from the $t = -\pi/2$ slice on the boundary to the center at $t = 0$.

indeed gives the same picture by a careful analysis. Of course, this is not surprising because the HKLL bulk reconstruction and the bulk reconstruction given in Appendix A are equivalent. The HKLL bulk reconstruction of the bulk local operator at the center is the following:

$$\phi(\rho = 0, t = 0) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt' \int_{S^{d-1}} d\Omega'\, K(\Omega', t')\mathcal{O}_\Delta(\Omega', t'), \tag{6}$$

where the smearing function $K(\Omega', t')$ for $d =$ odd is given by

$$K(\Omega, t) \sim \frac{1}{(\cos t)^{d-\Delta}} \tag{7}$$

up to a constant depending on Δ and d. For $d =$ even, the smearing function is $K(\Omega, t) \sim \frac{\log t}{(\cos t)^{d-\Delta}}$. If $d - \Delta \geq 1$, an integration $\int dt' K(\Omega', t')$ is divergent near $|t'| = \pi/2$, which means

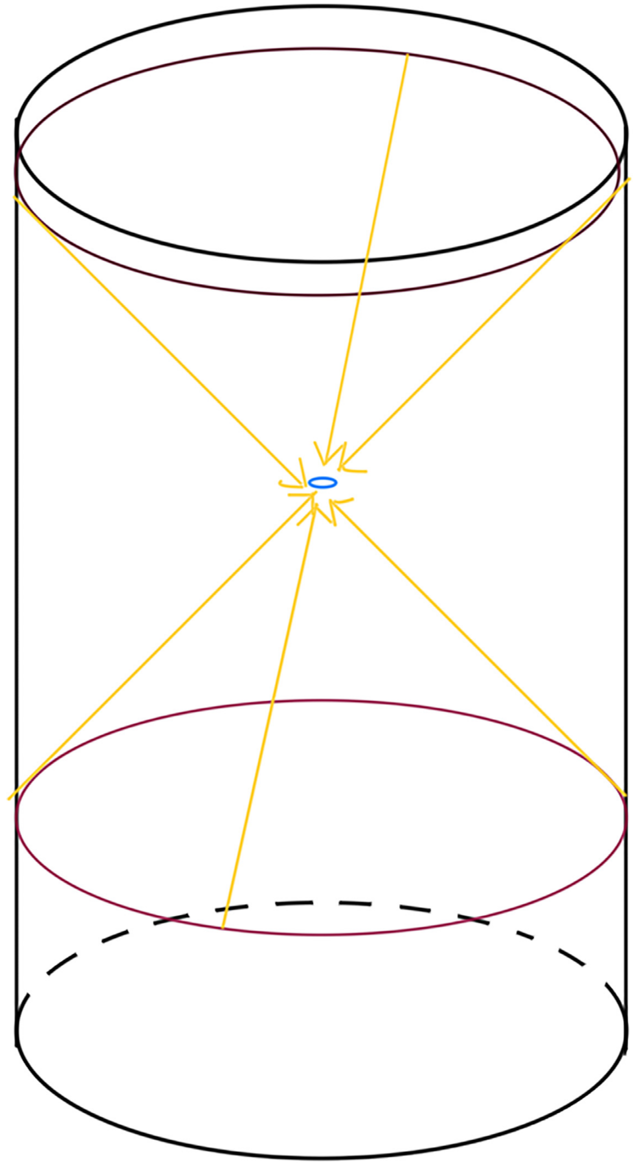

Fig. 3. The bulk reconstruction by Eq. (8). Instead of the interval $-\pi/2 \geq t \geq \pi/2$, only the contributions from the $t = \pm\pi/2$ slices are needed to reconstruct the bulk local operator.

that the contributions of the CFT local operator $\mathcal{O}_\Delta(\Omega', t')$ in the HKLL reconstruction (6) are negligible for $|t^{'}| < \pi/2$.[5] Thus the time integration is localized on $t^{'} = \pm\pi/2$:

$$\phi(\rho = 0, t = 0) \sim \int_{S^{d-1}} d\Omega' \, \big(\mathcal{O}_\Delta(\Omega', t = -\pi/2) + \mathcal{O}_\Delta(\Omega', t = \pi/2)\big) \,, \tag{8}$$

as shown in Fig. 3. (Again, this expression is valid only for $\Delta = d - 1$ and the general expression, which includes the time-derivatives, is given in Appendix A.) The previous picture is obtained for the bulk local state $\phi(\rho = 0, t = 0)|0\rangle$ if we notice that the CFT operator $\int_{S^{d-1}} d\Omega' \, \mathcal{O}^{+}_\Delta(\Omega', t)$, where $\mathcal{O}^{+}_\Delta(\Omega', t)$ is the positive energy modes of $\mathcal{O}_\Delta(\Omega', t)$, at $t = \pi/2$ is the same as the one at $t = -\pi/2$ up to an overall constant because of the time periodicity in the large N limit and the parity invariance.

[5]The divergence will be related to the divergence of the local operators as the operators acting on the Hilbert space and we need some regularization as discussed in Ref. [29].

Note that this localization of the integration is indeed required by the bulk causality and the BDHM relation $\mathcal{O}_\Delta(\Omega, t) \sim \phi(\rho = \pi/2, \Omega, t)$, i.e. the identification of the CFT primary operator as the bulk operator at the boundary. The bulk operator at the center $\phi(\rho = 0, t = 0)$ should be independent from the bulk operator at the spacetime point outside the light-cone of the center $\{\rho = 0, t = 0\}$; however, the boundary point $\{\rho = \pi/2, \Omega, t\}$ is obviously not within the light-cone for $|t| < \pi/2$. Thus, the bulk reconstruction of $\phi(\rho = 0, t = 0)$ should commute with any local operator constructed from $\mathcal{O}_\Delta(\Omega, t)$ for $|t| < \pi/2$. Thus, the CFT operators on $\mathcal{O}_\Delta(\Omega, t)$ for $|t| = \pi/2$ should be dominant for the reconstruction of the bulk local operator and our picture is natural and consistent with the bulk causality. In this sense, we can say that the HKLL reconstruction formula (6) is correct, but misleading.

For $d - \Delta < 1$, the HKLL reconstruction seems to give a completely different picture because the smearing function is zero on $t = \pm\pi/2$. However, even for this case, the above picture, in which only the CFT operators on an arbitrary small region containing $t = \pm\pi/2$ are relevant, is correct as we will see in Appendix A.3, essentially because $t = \pm\pi/2$ is the branch point, thus the singular point, of the hypergeometric function that appeared in the HKLL reconstruction.

Next, we will consider the bulk local operator which is not at the center, $\phi(\rho, \Omega, t = 0)$, and the CFT operator corresponding to it. For this, the HKLL reconstruction formula was obtained by the conformal map from the formula for the center [13]. Thus, the above picture is also true for this case. More explicitly, the reconstruction formula is the same as Eq. (6),

$$\phi(\rho, \Omega, t = 0) = \int dt' \int d\Omega'\, K(\rho, \Omega, \Omega', t')\mathcal{O}_\Delta(\Omega', t'), \tag{9}$$

where the integrations of t' and Ω' were taken over the boundary points which are space-like and separated from the bulk operator insertion point $\{\rho, \Omega, t = 0\}$ and the smearing function is

$$K(\rho, \Omega, \Omega', t') \sim \frac{1}{(d(\rho, \Omega, \Omega', t'))^{d-\Delta}}, \tag{10}$$

where $d(\rho, \Omega, \Omega', t')$ is the geodesic distance between the point $\{\rho, \Omega, t = 0\}$ and the boundary point $\{\rho = \pi/2, \Omega', t'\}$. The geodesic distances are zero for the the boundary points connected to the bulk operator insertion point by the light-like trajectories, then the integration is localized on such boundary points. This means that the bulk local operator is composed by the CFT primary operators on such boundary points[6] and the bulk reconstruction picture above is true for this case also.[7]

2.3. *Bulk wave packet moving in a particular direction*

We have seen that a bulk local state is represented by CFT primary states integrated on a space-like surface, which is regarded as a time-slice with an appropriate definition of a time. In this section we will consider CFT primary operators integrated on a small and an almost point-

[6]The integration is localized on the future and the past boundary points. If we consider the ϕ^+ or the corresponding state, the integrations on these two regions are the same up to a constant as for the case with the bulk local operator at the center. This is because the time translation $t \to t + \pi$ gives the factor $e^{i\pi(2n+l+\Delta)}$ and the parity transformation in space S^{d-1} gives $(-1)^l$ [29]. The combination of these gives only a phase $e^{i\pi\Delta}$.

[7]We have considered the one particle state. For multiparticle states, we have $\phi(\rho_1, \Omega_1, t = 0)\phi(\rho_2, \Omega_2, t = 0)\cdots\phi(\rho_q, \Omega_q, t = 0)|0\rangle = \phi^+(\rho_1, \Omega_1, t = 0)\phi^+(\rho_2, \Omega_2, t = 0)\cdots\phi^+(\rho_q, \Omega_q, t = 0)|0\rangle$ in the large N limit if $\{\rho_i, \Omega_i\}$ are different from each other. Thus, the bulk reconstruction of the state can be done from the CFT operators on only the past boundary points.

like region, instead of the space-like surface. We will see that the bulk local state moving in a particular direction[8] corresponds to a state obtained by such a CFT operator acting on the vacuum.

We will consider the following operator:

$$\phi_A \equiv \int_{S^{d-1}} d\Omega\, \mathrm{f}_A(\Omega)\, e^{i\frac{\pi}{2}H} \mathcal{O}_\Delta(\Omega) e^{-i\frac{\pi}{2}H}, \tag{11}$$

where $\mathrm{f}_A(\Omega)$ is a function (effectively) supported on a small region A in S^{d-1}. More explicitly, we take a small ball-shaped region with radius l_A for A and the Gaussian distribution for $\mathrm{f}_A(\Omega)$. If we average the position of A over the whole space S^{d-1}, this operator becomes the bulk local operator at the center, as we have seen. Precisely speaking, we need to apply some derivatives on $\mathcal{O}_\Delta(\Omega)$ in Eq. (11) as explained in Appendix A.1. Below, we will omit such derivatives for notational simplicity.

Thus, it is natural to think ϕ_A represents a part of the bulk local operator at the center. Indeed, in Ref. [29], it was shown that $\phi_A|0\rangle$ is a bulk (almost) local state or a wave packet at the center moving in the radial direction. Note that the bulk local operator should be smeared over a small region, but one larger than the Planck-scale region.

Furthermore, the following operator:

$$\phi_A(z) \equiv \int_{S^{d-1}} d\Omega\, \mathrm{f}_A(\Omega)\, e^{izH} \mathcal{O}_\Delta(\Omega) e^{-izH}, \tag{12}$$

was shown to be a bulk wave packet moving inward in the radial direction at $\rho = \pi/2 - z$, where $\pi/2 \geq z > 0$, and Ω is the location of the small region A [29].[9] Of course, this CFT operator is the time evolution of the CFT operator at $t = -z$ to $t = 0$ (Fig. 4).

This result also has a simple interpretation in the bulk picture as before. The CFT operator of ϕ_A is regarded as the bulk operator on the boundary smeared over region A by the Gaussian distribution, but not smeared for the radial direction. (More precisely, it should be smeared for the time-direction with some length scale which is very small, but larger than the Planck scale [29].) This is a kind of bulk wave packet. A usual wave packet is defined by choosing a particular momentum. For our case, the momentum for the angular direction is $\mathcal{O}(1/l_A)$, but the momentum for the radial direction is not constrained, then most of them are much larger than $1/l_A$. Thus, the operator is a linear combination of wave packets and most of them are moving in the radial direction. This means that at $t = 0$ the wave packet is at $\rho = -z$ and $\phi_A(z)$ represents a small wave packet localized at the center moving in the radial direction.

On the other hand, in the CFT picture, the operator localized in region A at $t = -\pi/2$ is spread out spherically (S^{d-2}) in space (S^{d-1}) at $t = 0$. Here, very large momentum modes up to $1/l_A$ are contained in the operator and this spreading is light-like [31].[10] Thus, the operator at $t = 0$ is localized in a "great circle" in S^{d-1}. For the $d = 2$ case, the "great circle" is two antipodal points in S^1 (Fig. 5). Even for $d > 2$, if we take A in the definition of ϕ_A as an ellipsoid which is squeezed in a particular direction, then the operator ϕ_A is localized on the two antipodal points in S^{d-1} in the CFT picture. Furthermore, this ϕ_A is still localized at the center in the

[8] More precisely, what we will consider is a kind of wave packets whose size is much larger than the cutoff (Planck) scale, but much smaller than any other length scale.

[9] For $\pi/2 \leq z < 0$, the corresponding bulk wave packet is at $\rho = \pi/2 - |z|$ and moving outward in the radial direction.

[10] The nontrivial CFT commutator in the Källén–Lehmann-like representation contains infinitely many massive modes [31] and one might think that this light-like spreading is not valid. In Appendix B, we will show that this is indeed valid.

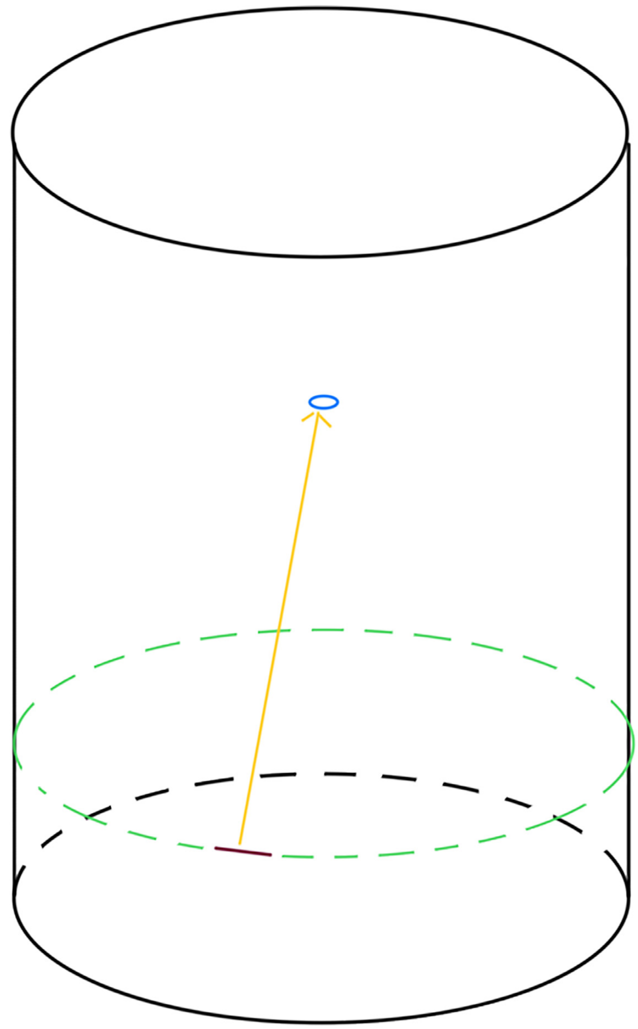

Fig. 4. The bulk wave packet moving toward the center. The small region A, which is represented by small red curves, is on the $t = -\pi/2$ slice on the boundary, which is represented by the green dotted circle. The yellow line to the center is the light-like trajectory from A and represents the wave packet.

bulk picture if we take the length scales of the ellipsoid to be much larger than the cutoff scale (smearing) of the local operator as per the ball-shaped region case.

This implies that this bulk localized wave packet is constructed as the entangled state in CFT, which can be written schematically as $(|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) \otimes |\text{others}\rangle$, where $|1\rangle$ is the one particle state, $|0\rangle$ is the no excitation state, and the first and second kets represent the states localized on the two antipodal points. On the other hand, a linear combination of CFT primary states (with a small number of derivatives) at two different points is a linear combination of bulk states localized on two points on the boundary, not inside, even in the bulk picture, although this is also an entangled state.[11]

[11] This implies that if we choose one of the antipodal points at $t = 0$ in Fig. 5 and extract the state localized there, it cannot be realized by the primary operators (with a small number of derivatives). The primary operators with a large number of derivatives on the antipodal points at $t = 0$ can realize this state, which is the local state there. However, this is effectively a nonlocal state in the low energy theory because of the derivatives, which can be regarded as a result of the Reeh–Schlieder theorem [29].

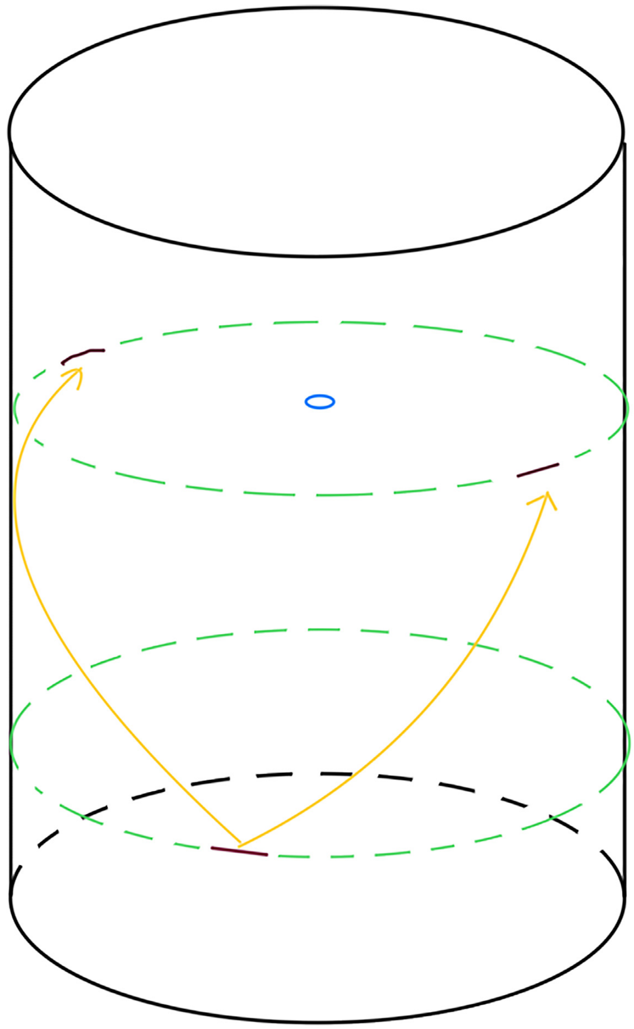

Fig. 5. The CFT picture of Fig. 4. Here, the two green dotted circles represent the $t = 0$ and $t = -\pi/2$ slices. The two yellow curves on the boundary from A to the antipodal points on the boundary are the light-like trajectories from A and represent the wave packet in the CFT picture.

We have seen that the bulk local operator moving in the radial direction is represented by $\phi_A(z)$ in the CFT picture. It is easy to generalize this to the bulk local operator moving in an arbitrary direction by considering a conformal map, which maps the bulk local operator at the center to an arbitrary point, say $\rho = \rho_1$, $\Omega = \Omega_1$, $t = t_1$ [13]. This conformal map changes the $t = -\pi/2$ slice (S^{d-1}) on the boundary to the slice on $t = -\pi/2 + \delta t(\Omega)$ where $\delta t(\Omega)$ represents the tilt in the time direction. Thus, if we generalize $\phi_A(z)$ to

$$\phi_A(z) \equiv \int_{S^{d-1}} d\Omega \, \mathrm{f}_A(\Omega) \, e^{izH} \mathcal{O}_\Delta(\Omega, t = \delta t(\Omega)) e^{-izH} \tag{13}$$

$$= \int_{S^{d-1}} d\Omega \, \mathrm{f}_A(\Omega) \, e^{i(z+\delta t(\Omega))H} \mathcal{O}_\Delta(\Omega) e^{-i(z+\delta t(\Omega))H}, \tag{14}$$

this $\phi_A(z)$ represents the bulk local operator moving from the small region A on the $t = -\pi/2$ slice to the point $\rho = \rho_1$, $\Omega = \Omega_1$, $t = t_1$ in a light-like way where z parametrizes the light-like trajectory. The wave packet is at the small region A for $z = 0$ and at the bulk point $\rho = \rho_1$, $\Omega = \Omega_1$, $t = t_1$ for $z = \pi/2 + t_1$. This picture is also checked in the operator formalism as we will see in Appendix A.2.

In summary, we have the following simple and intuitive picture of the reconstruction of the bulk local operator or state in the AdS/CFT correspondence around the vacuum. The bulk local state at a spacetime point P is represented by the CFT primary state integrated over C, where the region C is the intersection of the light rays emanating from P to the past and the boundary of AdS spacetime. The wave packet of the bulk local state at the point P is represented by the CFT primary state integrated over A, where the very small region A is the intersection of the trajectory of the wave packet to the past and the boundary of AdS spacetime. For the bulk local operator instead of the state, we just need to take into account the light rays to the future in addition to the ones to the past.

2.3.1. *Causalities and duality.* Here, we will check that the above picture is consistent with the causalities and the duality. For the light-like trajectory from a boundary point toward the center, as drawn in Figs. 4 and 5, it was already shown in Ref. [29] that the bulk causality is consistent with the CFT picture. Indeed, in both the CFT and bulk pictures, the (light-like) wave packets reach the boundary at the antipodal point at the same time. This result was already shown in Ref. [32] where, in AdS spacetime, any light-like trajectory from a boundary point at $t = 0$ reaches the antipodal point on the boundary at $t = \pi$. Such a light-like trajectory can be always on the boundary, which is realized in the CFT picture. Thus, any light-like trajectory from a boundary point in the bulk picture and the corresponding two light-like trajectories in our CFT picture reach the antipodal boundary point at the same time, therefore this picture is consistent with the causality. Furthermore, this is the only possibility for the consistent CFT dual to the bulk wave packet[12] if we require the causalities in both the bulk and the CFT pictures.[13] This is because the bulk wave packet on the boundary, which is realized when $t = 0$ or $t = \pi$, is regarded as a wave packet composed by the CFT primary field and the speed of the propagation is bounded by the speed of light in both pictures. This implies that in the CFT picture also, the corresponding wave packets should be on the light-like trajectories.

2.3.2. *Locality in radial direction.* Here, we will discuss how the locality in the radial direction appears in our picture of the bulk reconstruction. Note that the discussions here are only for the free theory limit in the bulk theory. For the spherical symmetric states (e.g. in Fig. 2), the bulk local state integrated over the sphere S^{d-1} at the radial coordinate $\rho = \pi/2 - z_0$ corresponds to the CFT primary state on the time $t = -z_0$ slice integrated over the sphere S^{d-1}. This implies that the radial locality is the same as the locality in the time direction in this setup. In order to realize this, it is necessary that the primary fields with different times are (almost) independent (for $-\pi/2 < t < \pi/2$). This is only possible for a theory with (almost) infinite degrees of freedom like the large N gauge theory. Indeed, for the free CFT or a theory with finite degrees of freedom, the fields with different time are related by the equation of motions and they are not independent.

[12] In this paper, we take $\Delta = \mathcal{O}(1) = \mathcal{O}(1/l_{AdS})$. This implies that if we consider the wave packet whose size should be much smaller than the AdS scale l_{AdS}, the mass is negligible and the trajectory of the wave packet is light-like.

[13] The causality constraints were also studied in Ref. [33] although the bulk reconstruction picture is different from ours.

For the nonspherical symmetric case, the radial location of the bulk local operator is related to the time coordinate of the CFT primary fields although the relation is not direct. The independence between CFT primary fields at different times is important for this case also. This independence means that the commutators of the CFT primary fields at different times vanish. For the free CFT, the commutators of two free fields do not vanish if one field is in the light-cone of the other field. Of course, this property is the same for any CFT; however, for the nontrivial CFT, a commutator of the CFT primary fields takes a nonzero value on the light-cone and is diverging at $\sin t = 0$ where t is the time difference between the two fields, as was explicitly shown for $\Delta = d/2$ [31]. This divergence is regularized by the smearing of the local operator with a cutoff and the normalized commutator by this cutoff is effectively zero for CFT primary fields at different times.

Note that this discussion can be applied for any nontrivial CFT. The holographic CFT[14] is special because it behaves like the generalized free field, which implies that the products of the CFT primary fields are independent of each other. In the commutator or the OPE (Operator product expansion) languages, this is reflected in the fact that the contribution proportional to the identity operator is dominant in the large N limit. Thus, in our picture, the bulk locality in the radial direction is essentially mapped to the locality in the time direction in the CFT.

2.4. *Bulk operators correspond to CFT operators in a region*

We have seen how bulk local operators are mapped to the CFT operators. We can also map the CFT operators supported in a region A in S^{d-1} on the $t = 0$ slice, $\{\mathcal{O}_A\}$, to the bulk operators on the $t = 0$ slice [29]. In particular, we will consider what is the (smallest) bulk region a corresponding to the CFT region A, where the bulk operators corresponding to $\{\mathcal{O}_A\}$ are supported in the bulk region a. (Note that if the subregion duality [35,39] is correct, these bulk operators should be equivalent to $\{\mathcal{O}_A\}$, i.e. any bulk operator supported in the region a should correspond to some $\mathcal{O}_A$, in the low energy approximation. Here we do not assume this version of the duality. We will call it the strong version of the subregion duality, in order to distinguish it from a version of the subregion duality where correspondence between the density matrices is required, as we will explain later.)

We note that any CFT operator (at $t = 0$) can be generated[15] by the CFT primary operator integrated over the spacetime[16] with a weight $\mathrm{f}(\Omega, t)$:

$$\phi_f \equiv \int dt \int_{S^{d-1}} d\Omega \, \mathrm{f}(\Omega, t) \mathcal{O}_\Delta(\Omega, t), \tag{15}$$

because derivatives of the primary field can be represented by taking the distribution $\mathrm{f}(\Omega, t)$ as derivatives of the (smeared) delta function.

Let us investigate which ϕ_f is a CFT operator supported in the region A. It is obvious that ϕ_f is supported in region A if $\mathrm{f}(\Omega, t)$ is nonzero only in the causal diamond of region A in the CFT picture because of the causality (Fig. 6). More generally, as we have seen in the previous section, a small wave packet of CFT local operators at (Ω_0, t_0) can behave like two light-rays emitted from there in opposite directions in space S^{d-1}. If both of the two light rays reach

[14] In the operator formalism, the holographic CFT is the CFT with the properties assumed in Ref. [26].

[15] In the large N limit, the bulk theory is free, i.e. the CFT we consider is the generalized free theory. Thus, properties of a product of ϕ_f's, which correspond to multiple particles, follow from the properties of ϕ_f, which corresponds to one particle.

[16] Of course, here we regard $\mathcal{O}_\Delta(\Omega, t) = e^{itH} \mathcal{O}_\Delta(\Omega) e^{-itH}$ as an operator at the $t = 0$ slice.

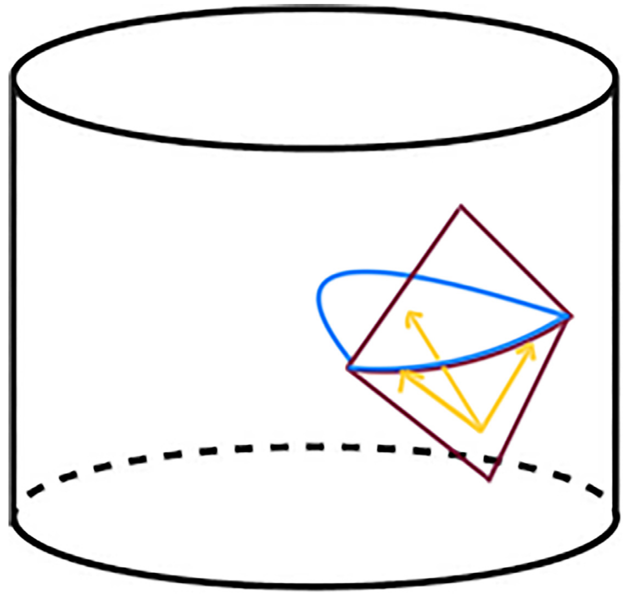

Fig. 6. The red lines represent the region A on the $t = 0$ slice and its causal diamond on the boundary. The causal wedge of A on the $t = 0$ slice is the bulk region inside of the blue curve. The yellow lines are light-like trajectories. The one of these lines heading toward the inside of the bulk represents the wave packet in the causal wedge. The other two are on the boundary and represent the wave packets in the CFT picture.

region A on the $t = 0$ slice, this wave packet of CFT local operators is supported in region A. Thus, by taking $\mathrm{f}(\Omega, t)$ as this wave packet at (Ω_0, t_0), ϕ_f is an operator supported in region A. It is difficult to expect other possibilities of choice of $\mathrm{f}(\Omega, t)$ to obtain that ϕ_f is an operator supported in region A because the propagation is generic other than the causality constraints. Thus, below we assume that ϕ_f is a CFT operator supported in region A only if $\mathrm{f}(\Omega, t)$ is a linear combination of such wave packets.

2.4.1. *Ball-shaped region.* First, we take the subregion A as a ball-shaped region in S^{d-1}. For this case, in the CFT picture, the above wave packets are in the causal diamond if the two light rays reach region A on $t = 0$. Thus, ϕ_f is supported in region A only if $\mathrm{f}(\Omega, t)$ is nonzero only in the causal diamond of region A (Fig. 6).

In the bulk picture, as we have seen in the previous section, the bulk operator corresponding to such ϕ_f represents a linear combination of wave packets moving in arbitrary directions in bulk space. Such wave packets in the bulk can reach only the causal wedge of A on $t = 0$. This means that any CFT operator supported in region A corresponds to a bulk operator supported in the causal wedge of A. Note that the causal wedge (on $t = 0$) is the bulk region inside the Ryu–Takayanagi surface of A [34], i.e. the entanglement wedge.[17]

[17] The entanglement entropy for a region a in a QFT (quantum field theory) is known to be proportional to the area of the region a. Thus, if we regard the perturbative bulk theory as a QFT, that the Ryu–Takayanagi surface appears in our setup might be natural. However, this area law depends on the UV (ultra violet) cutoff. Furthermore, e.g. for $d = 2$ CFT, the entanglement entropy depends on the central charge c only in the large c limit although the bulk theory with a fixed central charge can have any number of scalar fields which give different entanglement entropies as a QFT. Thus, it is not obvious that the relation between the entanglement entropy computation using the Ryu–Takayanagi surface and the low energy bulk states corresponds to the CFT region A.

2.4.2. *Subregion duality, error correction code, and AdS-Rindler reconstruction.* It is important that there exist bulk operators supported in the causal wedge of A which cannot be reconstructed from CFT operators supported in region A. Indeed, there are bulk light-rays or null-geodesics starting from a boundary point and heading to another boundary point through the causal wedge of A such that both of the two boundary points are not in the (CFT) causal diamond of A. Then, the wave packets along these cannot be reconstructed from CFT operators supported in region A. Note that in the large N (free bulk theory) limit we have considered in the paper, any state with a (Planck-scale) cutoff of the energy, which is realized by a smearing of the local operator, can be regarded as a low energy state, and states of the bulk free theory and the CFT (which is approximated as the generalized free theory) are identical. In other words, we restricted only the low energy states in this paper. Thus, there is no alternative reconstruction from CFT operators supported in region A.

This has implications for the subregion duality [35–39]. Indeed, this violates a strong version of the subregion duality, which claims that any bulk operator supported in the causal wedge of A can be reconstructed from the low energy CFT operators supported in region A, for this setup. (We consider only the low energy CFT operators in this paper.) Problems of the strong version of subregion duality related to such null-geodesics were already raised in Ref. [30]. In Ref. [30], it was stated that some nonlocal operators could solve the problems although there are no concrete arguments for this. In our setup there are no such operators solving the problem because the low energy CFT spectrum is given and the operators associated to the null-geodesics are explicitly given by the CFT primary operators, which are not supported in region A.[18] Thus, the strong version of the subregion duality is not valid.[19] Note that this duality is assumed in many papers, e.g. in Ref. [41] for the entanglement wedge reconstruction.

The strongest reason to believe this strong version of the subregion duality may be the HKLL AdS-Rindler reconstruction [13], by which the correlation function of the bulk local operators in the AdS-Rindler patch is reproduced from certain CFT operators in the corresponding subregion A [42]. This seems to implies that the bulk local operators can be reconstructed from CFT operators in a subregion, not in the whole space S^{d-1} as we have seen. However, the HKLL AdS-Rindler reconstruction assumes that the BDHM relation holds even in the AdS-Rindler patch. This assumption was recently shown to be violated because of the finite N effect [44], and the HKLL AdS-Rindler reconstruction is not correct. Thus, $\phi(x) \neq \phi_{\mathrm{Rindler}}(x)$, where we denote by $\phi_{\mathrm{Rindler}}(x)$ the part of the bulk local operator $\phi(x)$ which can be reconstructed from the CFT operators supported in the Rindler subregion A on the boundary.[20] Note that

[18] The precursor was introduced in Ref. [40] and this might comprise such operators. However, the energy of the wave packet considered in Ref. [40] is string scale, which is infinite in the approximation in this paper. Moreover, in our setup, we consider the state without introducing the source term and the bulk local state at the center at some time, which is bouncing by the boundary periodically by the time evolution, corresponds to the (nonlocal) CFT state (5), which is different from the vacuum at any time. Thus, there is no need to introduce the precursor.

[19] In other words, $\mathcal{A}_A^{CFT} \subset \mathcal{A}_a^{bulk}$, instead of the subregion duality $\mathcal{A}_A^{CFT} = \mathcal{A}_a^{bulk}$, in the low energy approximation, where $\mathcal{A}_A^{CFT}$ is the set of CFT operators supported on the ball-shaped region A and $\mathcal{A}_a^{bulk}$ is the set of bulk operators supported on the causal wedge a for region A.

[20] In the CFT picture, the modes of region A and region $\bar{A}$ will be complete as for the massless scalar in the Minkowski space. (It is complete if the primary operators with a large number of derivatives are regarded as local operators. In the low energy theory, such operators are not regarded as local operators [29] as mentioned in footnote 11.) One might think that this means that the modes of the corresponding two causal wedges of A and $\bar{A}$ are complete. This is not correct because the reconstruction of the bulk

the part of $\phi(x)$ which cannot be reconstructed from CFT on A is related to the horizon to horizon null-geodesics in Ref. [44].

Note that the quantum error correction code proposal [43] is partly based on the claim that $\phi(x) = \phi_{\mathrm{Rindler}}(x)$ in the low energy theory,[21] thus this proposal is not realized in holographic CFTs. Even from the following discussion using the HKLL bulk reconstruction (6), it is easy to see that $\phi(x) = \phi_{\mathrm{Rindler}}(x)$ is not valid. By inserting Eq. (A11) into Eq. (6) and integrating over the space S^{d-1} first, we can see that the bulk local operator at the center $\phi(\rho = 0)$ only contains the modes with $l = 0$, i.e. the spherical symmetric modes a_{n00}. These modes are generically low energy modes. On the other hand, it is obviously impossible to construct the operator, which only contains the spherical symmetric modes, from the CFT operators supported in a subregion in S^{d-1}. This means that $\phi_{\mathrm{Rindler}}(\rho = 0)$ of the AdS-Rindler reconstruction is different from $\phi(\rho = 0)$.

Of course, these discussions of the AdS-Rindler reconstruction are related to our picture in this paper. In particular, as we have seen, the wave packet reconstructed from a small subregion in CFT is a part of a bulk local operator reconstructed from a whole space S^{d-1}, i.e. the bulk local operator is obtained by assembling such wave packets for small subregions in whole space. The null-geodesics correspond to such wave packets. The AdS-Rindler reconstruction uses only the wave packets corresponding to the half of S^{d-1}.

The AdS-Rindler reconstruction and our bulk reconstruction picture lead to another version of the subregion duality, in which the (low energy) CFT operators supported in region A are dual to the part of bulk operators in the bulk theory in the AdS-Rindler patch M_A. We stress again that these are not equivalent to the bulk operators supported in the AdS-Rindler patch M_A in the bulk theory in the global AdS reconstruction. By tracing out the states supported in $\bar{A}$, this duality becomes the correspondence between the density matrix ρ_A of the region A in CFT and the density matrix ρ_{M_A} of the AdS-Rindler patch in the "bulk theory" which includes the part of the bulk local operators.

2.5. *Disjoint regions*

Now, we will take the subregion A as a space which is (topologically) different from the ball-shaped space. For the concreteness of the discussion, we will take A as a sum of two disjoint intervals A_1, A_2 for $d = 2$ CFT. (A ball-shaped region is an interval for $d = 2$.) For this case, we can apply the previous discussion on the ball-shaped region to each A_i. Then, we find that there are CFT operators supported in the region $A = A_1 \cup A_2$ that correspond to bulk operators supported in the causal wedge of A_1 or A_2. Other than these, there are other kinds of CFT operators supported in region A. Indeed, the time evolution of the CFT local operator at (Ω_0, t_0) to $t = 0$ is supported in region A if the intersection of the light-cone of (Ω_0, t_0) and the $t = 0$ slice in CFT is contained in $A = A_1 \cup A_2$. The bulk operators corresponding to such CFT operators are the wave packets emitting from (Ω_0, t_0) to the bulk in arbitrary directions. Then, these

local operators is nonlocal in the CFT picture (and the strong version of the subregion duality is not valid). Indeed, there exist the wave packets, which are almost bulk local operators, reconstructed from the CFT operators supported on both of the regions A and $\bar{A}$, for the null-geodesics. Note also that the bulk local operator itself is always supported in the whole space S^{d-1} in the CFT picture, as seen in Fig. 2.

[21] In this paper, we always consider the low energy theory and this equation is understood in the low energy theory.

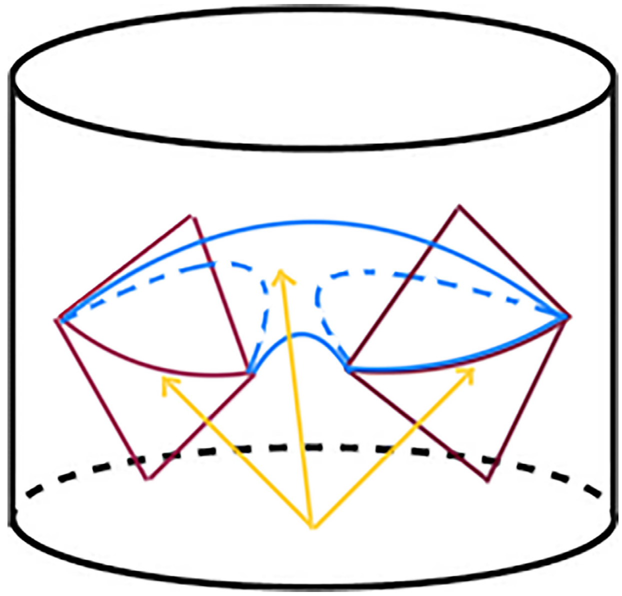

Fig. 7. Two intervals (region A) and their causal diamonds on the boundary are represented by red lines. Inside the blue curve is the entanglement wedge and inside the dotted blue curves are causal wedges. The two light rays, represented by the yellow lines on the boundary, reach region A at $t = 0$. Thus, the operator is supported in region A. The other yellow line, which represents the bulk picture of them, can reach the entanglement wedge outside the causal wedges at $t = 0$.

bulk operators are supported in the bulk region between the two minimal surfaces connecting A_1 and A_2 (Fig. 7) as shown in Ref. [29].

This region coincides with the entanglement wedge [35–37] if the sizes of A_i are sufficiently large compared with the distance between A_1 and A_2. It had been difficult to understand how to reconstruct the bulk operators outside the causal wedge.[22] It is interesting that we can explicitly reconstruct it here.

If the sizes of A_i are sufficiently small, it is believed that the entanglement wedge becomes the causal wedge and these bulk operators reconstructed from the CFT operators supported in A are outside the entanglement wedge of A. This phase transition–like behavior does not appear in our case and there seem to be some contradictions. However, the Ryu–Takayanagi surface appears in the computation of the entanglement entropy of region A using the replica trick in the path-integral, and there are no concrete arguments that this implies a relation between the Ryu–Takayanagi surface and the bulk region in which the bulk operators corresponding to A are supported. Furthermore, it is difficult to determine in which states the bulk modular Hamiltonian, which appears in the relative entropy [38], acts. Thus, there will be no contradictions between our results and the phase transition.[23]

[22]In Ref. [45], this problem was studied although there are some differences between our setup and the one in Ref. [45]. One such difference is that $\Delta \gg 1$ and the bulk local operator is smeared over the small length scale $1/\Delta$ in Ref. [45]. Here, the bulk local operator is smeared over a length scale much smaller than $1/\Delta$, thus the operator considered in Ref. [45] is regarded as a bulk nonlocal operator..

[23]The phase transition of the entanglement entropy occurs because of the exponential enhancement of the contribution of the minimal surfaces in the large N limit. However, in both the gravity computation [46] and CFT computation [48], it seems that the large N limit is taken first before taking the $n \to 1$ limit of the replica trick and these limits could not commute. Indeed, taking the $n \to 1$ limit first, the exponential suppression factor for the large N limit will disappear because the exponent is proportional to $n - 1$ also.

3. Generalization to asymptotic AdS spacetime

In this section, we will generalize our picture of the reconstruction of the bulk local state or operators to the asymptotic AdS spacetime. Here, we consider, instead of the vacuum, a CFT state $|g_{\mu\nu}\rangle$ which corresponds to the semiclassical asymptotic AdS spacetime background of the bulk theory. First, we will consider a part of the geometry in which the causal structure is trivial and there is no horizon for simplicity. We expect that the BDHM relation (4) holds for this case also and we assume it.[24] We also assume as usual that in the large N limit the fluctuations around this background is identical to free theory on this asymptotic AdS spacetime in the bulk picture. We only consider this perturbative approximation.

Let us consider a bulk local state at a point p on a time slice ($t = 0$) and the backward-time evolution of it. (Here, the time slice is defined by the dilatation operator, for example.) As in the AdS background, this evolution is on the past light-cone of the point p because the local operator contains arbitrary high-momentum modes. Then, the bulk local state is represented as the bulk local operators acting on $|g_{\mu\nu}\rangle$ integrated over a submanifold M_p^- on the boundary of the asymptotic AdS spacetime, where M_p^- is the intersection of this past light-cone and the boundary of the asymptotic AdS spacetime. Using the BDHM relation, we can replace the bulk local operators on the boundary as the CFT primary operators. Thus, the bulk local state at p is reconstructed from the CFT primary operators acting on $|g_{\mu\nu}\rangle$ integrated over M_p which we regard as a subregion in the spacetime for CFT.

Instead of the bulk local state $\phi(x)|g_{\mu\nu}\rangle$, we can consider the bulk local operator $\phi(x) = \phi(x)^+ + \phi(x)^-$ itself. Note that $\phi(x)|g_{\mu\nu}\rangle = \phi(x)^+|g_{\mu\nu}\rangle$ determines $\phi(x)^+$ up to an overall phase in the large N limit. In order to construct $\phi(x) = \phi(x)^+ + \phi(x)^-$, we need to give this phase. Note that the time evolution is symmetric for past and future. Thus, as for the AdS spacetime, we expect that the bulk local operator is reconstructed from the CFT primary operators integrated over $M_p = M_p^+ \cup M_p^-$ where M_p^+ and M_p^- are for the future and past light-cones, respectively.

The bulk operator corresponding to the wave packet moving along a light-like trajectory is also realized by a CFT operator, as for the AdS spacetime. Indeed, the positive energy parts of the CFT operator $\phi_A(z)$ defined by Eq. (14), which was constructed for AdS spacetime, represent the wave packet moving into the bulk in the direction determined by the tilt $\delta t(\Omega)$ near $z = 0$, i.e. on the boundary because near the boundary the BDHM relation is assumed to hold. Then, this represents the wave packet moving along the light-like trajectory of the asymptotic AdS geometry.

The causality and duality should hold for the above wave packet also. Assuming the null-energy condition for the bulk theory, by the Gao–Wald theorem [32], any "fastest null geodesic" connecting two points on the boundary must lie entirely within the boundary. On the other hand, the wave packets should reach the boundary point at the same time in both the bulk and CFT pictures because of the duality and the BDHM relation. Thus, the bulk wave packet,

As an illustration, $\sum_{i=1}^m e^{a_i(n-1)N^2} \to \sum_{i=1}^m a_i(n-1)N^2$ where a_i are constants. In the $n \to 1$ limit, the calculation becomes the infinitely small deformation of the geometry for which the partition function is proportional to the deformation of the action, not exponential of it. Thus, the phase transition itself may not be realized if we take the $n \to 1$ limit first, which is the correct way to obtain the entanglement entropy in the CFT which is dual to the quantum gravity on AdS. A similar modification of the Ryu–Takayanagi formula was proposed in Ref. [49].

[24] A generalization of [26] to general classical backgrounds was done in Ref. [50] and showed that the Einstein equation was derived with assumptions made in Ref. [26]. It is interesting to determine the spectrum around the background and derive the BDHM relation in this setup.

which is on a light-like trajectory, should correspond to time-like trajectories on the boundary in the CFT picture. This is explained by the group velocity of the wave packet in the CFT picture always being lower than the speed of light in a vacuum. In the CFT picture, the asymptotic AdS spacetime is represented as the nontrivial CFT state $|g_{\mu\nu}\rangle$ and the speed of the wave packet will become lower in this medium. Of course, in the bulk picture this effect of medium is included in the metric and the velocity of the wave packet in the bulk is the speed of light. Thus, our picture of the bulk local operators for the asymptotic AdS spacetime is consistent with the Gao–Wald theorem [32].[25]

3.1. *Black hole*

Let us consider the bulk reconstruction in a black hole in the asymptotic AdS spacetime. First, if we consider a (spherical symmetric) heavy star, then a bulk local state or a wave packet at the center of the star at $t = 0$ will be reconstructed by CFT primary operators integrated on a region in $t < -\pi/2$ because of the time delay. If the star is nearer the black hole, the time delay is larger and can be arbitrarily large, which means the whole past region ($t \leq 0$) of the CFT space S^{d-1} is needed to reconstruct the bulk local operators at $t = 0$. This means that an arbitrary large number of (approximately) independent fields are required because the bulk local operators at different points should be independent of each other and different. Of course, this is impossible because N is large, but finite, although N is formally infinite in the large N expansion.

For the black hole, the bulk local states or wave packets near the horizon need infinitely many independent fields. This implies that the CFT cannot reconstruct the bulk fields even outside the horizon. This is the same conclusion reached by the brick wall [52,53] where some hypothetical boundary conditions were imposed. Here, even though we have not not imposed such boundary conditions, we conclude that the shortage of the CFT states reconstructs the bulk fields near the horizon, which strongly indicates the existence of the brick wall [52] /fuzzball [54,55] /firewall [56] for the (single-sided) black hole. Note that it is important to include the finite N effects, which are the (strong) nonperturbative effects of the gravitational coupling from the bulk viewpoint, for this conclusion. Thus, this should be missed by the semiclassical analysis in the bulk.

Acknowledgements

S.T. would like to thank Sinya Aoki, Janos Balog, Yuya Kusuki, Takeshi Morita, Lento Nagano, Sotaro Sugishita, Kotaro Tamaoka, Tadashi Takayanagi, and Masataka Watanabe for useful discussions. S.T. would like to thank the Yukawa Institute for Theoretical Physics (YITP) at Kyoto University. Discussions during the YITP workshop YITP-W-20-03 on “Strings and Fields 2020” were useful to complete this work. This work was supported by JSPS KAKENHI Grant Number 17K05414.

Funding

Open Access funding: SCOAP^3.

Appendix A. Map between bulk and CFT operators

In this Appendix, we will review the explicit map between the bulk and CFT operators on the energy eigen value basis [29] and show how the bulk wave packet is represented by the CFT

[25] In Ref. [32] some energy conditions were assumed. In Ref. [51], they attempted to understand the bulk energy conditions by the causality constraint.

primary operators. We also will see that only the CFT operators on $t = \pm\pi/2$ are relevant in the global HKLL reconstruction.

We expand the quantized bulk field ϕ with the spherical harmonics $Y_{lm}(\Omega)$,

$$\phi(t, \rho, \Omega) = \sum_{n,l,m} \left(\hat{a}^\dagger_{nlm} e^{i\omega_{nl}t} Y_{lm}(\Omega) + \hat{a}_{nlm} e^{-i\omega_{nl}t} Y^*_{lm}(\Omega)\right) \psi_{nl}(\rho), \tag{A1}$$

where Ω represents the coordinates of S^{d-1} and the energy eigen value ω_{nl} is given by

$$\omega_{nl} = 2n + l + \Delta. \tag{A2}$$

The "wave function" for the radial direction is given by

$$\psi_{nl}(\rho) = \frac{1}{\mathcal{N}_{nl}} \sin^l(\rho) \cos^\Delta(\rho)\, P_n^{l+d/2-1, \Delta-d/2}\left(\cos(2\rho)\right), \tag{A3}$$

where $P_n^{\alpha,\beta}(x)$ is the Jacobi polynomial defined by

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}\left((1-x)^\alpha (1+x)^\beta (1-x^2)^n\right), \tag{A4}$$

and the normalization constant $\mathcal{N}_{nl}$ is given by

$$\mathcal{N}_{nl} = (-1)^n \sqrt{\frac{\Gamma(n+l+\frac{d}{2})\Gamma(n+1+\Delta-\frac{d}{2})}{\Gamma(n+l+\Delta)\Gamma(n+1)}}. \tag{A5}$$

Note that $(-1)^n \mathcal{N}_{nl} \to 1$ for $n \to \infty$ with a fixed l. We will decompose the local operator in the bulk description to positive and negative energy modes as

$$\phi(t, \rho, \Omega) = \phi^+(t, \rho, \Omega) + \phi^-(t, \rho, \Omega), \tag{A6}$$

where $\phi^-(t, \rho, \Omega) = (\phi^+(t, \rho, \Omega))^\dagger$.

Let us consider the scalar primary field $\mathcal{O}_\Delta(x)$, corresponding to $\phi(t, \rho, \Omega)$, in CFT_d on $\mathbf{R} \times S^{d-1}$ where $\mathbf{R}$ is the time direction and the radius of S^{d-1} is taken to be one. For a review of the CFT_d, see, e.g. Refs. [57–59]. We define the operator

$$\hat{\mathcal{O}}_\Delta = \lim_{x \to 0} \mathcal{O}^+_\Delta(x), \tag{A7}$$

where $\mathcal{O}^+_\Delta(x)$ is the regular parts of $\mathcal{O}_\Delta(x)$ in $x^\mu \to 0$ limit[26] which can be expanded by the polynomial of x^μ.

The identification of the CFT states to the states of the Fock space of the scalar fields in AdS is explicitly given by the identification of the creation operators as

$$\hat{a}^\dagger_{nlm} = c_{nl}\, s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_\Delta \tag{A8}$$

where c_{nl} is the normalization constant, which was given in Ref. [26], the translation operation P^μ acts on an operator such that $P^\mu \phi = [\hat{P}^\mu, \phi]$, and $s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)}$ is the normalized rank l symmetric traceless constant tensor such that $Y_{lm}(\Omega) = s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} x_{\mu_1} x_{\mu_2} \cdots x_{\mu_l}/|x|^l$.

Then, the bulk local operator is expressed as

$$\begin{aligned} \phi^+(t=0, \rho, \Omega) &= \sum_{n,l,m} \psi_{nl}(\rho) Y_{lm}(\Omega) \hat{a}^\dagger_{nlm} \\ &= \sum_{n,l,m} \psi_{nl}(\rho) Y_{lm}(\Omega) c_{nl} s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_\Delta, \end{aligned} \tag{A9}$$

where only the CFT operators appear in the last line. It is convenient to write both of the bulk and CFT operators in the creation and annihilation operators, which are energy eigen operators.

[26] This should be done after taking the large N limit. More precisely, $\hat{\mathcal{O}}_\Delta$ is the sum of the operators of dimension Δ up to $1/N$ corrections in $\mathcal{O}_\Delta(x)$.

Indeed, from the BDHM relation [4] in our normalization:

$$\lim_{\rho\to\pi/2}\frac{\phi(\rho,\Omega)}{\cos^{\Delta}(\rho)}=\sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}\mathcal{O}_{\Delta}(\Omega), \tag{A10}$$

we find

$$\begin{aligned}\mathcal{O}_{\Delta}(\Omega,t)&=\sum_{n,l,m}\psi_{nl}^{CFT}Y_{lm}(\Omega)e^{i\omega_{nl}t}\hat{a}^{\dagger}_{nlm}+h.c.\\&=\sum_{n,l,m}\psi_{nl}^{CFT}\left(Y_{lm}(\Omega)e^{i\omega_{nl}t}\hat{a}^{\dagger}_{nlm}+Y^{*}_{lm}(\Omega)e^{-i\omega_{nl}t}\hat{a}_{nlm}\right),\end{aligned} \tag{A11}$$

where

$$\psi_{nl}^{CFT}\equiv\sqrt{\frac{2}{\pi}\frac{\Gamma(d/2)}{\Gamma(\Delta)\Gamma(\Delta+1-d/2)}}\sqrt{\frac{\Gamma(n+\Delta+1-d/2)\Gamma(n+l+\Delta)}{\Gamma(n+1)\Gamma(n+l+d/2)}}, \tag{A12}$$

which reduces to a constant $\psi_{nl}^{CFT}=\sqrt{\frac{2}{\pi}}$ for $\Delta=d/2$. For $n\to\infty$ with a fixed l, we have $\psi_{nl}^{CFT}\to\sqrt{\frac{2}{\pi}\frac{\Gamma(d/2)}{\Gamma(\Delta)\Gamma(\Delta+1-d/2)}}n^{\Delta-d/2}$.

Below we will see the relation between the bulk local operator at the origin and the CFT primary operator integrated over S^{d-1}. These two operators are given by

$$\phi(t=0,\rho=0)=\sum_{n,l,m}\left(\hat{a}^{\dagger}_{nlm}Y_{lm}(\Omega)+\hat{a}_{nlm}Y^{*}_{lm}(\Omega)\right)\psi_{nl}(0) \tag{A13}$$

$$=\sum_{n}(-1)^{n}\sqrt{\frac{\Gamma(n+\Delta)\Gamma(n+d/2)}{\Gamma(n+1)\Gamma(n+1+\Delta-d/2)}}\left(\hat{a}^{\dagger}_{n00}+\hat{a}_{n00}\right), \tag{A14}$$

and

$$\int d\Omega\,\mathcal{O}_{\Delta}(\Omega,t)=\frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)}\sum_{n}\psi_{n0}^{CFT}\left(e^{i(2n+\Delta)t}\hat{a}^{\dagger}_{n00}+e^{-i(2n+\Delta)t}\hat{a}_{n00}\right), \tag{A15}$$

where we have used $Y_{00}(\Omega)=1$ and $\int d\Omega=2\pi^{d/2}/\Gamma(d/2)$. In particular, for $t=\pm\pi/2$, the latter becomes

$$\begin{aligned}\int d\Omega\,\mathcal{O}_{\Delta}(\Omega,t=\pm\pi/2)=\sum_{n}&\sqrt{\frac{8\pi^{d-2}}{\Gamma(d/2)\Gamma(\Delta)\Gamma(\Delta+1-d/2)}}\sqrt{\frac{\Gamma(n+\Delta)\Gamma(n+\Delta+1-d/2)}{\Gamma(n+1)\Gamma(n+d/2)}}\\&\times(-1)^{n}\left(e^{\pm i\frac{\pi}{2}\Delta}\hat{a}^{\dagger}_{n00}+e^{\mp i\frac{\pi}{2}\Delta}\hat{a}_{n00}\right).\end{aligned} \tag{A16}$$

Comparing these two expressions, we find

$$\phi^{+}(t=0,\rho=0)=C_{\Delta}e^{\mp i\frac{\pi}{2}\Delta}\int d\Omega\left(\mathrm{F}_{\Delta}(\partial_{\mathrm{t}})\,\mathcal{O}^{+}_{\Delta}(\Omega,\mathrm{t})\right)|_{t=\pm\frac{\pi}{2}}, \tag{A17}$$

where

$$C_{\Delta}=\sqrt{\frac{\Gamma(d/2)\Gamma(\Delta)\Gamma(\Delta+1-d/2)}{8\pi^{d-2}}}, \tag{A18}$$

$$\mathrm{F}_{\Delta}(\mathrm{x})=\frac{\Gamma((-\mathrm{ix}-\Delta+\mathrm{d})/2)}{\Gamma((-\mathrm{ix}+\Delta-\mathrm{d}+2)/2)}. \tag{A19}$$

We have used $\mathrm{F}_{\Delta}(\partial_{\mathrm{t}})\mathrm{e}^{\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}=\frac{\Gamma(\mathrm{n}+\mathrm{d}/2)}{\Gamma(\mathrm{n}+\Delta-\mathrm{d}/2+1)}\mathrm{e}^{\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}$. On the other hand, we find $\mathrm{F}_{\Delta}(\partial_{\mathrm{t}})\mathrm{e}^{-\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}=\frac{\Gamma(-\mathrm{n}-\Delta+\mathrm{d}/2)}{\Gamma(-\mathrm{n}-\mathrm{d}/2+1)}\mathrm{e}^{-\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}$ for the annihilation modes.

From Eq. (A17), we obtain the following relation between the bulk local state and the CFT state:

$$\phi(t=0,\rho=0)|0\rangle = C_\Delta e^{\mp i\frac{\pi}{2}\Delta}\int d\Omega\, (\mathrm{F}_\Delta(\partial_\mathrm{t})\,\mathcal{O}_\Delta(\Omega,\mathrm{t}))\,|_{t=\pm\frac{\pi}{2}}|0\rangle. \tag{A20}$$

This relation can be simplified for $\Delta = d-1$, which implies $\mathrm{F}_\Delta(x)=1$, as

$$\phi(t=0,\rho=0)|0\rangle = \sqrt{\frac{\Gamma(d/2)^2\Gamma(d-1)}{8\pi^{d-2}}}e^{\mp i\frac{\pi}{2}(d-1)}\int d\Omega\,\mathcal{O}_\Delta(\Omega, t=\pm\pi/2)|0\rangle. \tag{A21}$$

This expression, which implies Eq. (5) and the picture given in Fig. 2, was obtained in Ref. [29]. Similarly, for $\Delta = d-1-m$ where m is a positive integer, the relation can be also explicitly expressed as the integration localized on $t=\pm\pi/2$ because $\mathrm{F}_\Delta(x)$ is a polynomial:

$$\mathrm{F}_{\Delta=\mathrm{d}-1-\mathrm{m}}(\mathrm{x}) = \prod_{\mathrm{m}'=0}^{\mathrm{m}-1}((-\mathrm{ix}-1+\mathrm{m}-2\mathrm{m}')/2). \tag{A22}$$

In Ref. [62], the corresponding results for the bulk local operators were obtained using the HKLL bulk reconstruction formula. For $\Delta = d-1+m$ where m is a positive integer, we find the following simple relation:

$$\mathrm{F}_\Delta^{-1}(\partial_\mathrm{t})\phi(\mathrm{t}=0,\rho=0)|0\rangle = \mathrm{C}_\Delta \mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\Delta}\int \mathrm{d}\Omega\,\mathcal{O}_\Delta(\Omega,\mathrm{t}=\pm\pi/2)|0\rangle, \tag{A23}$$

where $\mathrm{F}_\Delta^{-1}(\mathrm{x}) = \prod_{\mathrm{m}'=0}^{\mathrm{m}-1}((-\mathrm{ix}-1+\mathrm{m}-2\mathrm{m}')/2)$ is a polynomial of x.

We will consider the bulk reconstruction for generic values of Δ and see that the bulk local state is represented as the CFT primary operators only on $t=\pm\pi/2$ as in Fig. 3 for these cases also, even though F_Δ may contain infinitely many derivatives.[27] We would like to express the right-hand side of Eq. (A20), $\int d\Omega\,(\mathrm{F}_\Delta(\partial_\mathrm{t})\,\mathcal{O}_\Delta(\Omega,\mathrm{t}))\,|_{t=-\frac{\pi}{2}}|0\rangle$, by $\int d\Omega\,\mathcal{O}_\Delta(\Omega,t)|0\rangle$ where $-\pi/2 \le t < \pi/2$. In order to do this, first, we introduce a UV (ultra violet) energy cutoff Λ and consider the (one particle) spherical symmetric state, $a^\dagger_{n00}|0\rangle$, only for $n=0,1,\ldots,\Lambda$. Denoting $\mathcal{O}^\Lambda_\Delta(\Omega',t')$ as $\mathcal{O}_\Delta(\Omega',t')$ with a UV energy cutoff Λ, we also introduce $|\partial^q\mathcal{O};t\rangle \equiv \int d\Omega(\partial_t)^q\mathcal{O}^\Lambda_\Delta(\Omega,t)|0\rangle$ for $t=t_s\equiv -\frac{\pi}{2}(1-2s/(\Lambda+1))$ with $s=0,1,\ldots,\Lambda$. These form a (not-orthonormal) basis of the cutoff spherical symmetric states for a fixed non-negative integer m, q. For $q=0$ we will use $|\mathcal{O};t\rangle = \left|\partial^0\mathcal{O};t\right\rangle$.[28] Denoting the inner products of them as $g_{s,s'} = \langle\partial^q\mathcal{O};t_{s'}|\mathcal{O};t_s\rangle$, we can express a state $|v\rangle$ in the cutoff space as $|v\rangle = \sum_s|\mathcal{O};t_s\rangle\sum_{s'}(g^{-1})^{ss'}\langle\partial^q\mathcal{O};t_{s'}|v\rangle$.

In order to estimate this, we note that, for $x\to\infty$, $\mathrm{F}(x)\to(-ix/2)^{-\Delta+d-1}$ which is a power function of x. We also note that $\int d\Omega\,\mathcal{O}_\Delta(\Omega,t)\sim\sum_n e^{i(2n+\Delta)t}n^{\Delta-d/2}a^\dagger_{n00}$, where we have approximated the coefficients of $a^\dagger_{n00}$ as the large n leading terms. Using these, we find

$$g_{s,s'} = \left\langle\partial^q\mathcal{O};t_{s'}\middle|\mathcal{O};t_s\right\rangle \sim \textstyle\sum_{n=0}^\Lambda n^{2\Delta-d+q}e^{i2n(t_s-t_{s'})}, \tag{A24}$$

for large n. Thus, for $|t_s-t_{s'}| \bmod \pi \le 1/\Lambda$, we find $g_{s,s'} = \mathcal{O}(\Lambda^{2\Delta-d+q+1})$, which is divergent in the $\Lambda\to\infty$ limit for $\Delta > (d-1-q)/2$, and $g_{s,s'}=\mathcal{O}(\Lambda^0)$ for $|t_s-t_{s'}| \bmod \pi \gg 1/\Lambda$. This implies that $|\mathcal{O};t_s\rangle$ is (almost) orthogonal to $|\mathcal{O};t_{s'}\rangle$ for $|t_s-t_{s'}| \bmod \pi \gg 1/\Lambda$. Similarly, we

[27] The infinitely many derivatives do not always mean nonlocality. As an example, let us consider a delta-function on S^1 in momentum space, $e^{in\theta}$, and the derivative operator $(1/(1-(\partial_\theta)^2))^\eta$ acting on it, $(1/(1+n^2))^\eta e^{in\theta}$, where $\eta>0$. With a momentum cutoff Λ, the inner product of these with different θ, θ', $\sum_{n=-\Lambda}^\Lambda e^{-in\theta'}(1/(1+n^2))^\eta e^{in\theta}$, is $\mathcal{O}(\Lambda^0)$ for $|\theta-\theta'|\gg 1/\Lambda$. It is $\mathcal{O}(\Lambda^{1-2\eta})$ for $|\theta-\theta'|\lessapprox 1/\Lambda$ which is larger than $\Lambda^{1/2}$ if $\eta<1/4$. Thus, for $\eta<1/4$ the derivative operator keeps the locality effectively for the cutoff theory.

[28] Below, we can take any q. In particular, we can take $q=0$ and consider $|\mathcal{O};t\rangle$ only.

find

$$\left\langle \partial^q \mathcal{O}; t_{s'} \right| \int d\Omega \left(\mathrm{F}_\Delta(\partial_\mathrm{t})\, \mathcal{O}^\Lambda_\Delta(\Omega, \mathrm{t}) \right)|_{t=-\frac{\pi}{2}} |0\rangle \sim \sum_{n=0}^{\Lambda} n^{\Delta+q-1} e^{-i2n(t'+\pi/2)}, \tag{A25}$$

which is $\mathcal{O}(\Lambda^{\Delta+q})$ for $|t_{s'}+\pi/2| \bmod \pi \leq 1/\Lambda$, and it is $\mathcal{O}(\Lambda^0)$ for $|t_{s'}+\pi/2| \bmod \pi \gg 1/\Lambda$. Now, we set $|v\rangle = c_v \int d\Omega \left(\mathrm{F}_\Delta(\partial_\mathrm{t})\, \mathcal{O}^\Lambda_\Delta(\Omega, \mathrm{t}) \right)|_{t=-\frac{\pi}{2}} |0\rangle$ where the constant c_v is determined by $|||v\rangle|| = 1$, then we find $c_v = \mathcal{O}(\Lambda^{(1-d)/2})$ from Eq. (A14). We also introduce $c_b = |||\mathcal{O}; t_s\rangle||^{-1} = \mathcal{O}(\Lambda^{(d-2\Delta-1)/2})$, which vanishes in the large Λ limit for $\Delta > (d-1)/2$. Here we assume $\Delta > (d-1)/2$ for simplicity. Using these estimations, we obtain $|v\rangle = \sum_s c_s (c_b |\mathcal{O}; t_s\rangle)$ where c_s is $\mathcal{O}(1)$ for $\pi/2 - |t_s| \leq 1/\Lambda$. Here, we have used the fact that $c_b|\mathcal{O}; t_s\rangle$ is (almost) orthonormal. On the other hand, c_s is $\mathcal{O}(\Lambda^{-\Delta})$ for $\pi/2 - |t_s| \gg 1/\Lambda$. This means that $\int d\Omega \left(\mathrm{F}_\Delta(\partial_\mathrm{t})\, \mathcal{O}_\Delta(\Omega, \mathrm{t}) \right)|_{t=-\frac{\pi}{2}} |0\rangle$ is localized to $t = -\pi/2$.

Instead of the bulk local state, we can reconstruct the bulk local operator. In order to do this, we first note that

$$\mathrm{F}_\Delta(-\mathrm{x}) = \mathrm{F}_\Delta(\mathrm{x}) \frac{\sin((\mathrm{ix} + \Delta - \mathrm{d} + 2)\pi/2)}{\sin((\mathrm{ix} - \Delta + \mathrm{d})\pi/2)}, \tag{A26}$$

where we have used $\Gamma(x)\Gamma(-x) = \pi/\sin(\pi x)$. This implies

$$\mathrm{F}_\Delta(-\partial_\mathrm{t}) \mathrm{e}^{\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}} = \frac{\sin(\mathrm{d}\pi/2)}{\sin((\mathrm{d}-2\Delta)\pi/2)} \mathrm{F}_\Delta(\partial_\mathrm{t}) \mathrm{e}^{\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}, \tag{A27}$$

and $\mathrm{F}_\Delta(\partial_\mathrm{t}) \mathrm{e}^{-\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}} = \frac{\sin(\mathrm{d}\pi/2)}{\sin((\mathrm{d}-2\Delta)\pi/2)} \mathrm{F}_\Delta(-\partial_\mathrm{t}) \mathrm{e}^{-\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}$. We define

$$\mathrm{F}^+_\Delta(\mathrm{x}) = \frac{\sin((\mathrm{d}-2\Delta)\pi/2)}{\sin(\mathrm{d}\pi/2) + \sin((\mathrm{d}-2\Delta)\pi/2)} \left(\mathrm{F}_\Delta(\mathrm{x}) + \mathrm{F}_\Delta(-\mathrm{x}) \right), \tag{A28}$$

which satisfies $\mathrm{F}^+_\Delta(\partial_\mathrm{t}) \mathrm{e}^{\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}} = \mathrm{F}_\Delta(\partial_\mathrm{t}) \mathrm{e}^{\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}$ and $\mathrm{F}^+_\Delta(-\partial_\mathrm{t}) \mathrm{e}^{-\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}} = \mathrm{F}_\Delta(-\partial_\mathrm{t}) \mathrm{e}^{-\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}$. We also define

$$\mathrm{F}^-_\Delta(\mathrm{x}) = \frac{\sin((\mathrm{d}-2\Delta)\pi/2)}{-\sin(\mathrm{d}\pi/2) + \sin((\mathrm{d}-2\Delta)\pi/2)} \left(\mathrm{F}_\Delta(\mathrm{x}) - \mathrm{F}_\Delta(-\mathrm{x}) \right), \tag{A29}$$

which satisfies $\mathrm{F}^-_\Delta(\partial_\mathrm{t}) \mathrm{e}^{\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}} = \mathrm{F}_\Delta(\partial_\mathrm{t}) \mathrm{e}^{\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}$ and $\mathrm{F}^-_\Delta(-\partial_\mathrm{t}) \mathrm{e}^{-\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}} = -\mathrm{F}_\Delta(-\partial_\mathrm{t}) \mathrm{e}^{-\mathrm{i}(2\mathrm{n}+\Delta)\mathrm{t}}$. Then, using the following relations,

$$\phi^+(t=0, \rho=0) = C_\Delta e^{\mp i\frac{\pi}{2}\Delta} \int d\Omega \left(\mathrm{F}_\Delta(\partial_\mathrm{t})\, \mathcal{O}^+_\Delta(\Omega, \mathrm{t}) \right)|_{t=\pm\frac{\pi}{2}}, \tag{A30}$$

and

$$\phi^-(t=0, \rho=0) = C_\Delta e^{\pm i\frac{\pi}{2}\Delta} \int d\Omega \left(\mathrm{F}_\Delta(-\partial_\mathrm{t})\, \mathcal{O}^-_\Delta(\Omega, \mathrm{t}) \right)|_{t=\pm\frac{\pi}{2}}, \tag{A31}$$

we find the reconstruction formula of the bulk local operator:

$$\begin{aligned} &\phi(t=0, \rho=0) \\ &= \frac{C_\Delta}{2\cos(\frac{\pi}{2}\Delta)} \left(\int d\Omega \left(\mathrm{F}^+_\Delta(\partial_\mathrm{t})\, \mathcal{O}_\Delta(\Omega, \mathrm{t}) \right)|_{t=\frac{\pi}{2}} + \int d\Omega \left(\mathrm{F}^+_\Delta(\partial_\mathrm{t})\, \mathcal{O}_\Delta(\Omega, \mathrm{t}) \right)|_{t=-\frac{\pi}{2}} \right), \end{aligned} \tag{A32}$$

and

$$\begin{aligned} &\phi(t=0, \rho=0) \\ &= \frac{C_\Delta}{2\sin(\frac{\pi}{2}\Delta)} \left(\int d\Omega \left(\mathrm{F}^-_\Delta(\partial_\mathrm{t})\, \mathcal{O}_\Delta(\Omega, \mathrm{t}) \right)|_{t=\frac{\pi}{2}} - \int d\Omega \left(\mathrm{F}^-_\Delta(\partial_\mathrm{t})\, \mathcal{O}_\Delta(\Omega, \mathrm{t}) \right)|_{t=-\frac{\pi}{2}} \right). \end{aligned} \tag{A33}$$

There are different expressions for the bulk local operator because $\mathcal{O}_\Delta(\Omega, t=\pi/2)$ and $\mathcal{O}_\Delta(\Omega, t=-\pi/2)$ are not independent in the large N limit. In fact, there are infinitely many expressions which include the following one, which only uses the CFT primary operators on $t = \pi/2$:

$$\phi(t=0,\rho=0) = C_\Delta \int d\Omega \left(\left(\alpha \mathrm{F}_\Delta(\partial_t) + \bar{\alpha} \mathrm{F}_\Delta(-\partial_t)\right) \mathcal{O}_\Delta(\Omega, t)\right)|_{t=\frac{\pi}{2}}, \tag{A34}$$

and this one, which only uses the CFT primary operators on $t = -\pi/2$:

$$\phi(t=0,\rho=0) = C_\Delta \int d\Omega \left(\left(\bar{\alpha} \mathrm{F}_\Delta(\partial_t) + \alpha \mathrm{F}_\Delta(-\partial_t)\right) \mathcal{O}_\Delta(\Omega, t)\right)|_{t=-\frac{\pi}{2}}, \tag{A35}$$

where

$$\alpha = \frac{\cos(\Delta\pi/2)}{1+\frac{\sin(d\pi/2)}{\sin((d-2\Delta)\pi/2)}} - i\frac{\sin(\Delta\pi/2)}{1-\frac{\sin(d\pi/2)}{\sin((d-2\Delta)\pi/2)}},\quad \bar{\alpha} = \frac{\cos(\Delta\pi/2)}{1+\frac{\sin(d\pi/2)}{\sin((d-2\Delta)\pi/2)}} + i\frac{\sin(\Delta\pi/2)}{1-\frac{\sin(d\pi/2)}{\sin((d-2\Delta)\pi/2)}}. \tag{A36}$$

A.1. Bulk wave packets

Let us consider the bulk local operator at $\rho = \rho_0$, $\Omega = \Omega_0$, and $t = 0$, which is $\phi^+(t = 0, \rho_0, \Omega_0) = \sum_{n,l,m} \psi_{nl}(\rho_0) Y_{lm}(\Omega_0) \hat{a}^\dagger_{nlm}$. If we smear this bulk local operator only for the angular direction Ω for a very small short-distance scale $1/M_c$ where $M_c \gg 1$, we have bulk wave packets moving in the radial direction ρ as

$$\phi^+_{\text{smear}}(\rho_0, \Omega_0) = \sum_{n,l,m}^{l<M_c} \psi_{nl}(\rho_0) Y_{lm}(\Omega_0) \hat{a}^\dagger_{nlm}, \tag{A37}$$

where the smearing was represented by the restriction of the angular momentum l because the precise form of the smearing is not important. The dominant contributions in the summation over n, l are those for $M_c \ll n$, which implies $l \ll n$. The asymptotic behavior of $\psi_{nl}(\rho)$ for large n (with l fixed) is computed, using the asymptotic behavior of the Jacobi polynomial [61], as

$$\psi_{nl}(\rho) = \frac{1}{\sqrt{\pi n}} (\tan z)^{\frac{d-1}{2}} \cos\left((2n + l + \Delta) z - \frac{\pi}{2}\left(\Delta - \frac{d}{2} + \frac{1}{2}\right)\right) + \mathcal{O}(n^{-3/2}), \tag{A38}$$

where

$$z = \pi/2 - \rho, \tag{A39}$$

and the boundary is at $z = 0$. This includes both the ingoing and outgoing waves, which correspond to the two exponentials in $\cos\left((2n+l+\Delta)z - \frac{\pi}{2}(\Delta - \frac{d}{2} + \frac{1}{2})\right) = \frac{1}{2} e^{i((2n+l+\Delta)z - \frac{\pi}{2}(\Delta-\frac{d}{2}+\frac{1}{2}))} + \frac{1}{2} e^{-i((2n+l+\Delta)z - \frac{\pi}{2}(\Delta-\frac{d}{2}+\frac{1}{2}))}$.

We will compare this with the same smearing of the CFT primary operator $\mathcal{O}_\Delta(\Omega_0, t) = \sum_{n,l,m} \psi^{CFT}_{nl} Y_{lm}(\Omega_0) e^{i\omega_{nl} t} \hat{a}^\dagger_{nlm} + h.c.$ for the angular direction Ω_0, i.e.

$$\mathcal{O}^{\text{smear}}_\Delta(\Omega_0, t) \equiv \sum_{n,l,m}^{l<M_c} \psi^{CFT}_{nl} Y_{lm}(\Omega_0) e^{i\omega_{nl} t} \hat{a}^\dagger_{nlm} + h.c \sim \sum_{n,l,m}^{l<M_c} n^{\Delta - d/2} Y_{lm}(\Omega_0) e^{i(2n+l+\Delta)t} \hat{a}^\dagger_{nlm} + h.c. \tag{A40}$$

For $0 < t < \pi/2$, we find that this smeared CFT primary operator with the time-derivatives, $(\frac{1}{2i}\frac{\partial}{\partial t})^{-\Delta + d/2 - 1/2} \mathcal{O}^{+\,\text{smear}}_\Delta(\Omega_0, t)$, is almost equivalent to the ingoing part of a bulk wave packet at $z = t$ and $\Omega = \Omega_0$, i.e. $\phi^+_{\text{smear}}(\rho, \Omega_0)|_{\rho=\pi/2-t}$, up to the overall normalizations.[29] The differences are negligible if the cutoff M_c (and the cutoff for n which is not explicitly introduced in this

[29] Here, the derivatives $(\frac{1}{2i}\frac{\partial}{\partial t})^{-\Delta+d/2-1/2}$ are the asymptotic form of a function like the $\mathrm{F}_\Delta(x)$ in Eq. (A19), which includes the contributions we neglected in the large n limit. In this paper, we do not determine this function; however, the asymptotic form in the large n limit is enough for reconstruction of the bulk operator corresponding to the local wave packet, as we have seen for the bulk local operator.

paper, but needed for the regularization of the local operator) is large. For $0 > t > -\pi/2$, it is the outgoing part of the bulk wave packet.

Note that the positive energy part of the bulk local operator at the center (A32) is represented as $\int_{S^{d-1}} d\Omega (\frac{1}{2i}\frac{\partial}{\partial t})^{-\Delta+d-1}\mathcal{O}^{+}_{\Delta}(\Omega, t)|_{t=\pi/2}$. On the other hand, for the wave packet, we have $\phi^{+}_{\rm smear}(\rho, \Omega_0)|_{\rho=\pi/2-t} \sim (\frac{1}{2i}\frac{\partial}{\partial t})^{-\Delta+d/2-1/2}\mathcal{O}^{\rm smear+}_{\Delta}(\Omega, t)$ for $0 < t < \pi/2$. The additional kinematical factor $(\frac{1}{2i}\frac{\partial}{\partial t})^{(1-d)/2}$ is needed because the bulk local operator is spherical and the wave packets have definite directions.

A.2. Bulk wave packet not at the center

We will study Eq. (14) more explicitly for the $d = 2$ case. For this case, the angular direction Ω is parametrized by φ where $-\pi \leq \varphi < \pi$. Let us take $\mathrm{f}_A(\Omega) \sim e^{-\varphi^2/(2L^2_{\rm smearing})}$ where $l_{\rm planck} \ll L_{\rm smearing} \ll 1$. For this small region A around $\varphi = 0$, we can approximate $\delta t(\Omega) \sim a\varphi$, where a is a constant satisfying $|a| < 1$ and we will assume $a > 0$ for simplicity. We expand $\mathcal{O}_\Delta(\Omega)$ in the energy eigen basis and the spherical harmonics $e^{i\varphi m}$ for S^1 where $m \in \mathbf{Z}$. Then, the Ω integration in Eq. (14) for the creation operator is evaluated as

$$\int d\varphi\, e^{-\varphi^2/(2L^2_{\rm smearing})+ia\varphi(2n+|m|+\Delta)+i\varphi m} \sim e^{-(a(2n+|m|+\Delta)+m)^2(L_{\rm smearing})^2/2}, \tag{A41}$$

where n is an analog of the momentum for the radial direction. This is exponentially suppressed for large n or m except the following two conditions are satisfied: $m < 0$ and

$$2an - (1-a)|m| \ll (L_{\rm smearing})^{-1}. \tag{A42}$$

For $a = 0$ which corresponds to the bulk local operator at the center $\rho_1 = 0$, these conditions imply that m is small compared with $(L_{\rm smearing})^{-1}$, thus the angular momentum is small. For $a \to 1$ which corresponds to considering the bulk local operator close to the boundary, these conditions imply that n is small, thus the wave packet is along the boundary. Between the two special cases, only the modes which satisfy $2n/m = -(1-a)/a$ approximately are dominant and we expect this represents the wave packet along the corresponding light-like trajectory using the asymptotic behaviour of the Jacobi polynomials [63].

A.3. HKLL bulk reconstruction for $d - \Delta < 1$

In this subsection, we will consider the HKLL reconstruction (6),

$$\begin{aligned}\phi(\rho = 0, t = 0) &\sim \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt' \int_{S^{d-1}} d\Omega' \frac{1}{(\cos t')^{d-\Delta}} \mathcal{O}_\Delta(\Omega', t') \\ &\sim \sum_n \psi^{CFT}_{n0} \hat{a}^\dagger_{n00} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt' \frac{1}{(\cos t')^{d-\Delta}} e^{i(2n+\Delta)t'} + h.c.,\end{aligned} \tag{A43}$$

for $d =$ odd, but not assuming $d - \Delta \geq 1$, and will see that only the CFT operators on an arbitrary small region containing $t = \pm\pi/2$ are relevant. This localization has been seen in Appendix A already, and here, we will confirm it from the different, but equivalent expression (A43).

In Eq. (A43), the t'-integration can be written as a Fourier transformation:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt' \frac{1}{(\cos t')^{d-\Delta}} e^{i(2n+\Delta)t'} = \int_0^{2\pi} dx \frac{1}{2|\sin(x/2)|^{d-\Delta}} e^{i(n+\Delta/2)(x-\pi)} \tag{A44}$$

$$= \int_0^{2\pi} dx e^{in(x-\pi)} \left(\frac{1}{2|\sin(x/2)|^{d-\Delta}} e^{i\frac{\Delta}{2}(x-\pi)} \right), \tag{A45}$$

where $t^{'} = (x - \pi)/2$. If we regard the integrand as a periodic function, it is singular at $x = 0$. Note that it is regular if $(\Delta - d)/2$ is a non-negative integer and $e^{i\pi\Delta} = 1$, but these two conditions are not consistent with $d =$ odd. Thus, for large n the contribution from the region near $x = 0$ is dominant because the contributions from the smooth function are exponentially suppressed for large n. Because the integrand behaves near $x = 0$ like $(\Theta(x) + const.)|x|^{\Delta - d}$ for $e^{i\pi\Delta} \neq 1$ or $|x|^{\Delta - d}$ for $e^{i\pi\Delta} = 1$, the contribution from the region near $x = 0$ for large n is $\mathcal{O}(1/n^{\Delta - d + 1})$. Combining these with $\psi^{CFT}_{nl} \to n^{\Delta - d/2}$, we find $\phi(\rho = 0, t = 0) = \sum_n \mathcal{O}(n^{d/2-1})\, \hat{a}^{\dagger}_{n00} + h.c.$. The summation $\sum_n \mathcal{O}(n^{d/2-1})$ diverges, and then the time integral of Eq. (A43) is localized on an arbitrary small region containing $t = \pm\pi/2$.

For $d =$ even, the smearing function includes the $\log\cos t^{'}$ factor and it is singular at $t^{'} = \pm\pi/2$ as a periodic function. Then, for this case also, we can see that the discussions for $d =$ odd can be applied.

Appendix B. Commutator of CFT

In this appendix, we will see that the propagation of a scalar of a nontrivial CFT in the large N limit is light-like as for the free theory. First, let us consider the $\Delta = d/2$ case. For this case, the v.e.v (vacuum expectation value) of the commutator, only that which is relevant in the large N limit, is given in Ref. [31] as

$$\langle 0|[\mathcal{O}(t_1, \Omega_1), \mathcal{O}(t_2, \Omega_2)]|0\rangle = \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{i \sin t_{12}} \sum_{l=0}^{\infty} \cos((d/2 - 1 + l)t_{12}) \sum_m Y_{lm}(\Omega_1) Y_{lm}(\Omega_2), \tag{B1}$$

for $e^{2it_{12}} \neq 1$. Comparing with the free scalar $\mathcal{O}^{\rm free}(t_1, \Omega_1)$ commutator on the cylinder [31], we find

$$\langle 0|[\mathcal{O}(t_1, \Omega_1), \mathcal{O}(t_2, \Omega_2)]|0\rangle = \frac{d-2}{4} \frac{1}{\sin t_{12}} [\frac{\partial}{\partial t_1} \mathcal{O}^{\rm free}(t_1, \Omega_1), \mathcal{O}^{\rm free}(t_2, \Omega_2)], \tag{B2}$$

thus, up to the $\frac{1}{\sin t_{12}}$ factor, it is same as the one for the free scalar. This implies that the propagation of a scalar of a nontrivial CFT is light-like.[30]

Note that the $\Delta = d/2$ case is only special because of the computational simplicity here, thus this light-like property is expected to hold for other values of the conformal dimension Δ. Indeed, for the primary operator (A11) in our approximation, we have $\mathcal{O}(t, \Omega)|0\rangle = e^{-i\pi\Delta}\mathcal{O}(t + \pi, \bar{\Omega})|0\rangle$ where $\bar{\Omega}$ is the antipodal point of Ω in S^{d-1}, This means that the propagation is light-like.

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[30] The free scalar commutator itself is not nonzero for a time-like separation; however, it is singular on the light-cone, which dominates the commutator if we regularize the local operators by a smearing.

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