Statistical laws of a one-dimensional model of turbulent flows subject to an external random force

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Abstract. This paper provides mathematical and numerical analysis of a one-dimensional model of turbulent flow generating the anomalous cascade of the inviscid conserved quantity. The model is based on the generalized Constantin-Lax-Majda-DeGregorio (gCLMG) equation with viscous dissipation under a large-scale forcing. Suppose that the forcing function and the initial data are random variables defined on a certain probability space. Then, the equation is regarded as a random partial differential equation. We prove the global existence of a unique solution to the gCLMG equation, from which a stochastic process is defined. In addition, by approximating the solutions numerically by Galerkin approximation of random variables with generalized Polynomial Chaos, we confirm the existence of a steady distribution. We find that the steady distribution reproduces qualitatively the same cascades of the energy and the enstrophy spectra as those of a turbulent flow generated by randomly moving pulse [14]. We also investigate the structure functions, showing intermittency.

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1. Introduction

The cascade of energy is one of the characteristic properties observed in three-dimensional turbulence. The energy input from large-scale transfers at a constant rate in the middle scales and it then dissipates at smaller scales where viscous dissipation becomes dominant. Let $\boldsymbol{u}(t, \boldsymbol{x})$ denote the velocity field of the flow at time t and the position $\boldsymbol{x}$ in the domain $\mathbb{T}^3 \simeq (\mathbb{R}/2\pi\mathbb{Z})^3$ with the periodic boundary condition. The energy spectra of the scale $k \in [0, \infty)$ is then defined by

$$E(t, k) := \frac{1}{2} \sum_{|\boldsymbol{k}'|=k} |\widehat{\boldsymbol{u}}(t, \boldsymbol{k}')|^2 ,$$

where $\widehat{\boldsymbol{u}}(t, \boldsymbol{k}')$ represents the Fourier coefficient of the velocity field. The total energy of the flow is given by $E(t) = \int_0^\infty E(t, k)dk$ and the time derivative of the total energy, say ε, is called the energy dissipation rate. Kolmogorov [10, 11] claimed that the ensemble average of the energy spectra $\langle E(k)\rangle$ follows the scaling law,

$$\langle E(k)\rangle \simeq \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}},$$

in the zero viscous limit under the assumption that turbulent flows are statistically stationary, spatially homogeneous, and isotropic. This is known as the 5/3 law of turbulence. In real viscous fluid flows with high Reynolds numbers, there appears a range of middle-scale wavenumbers, called the inertial range, along which the energy cascades following the scaling law [6]. On the other hand, we must notice that the energy is a conserved quantity for flows with exactly zero viscosity. Hence, the energy cascade suggests a singular discrepancy between non-viscous flows and those in the zero-viscous limit. A similar cascade phenomenon is observed in 2D turbulent flows [12, 13]: Whereas the enstrophy, which is the L^2 norm of the vorticity, is a conserved quantity for inviscid and incompressible 2D flows, the enstrophy cascade yields the emergence of the inertial range satisfying $\langle E(k)\rangle \simeq \eta^{\frac{2}{3}} k^{-3}$ for the enstrophy dissipation rate η.

Kolmogorov's turbulence theory is based on the dimensional analysis of physical quantities without specifying their governing equations explicitly. Hence, it is theoretically important to describe the cascade phenomena in terms of the solutions of the hydrodynamic equations. In many theoretical and numerical investigations of turbulence, incompressible and viscous flows with high Reynolds numbers subject to random large-scale forcing are considered to be models of turbulence. Hence, we attempt to explain the scaling law of the energy cascade from the incompressible Navier-Stokes (NS) equation. However, since the existence of a solution to the NS equation in three-dimensional space has not yet been established, it is far from a complete understanding of the cascade phenomenon. Accordingly, to get insights about this phenomenon, we deal with hydrodynamic equations that model the NS equations such as Burgers equation, the surface quasi-geostrophic equation [4, 5] and Constantin-Lax-Majda equation [3]. In the study of the Burgers equation subject to a stochastic forcing, the existence of a uniform invariant measure has been established [17]. In addition, it has also been shown in [2] that there appears an inertial range on which the energy cascade follows the scaling law k^{-2}. The stochastic Burgers equation is a good model to realize the turbulent state generating the energy cascade, but it is a little bit different situation from that in the energy cascade of turbulence.

Another hydrodynamics model is based on the generalized Constantin-Lax-Majda (CLM) equation for the functions $u(t,x)$ and $\omega(t,x)$,

$$\omega_t + au\omega_x - u_x\omega = 0, \qquad u_x = \mathcal{H}(\omega),$$

where $a \in \mathbb{R}$ is a parameter and $\mathcal{H}$ represents the Hilbert transform. For a smooth function $\omega \in C^\infty$ on $\mathbb{S}^1 \simeq (\mathbb{R}/2\pi\mathbb{Z})$ with a periodic boundary condition, $\mathcal{H}(\omega)$ is defined by $\mathcal{H}(\omega) = -\mathcal{F}^{-1} i\mathrm{sgn}(n)\mathcal{F}\omega$, where $\mathcal{F}u = \widehat{u}$ denotes the Fourier transform of u and $\mathrm{sgn}\colon \mathbb{R} \to \mathbb{R}$ is the sign function, i.e., $\mathrm{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$ and $\mathrm{sgn}(x) = 0$ for $x = 0$. The nonlinear term $u_x\omega$ in the first equation models the vortex stretching term in the 3D vorticity equation and the second equation corresponds to the Biot-Savart law, which was originally proposed by Constantin, Lax, and Majda [3]. The model advection term $u\omega_x$ was added by De Gregorio [7, 8]. Okamoto et al. [16] introduced the parameter $a \in \mathbb{R}$ to investigate the role of the balance between the advection term and the vortex stretching term in the existence of the solution, and they have shown that a unique solution exists locally in time. While the existence of a global solution has not been established, a sufficient condition of the Beale-Kato-Majda type for global existence has been obtained. Furthermore, it is found that the L^p norm of the solution, i.e., $\|\omega(t,\cdot)\|_{L^p}$ with $p = -a$, is a conserved quantity for $a \leqq -1$. This indicates that the gCLMG equation shares similar mathematical properties with the 3D Euler equations.

Inspired by the mathematical studies, we propose the following one-dimensional partial differential equations as a model of turbulent flows.

$$\omega_t + au\omega_x - u_x\omega = \nu\omega_{xx} + f, \qquad u_x = \mathcal{H}(\omega), \qquad \omega(0,x) = \omega_0(x), \tag{1}$$

where $\nu > 0$ is the viscous coefficient, ω_0 is the initial data and f denotes a forcing term. The choice of the forcing function f is important. In [14, 15], the forcing function is given by a stochastic process, whose Fourier coefficient $\widehat{f}(k,t)$ with the large-scale wavenumbers $k = \pm 1$ are set to Gaussian, δ-correlated-in-time, and independent random variables with zero mean. Mathematically, the equation (1) in this case becomes an infinite-dimensional stochastic partial differential equation. Numerical investigations of the long-time evolution of the solution to the equation (1) with $a = -2$ have revealed that the cascade of the conserved quantity $\|\omega(t,\cdot)\|_{L^2}$ exists over the inertial range. The scaling law almost agrees with that proposed by the dimensional analysis, though a small correction is required. In addition, it is found that the scaling law coincides with the energy spectra of a stationary solution to the equation (1) with a deterministic forcing. Some higher-order moments of the solutions are computed, but their scaling laws are not well-identified due to random behaviors of the evolutions of sample solutions.

In the meantime, we can consider another type of forcing function to investigate the statistical properties of solutions. That is to say, when the initial data ω_0 and the forcing functions f are regarded as random variables defined on a certain probability space Ω, we introduce a continuous mapping $\mathcal{M}_t$ from the pair of the random variables (ω_0, f) to the solution $\omega(t)$ to the equation (1). In this formulation, it is regarded as a random partial differential equation, and the solution $\omega(t)$ becomes a stochastic process. Then the following questions are to be considered: (i) the global existence of the mapping $\mathcal{M}_t$ and its uniqueness, (ii) the evolution of solutions under the action of $\mathcal{M}_t$, and (iii) the scaling law of the statistical quantities associated with the solutions. The purpose of the present study is to answer these questions mathematically as well as numerically.

The construction of the paper is as follows. In Section 2, we prove the existence of a unique solution when the initial data and the forcing function are fixed. Using this global solution, we show that the mapping $\mathcal{M}_t$ is well-defined when the initial data and the forcing function are random variables chosen from a probability space and that the solution $\omega(t)$ becomes a stochastic process. In Section 3, we investigate the statistical properties of solutions numerically. The solution is approximated by the pseudo-spectral method in the function space and the Galerkin method using the generalized Polynomial Chaos (gPC) in the probability space [18, 20]. The use of gPC has an advantage over the numerical investigation of stochastic partial differential equations since the statistical quantities are explicitly represented by the numerical solutions without taking the ensemble average of the long-time randomly moving solutions as shown in [19, 20]. Section 4 gives a conclusion and we mention some future directions.

2. Well-posedness of the gCLMG eq with a random forcing

We consider the gCLMG equation with a forcing term (1) in $\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$ with the periodic boundary condition. For $m \in \mathbb{N} \cup \{0\}$, we introduce the function space,

$$\dot{H}^m := \left\{ u \in H^m(\mathbb{S}^1) \;\middle|\; \int_0^{2\pi} u(x)dx = 0 \right\},$$

endowed with the inner product $\langle f, g\rangle_{\dot{H}^m} := \langle \partial_x^m f, \partial_x^m g\rangle_{L^2(\mathbb{S}^1)}$ and the norm $\|u\|_{\dot{H}^m} = \|\partial_x^m u\|_{L^2(\mathbb{S}^1)} = \||n|^m \mathcal{F}u\|_{\ell^2} = \sum_{n\neq 0} |n|^{2m} |\widehat{u}(n)|^2$. Note that this norm is equivalent to that of the standard Sobolev space $H^m(\mathbb{S}^1)$, and that $\dot{H}^m$ is a closed subspace of $H^m(\mathbb{S}^1)$. For $\omega \in \dot{H}^m$, the Hilbert transform $u_x = \mathcal{H}(\omega)$ is represented by $u = -(-\triangle)^{-\frac{1}{2}}\omega$, where $(-\triangle)^{-\frac{\gamma}{2}}$ is defied by $\mathcal{F}^{-1} |n|^{-\gamma} \mathcal{F}$ for $\gamma \geqq 0$.

For $0 < T < \infty$, the set of continuous functions from $[0, T]$ to $\dot{H}^m$ is denoted by X_T^m. With the norm

$$\|u\|_{X_T^m} := \sup_{0\leqq t\leqq T} \|u(t)\|_{\dot{H}^m},$$

X_T^m becomes a Banach space. We also define $X_\infty^m := C([0,\infty); \dot{H}^m)$ with the distance induced by the seminorms $p_n(f) := \|f\|_{X_n^m}$ of $C([0,\infty); \dot{H}^m)$. Throughout this paper, by $C(a, b, c, \ldots)$, we mean the constant C depends on parameters $a, b, c, \ldots$. Also, to avoid notational complications, we use the same symbol C for constants in the following estimates, although they are different.

Let us first introduce the notions of the solution to the equation (1) as follows.

Definition 2.1. Let $m \in \mathbb{N}$ and $0 < T < \infty$.

- For the initial data $\omega_0 \in \dot{H}^m$ and the forcing function $f \in X_T^m$, $\omega \in X_T^m$ is called the mild solution to the equation (1), if

$$\omega(t) = e^{\nu t \triangle}\omega_0 + \int_0^t e^{\nu(t-s)\triangle}\left\{-a(u\omega)_x(s) + (1+a)(u_x\omega)(s) + f(s)\right\} ds \qquad (2)$$

  holds in $\dot{H}^m$ for $t \in [0,T]$, where $e^{\nu t\triangle} = \mathcal{F}^{-1}e^{-t\nu n^2}\mathcal{F}$ for $t \geqq 0$ represents the heat semi-group.

- For the initial data $\omega_0 \in \dot{H}^{m+2}$ and the forcing function $f \in X_T^{m+2}$, we call $\omega \in C([0,T];\dot{H}^{m+2}) \cap C^1((0,T];\dot{H}^m)$ is the strong solution to the equation (1), if the following equation holds in $\dot{H}^m$.

$$\omega_t + au\omega_x - u_x\omega = \nu\omega_{xx} + f, \quad u_x = \mathcal{H}(\omega), \quad \omega(0) = \omega_0.$$

- For the initial data $\omega_0 \in \dot{H}^{m+4}$ and the forcing function $f \in X_T^{m+4}$, we call $\omega \in C([0,T];C^{m+2}) \cap C^1((0,T];C^m)$ is the classical solution to the equation (1), if the following equation holds at each t and x.

$$\omega_t + au\omega_x - u_x\omega = \nu\omega_{xx} + f, \quad u_x = \mathcal{H}(\omega), \quad \omega(0) = \omega_0$$

- For the initial data $\omega_0 \in \dot{H}^m$ and the forcing function $f \in X_\infty^m$, $\omega \in X_\infty^m$ is said to be the global mild solution to (1), if $\omega|_{[0,T]} \in X_T^m$ for any $0 < T < \infty$ is the mild solution to the equation (1) for the initial data $\omega_0 \in \dot{H}^m$ and the forcing function $f|_{[0,T]} \in X_T^m$.

Remark that if the mild solution $\omega \in X_T^m$ to the equation (1) satisfies $\omega \in C^1((0,T];\dot{H}^m) \cap C([0,T];\dot{H}^{m+2})$, the Fourier transformation of (2) gives rise to the strong solution. In addition if the mild solution belongs to $\omega \in X_T^{m+4}$, it becomes a classical solution in $C([0,T];C^{m+2}) \cap C^1((0,T];C^m)$.

We first show the existence of a unique mild solution in X_T^m locally in time.

Theorem 2.1. *Let $a \in \mathbb{R}$, $\nu > 0$ and $m \in \mathbb{N}$. Suppose that $f \in X_\infty^m$ and $\omega_0 \in \dot{H}^m$. Then, there exists $T > 0$ such that the equation* (1) *has a unique mild solution $\omega \in X_T^m$.*

Proof. For fixed $T > 0$ and $\omega \in X_T^m$, let us define the operator $\Phi(\omega)$ by

$$\Phi(\omega)(t) \coloneqq e^{\nu t\triangle}\omega_0 + \int_0^t e^{\nu(t-s)\triangle}\left\{-a(u\omega)_x(s) + (1+a)(u_x\omega)(s) + f(s)\right\} ds.$$

When $\omega(0) = \omega_0 \neq 0$, we set $X_T \coloneqq \{\omega \in X_T^m | \, \|\omega\|_{X_T^m} \leqq 2\|\omega_0\|_{\dot{H}^m}\}$. We then show that Φ becomes a contraction mapping from X_T to X_T for sufficiently small $T > 0$.

First, for $\omega \in X_T$, we confirm that

$$\left\|e^{\nu t\triangle}\omega(t)\right\|_{\dot{H}^m} = \left\|\mathcal{F}^{-1}e^{-\nu t n^2}|n|^m\mathcal{F}\omega(t)\right\|_{L^2} = \left\||n|^m e^{-\nu t n^2}\mathcal{F}\omega(t)\right\|_{\ell^2} \leqq \||n|^m\mathcal{F}\omega(t)\|_{\ell^2} = \|\omega(t)\|_{\dot{H}^m},$$

$$\left\|e^{\nu t\triangle}\partial_x\omega(t)\right\|_{\dot{H}^m} \lesssim \left\||n|^m|n|e^{-\nu t n^2}\mathcal{F}\omega(t)\right\|_{\ell^2} \lesssim \sup_{n\in\mathbb{Z}}|n|e^{-\nu t n^2} \times \||n|^m\mathcal{F}\omega(t)\|_{\ell^2} \lesssim \nu^{-\frac{1}{2}}t^{-\frac{1}{2}}\|\omega(t)\|_{\dot{H}^m}.$$

Let us also notice that $u \in X_T^m$ holds for $\omega \in X_T^m$, since $u_x = \mathcal{H}\omega$ and

$$\|u(t)\|_{\dot{H}^m} = \|u_x(t)\|_{\dot{H}^{m-1}} = \|\mathcal{H}\omega(t)\|_{\dot{H}^{m-1}} = \|\omega(t)\|_{\dot{H}^{m-1}} \lesssim \|\omega(t)\|_{\dot{H}^m} \lesssim \|\omega\|_{X_T^m}.$$

From these estimates and (11), for $\omega \in X_T$, we have

$$\begin{aligned}
\left\|e^{\nu t\triangle}\omega_0\right\|_{\dot{H}^m} &\leqq \|\omega_0\|_{\dot{H}^m}, \\
\left\|e^{\nu(t-s)\triangle}(u\omega)_x(s)\right\|_{\dot{H}^m} &\lesssim \nu^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}\|(u\omega)(s)\|_{\dot{H}^m} \lesssim \nu^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}\|u(s)\|_{\dot{H}^m}\|\omega(s)\|_{\dot{H}^m} \\
&\lesssim \nu^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}\|\omega(s)\|_{\dot{H}^{m-1}}\|\omega(s)\|_{\dot{H}^m} \leqq C(m)\nu^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}\|\omega_0\|_{\dot{H}^m}^2, \\
\left\|e^{\nu(t-s)\triangle}(u_x\omega)(s)\right\|_{\dot{H}^m} &\leqq \|(u_x\omega)(s)\|_{\dot{H}^m} \lesssim \|u_x(s)\|_{\dot{H}^m}\|\omega(s)\|_{\dot{H}^m} \leqq C(m)\|\omega_0\|_{\dot{H}^m}^2, \\
\left\|e^{\nu(t-s)\triangle}f(s)\right\|_{\dot{H}^m} &\leqq \|f(s)\|_{\dot{H}^m} \leqq \|f\|_{X_T^m}.
\end{aligned}$$

Note that the constants in the above upper bounds depend on the regularity of the function space m and the viscous coefficient ν. Hence, we have

$$\begin{aligned}
\|\Phi(\omega)(t)\|_{\dot{H}^m} &\leqq \left\|e^{\nu t\triangle}\omega_0\right\|_{\dot{H}^m} + |a|\int_0^t \left\|e^{\nu(t-s)\triangle}(u\omega)_x(s)\right\|_{\dot{H}^m} ds \\
&\quad + |1+a|\int_0^t \left\|e^{\nu(t-s)\triangle}(u_x\omega)(s)\right\|_{\dot{H}^m} ds + \int_0^t \|f(s)\|_{\dot{H}^m} ds \\
&\leqq \|\omega_0\|_{\dot{H}^m} + C(m,a,\nu)\|\omega_0\|_{\dot{H}^m}^2 T^{\frac{1}{2}} + C(m,a)T\|\omega_0\|_{\dot{H}^m}^2 + T\|f\|_{X_T^m}.
\end{aligned}$$

Since the right-hand side is independent of t, by taking $\sup_{0\leqq t\leqq T}$, we have

$$\|\Phi(\omega)\|_{X_T^m} \leqq \|\omega_0\|_{\dot{H}^m} + C(m,a,\nu)T^{\frac{1}{2}}\|\omega_0\|_{\dot{H}^m}^2 + C(a)T\|\omega_0\|_{\dot{H}^m}^2 + T\|f\|_{X_T^m}.$$

Hence, we can choose a sufficiently small $T = T(m,a,\nu,\|\omega_0\|_{\dot{H}^m},\|f\|_{X_T^m}) > 0$ so that $\|\Phi(\omega)\|_{X_T^m} \leqq 2\|\omega_0\|_{\dot{H}^m}$, which means that Φ defines a mapping from X_T to X_T.

Next, we show that Φ is a contraction mapping. For $\omega_1, \omega_2 \in X_T$, we have

$$\begin{aligned}
\|(u_1\omega_1 - u_2\omega_2)(t)\|_{\dot{H}^m} &\leqq \|u_1\omega_1 - u_2\omega_2\|_{X_T^m} \\
&\leqq \|u_1\omega_1 - u_1\omega_2\|_{X_T^m} + \|u_1\omega_2 - u_2\omega_2\|_{X_T^m} \\
&\leqq \|u_1\|_{X_T^m}\|\omega_1-\omega_2\|_{X_T^m} + \|\omega_2\|_{X_T^m}\|u_1-u_2\|_{X_T^m} \\
&\lesssim (\|\omega_1\|_{X_T^m} + \|\omega_2\|_{X_T^m})\|\omega_1-\omega_2\|_{X_T^m} \lesssim \|\omega_0\|_{\dot{H}^m}\|\omega_1-\omega_2\|_{X_T^m}
\end{aligned}$$

and similarly

$$\|(u_{1x}\omega_1 - u_{2x}\omega_2)(t)\|_{\dot{H}^m} \leqq \|u_{1x}\omega_1 - u_{2x}\omega_2\|_{X_T^m} \lesssim \|\omega_0\|_{\dot{H}^m}\|\omega_1-\omega_2\|_{X_T^m}.$$

Hence, it follows from

$$\begin{aligned}\|\Phi(\omega_1)(s)-\Phi(\omega_2)(s)\|_{\dot H^m} &\leqq C(a)\int_0^t \left\|e^{\nu(t-s)\triangle}(u_1\omega_1-u_2\omega_2)_x(s)\right\|_{\dot H^m} ds\\ &\quad + C(a)\int_0^t \left\|e^{\nu(t-s)\triangle}(u_{1x}\omega_1-u_{2x}\omega_2)(s)\right\|_{\dot H^m} ds\\ &\leqq C(a,\nu)\int_0^t (t-s)^{-\frac12} ds\,\|u_1\omega_1-u_2\omega_2\|_{X_T^m} + C(\nu)T\,\|u_{1x}\omega_1-u_{2x}\omega_2\|_{X_T^m}\\ &\leqq C(m,a,\nu)(T^{\frac12}+T)\,\|\omega_0\|_{\dot H^m}\,\|\omega_1-\omega_2\|_{X_T^m}\end{aligned}$$

that we obtain

$$\|\Phi(\omega_1)-\Phi(\omega_2)\|_{X_T^m} \leqq C(m,a,\nu)(T^{\frac12}+T)\,\|\omega_0\|_{\dot H^m}\,\|\omega_1-\omega_2\|_{X_T^m}.$$

This indicates that $\Phi\colon X_T\to X_T$ becomes a contraction mapping for a sufficiently small $T=T(m,a,\nu,\|\omega_0\|_{H^m})>0$ with $\omega_0\neq 0$. On the other hand, when $\omega_0=0$, by simply setting $X_T:=X_T^m$, we can similarly show that $\Phi\colon X_T\to X_T$ is a contraction mapping. Consequently, by Banach fixed-point theorem, there exists a unique $\omega\in X_T$ satisfying $\omega=\Phi(\omega)$.

We have obtained the unique mild solution in X_T which is a subspace of X_T^m. This is the unique mild solution in the whole space X_T^m. Suppose that $\omega_1,\omega_2\in X_T^m$ are the solutions to the equation (1). Then we similarly obtain

$$\begin{aligned}\|\omega_1(t)-\omega_2(t)\|_{\dot H^m} &= \|\Phi(\omega_1)(t)-\Phi(\omega_2)(t)\|_{\dot H^m}\\ &\leqq C(m,a,\nu,\|\omega_0\|_{\dot H^m})\int_0^t (1+(t-s)^{-\frac12})\,\|\omega_1(s)-\omega_2(s)\|_{\dot H^m} ds.\end{aligned}$$

By Gronwall's inequality, we have $\omega_1=\omega_2$ in X_T^m. □

We show that the unique local mild solution to the equation (1) in Theorem 2.1 depends continuously on the initial data $\omega_0\in\dot H^m$ and the forcing function $f\in X_T^m$.

Theorem 2.2. *Let $0<T<\infty$, $a\in\mathbb{R}$, $\nu>0$ and $m\in\mathbb{N}$. Suppose that $\omega_i\in X_T^m$, $i=1,2$ represents the mild solution of (1) for the forcing function $f_i\in X_T^m$ and the initial data $\omega_{0i}\in\dot H^m$. Then, there exists a constant $C(a,\nu,T,\|\omega_1\|_{X_T^m},\|\omega_2\|_{X_T^m})>0$ such that the following inequality holds.*

$$\|\omega_1-\omega_2\|_{X_T^m} \leqq C(\|f_1-f_2\|_{X_T^m}+\|\omega_{01}-\omega_{02}\|_{\dot H^m}). \tag{3}$$

Proof. In a similar way as in the proof of Theorem 2.1, since

$$\begin{aligned}\|\omega_1(t)-\omega_2(t)\|_{\dot H^m} &= \|\Phi(\omega_1)-\Phi(\omega_2)\|_{\dot H^m}\\ &\leqq \|f_1-f_2\|_{X_T^m}+\|\omega_{01}-\omega_{02}\|_{\dot H^m}\\ &\quad + C(\nu,a,\|\omega_1\|_{X_T^m},\|\omega_2\|_{X_T^m})\int_0^t (1+(t-s)^{-\frac12})\,\|\omega_1(s)-\omega_2(s)\|_{\dot H^m} ds,\end{aligned}$$

Grownwall's inequality yields

$$\|\omega_1(t)-\omega_2(t)\|_{\dot{H}^m} \leqq C(\|f_1-f_2\|_{X_T^m}+\|\omega_{01}-\omega_{02}\|_{\dot{H}^m}).$$

By taking $\sup_{0\leqq t\leqq T}$, we finish the proof. □

The following theorem assures the persistence of regularity for the unique mild solution.

Theorem 2.3. *Let $m_*, m \in \mathbb{N}$ with $m_* > m$, $f \in X_\infty^{m_*}$ and $\omega_0 \in \dot{H}^{m_*}$. Suppose that $\omega \in X_T^m$ is a unique mild solution to the equation* (1) *when we regard f and ω_0 are elements in $X_T^m \subset X_\infty^{m_*}$ and $\dot{H}^m \subset \dot{H}^{m_*}$ respectively. Then, the solution ω belongs to $X_T^{m_*}$.*

Proof. According to Theorem 2.1, there exists a constant $T^* > 0$ such that we have the mild solution $\widetilde{\omega}$ in $X_{T_*}^{m_*}$. When $T_* \geqq T$, we have $\omega = \widetilde{\omega}$ by the uniqueness in this time range. For $T_* < T$, we then obtain

$$\begin{aligned}\|\widetilde{\omega}(t)\|_{\dot{H}^{m_*}} &= \|\Phi(\widetilde{\omega})(t)\|_{\dot{H}^{m_*}} \\ &\leqq \|\omega_0\|_{\dot{H}^{m_*}} + \|f\|_{X_T^m} T + C(a,\nu,\|\omega_0\|_{\dot{H}^m}) \int_0^t (1+(t-s)^{-\frac{1}{2}}) \|\widetilde{\omega}(s)\|_{\dot{H}^{m_*}} \, ds.\end{aligned}$$

Gronwall's inequality yields

$$\|\widetilde{\omega}\|_{X_{T_*}^{m_*}} \leqq C(a,\nu,T,\|\omega_0\|_{\dot{H}^m})(\|\omega_0\|_{\dot{H}^{m_*}}+\|f\|_{X_T^m}).$$

Owing to this upper bound, we can extend the existence time of the unique mild solution in $\dot{H}^{m_*}$ as long as $T_* < T$. By repeating this step finitely until $T_* \geqq T$, we have $\omega = \tilde{\omega}$ in $X_T^{m_*}$. □

In what follows, we fix $a = -2$ when the L^2 norm of ω is conserved. We then prove the existence of the mild solution globally in time. The first step is to obtain an a priori estimate.

Lemma 2.4 (L^2 estimate)**.** *Let $a = -2$, $\nu > 0$, $m \in \mathbb{N}$, $f \in X_\infty^{m+4}$ and $\omega_0 \in \dot{H}^{m+4}$. If there exists the classical solution $\omega \in C^1((0,T];C^m) \cap C([0,T];C^{m+2})$ to the equation* (1) *for any T, we have*

$$\|\omega(t)\|_{L^2} \leqq \|f\|_{C([0,T];L^2)} \, t + \|\omega_0\|_{L^2}.$$

Proof. Integrating the both sides of the equation (1) multiplied by ω and using the Hölder inequality, we obtain

$$\|\omega(t)\|_{L^2} \frac{d}{dt} \|\omega(t)\|_{L^2} \leqq \frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + \nu \|\omega(t)\|_{\dot{H}^1}^2 = \langle f, \omega \rangle_{L^2} \leqq \|f\|_{C([0,T];L^2)} \|\omega(t)\|_{L^2}.$$

Hence, we have

$$\frac{d}{dt}\|\omega(t)\|_{L^2} \leqq \|f\|_{C([0,T];L^2)}.$$

Integrating it with respect to t, we have the estimate as desired.

$$\|\omega(t)\|_{L^2} \leqq \|f\|_{C([0,T];L^2)} t + \|\omega_0\|_{L^2} .$$

□

Lemma 2.5 ($\dot{H}^m$ estimate)**.** *Let $a = -2$, $\nu > 0$, $m \in \mathbb{N}$, $f \in X^m_\infty$ and $\omega_0 \in \dot{H}^m$. Suppose that there exists a classical solution $\omega \in C^1([0,T];C^m) \cap C([0,T];C^{m+2})$ to the equation (1) for any $T > 0$. Then the solution ω satisfies the following estimate.*

$$\|\omega\|^2_{X^m_T} \leqq C(m,\nu,T)(P_m(\|\omega_0\|^2_{\dot{H}^m}) + Q_m(\|f\|^2_{X^m_T})), \tag{4}$$

where $P_m(x)$ and $Q_m(x)$ denote polynomials of degree $3m$ having non-negative coefficients that are independent of ν, T, ω_0 and f.

Proof. Let us rewrite the nonlinear terms of (1) as $-2u_x\omega - u_x\omega = -2\partial_x(u\omega) + u_x\omega$. Then acting ∂_x^m on the both sides of the equation (1), multiplying it by $\partial_x^m\omega$ and integrating it with respect to x, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2_{\dot{H}^m} + \nu\|\omega(t)\|^2_{\dot{H}^{m+1}} = -2\left\langle \partial_x^m(u\omega), \partial_x^{m+1}\omega\right\rangle_{L^2} - \langle \partial_x^m(u_x\omega), \partial_x^m\omega\rangle_{L^2} + \langle \partial_x^m f, \partial_x^m\omega\rangle_{L^2} . \tag{5}$$

We separately estimate the three terms on the right-hand side of (5). To avoid the notational complications, we omit the time dependence in the norm of $\dot{H}^m$ as long as no confusion occurs. The first term is estimated using the equality (11), the isometric property of the Hilbert transform, Sobolev interpolation inequality (10) with $\theta = 1$, $m_1 = m-1$ and $m_2 = m+1$.

$$\begin{aligned}\left|-2\left\langle \partial_x^m(u\omega), \partial_x^{m+1}\omega\right\rangle_{L^2}\right| &\leqq C(m)\|u\omega\|_{\dot{H}^m}\|\omega\|_{\dot{H}^{m+1}} \leqq C(m)\|u\|_{\dot{H}^m}\|\omega\|_{\dot{H}^m}\|\omega\|_{\dot{H}^{m+1}} \\ &\leqq C(m)\|u_x\|_{\dot{H}^{m-1}}\|\omega\|_{\dot{H}^m}\|\omega\|_{\dot{H}^{m+1}} \leqq C(m)\|\omega\|^{\frac{3}{2}}_{\dot{H}^{m-1}}\|\omega\|^{\frac{3}{2}}_{\dot{H}^{m+1}} \\ &\leqq \frac{\nu}{6}\|\omega\|^2_{\dot{H}^{m+1}} + C(m,\nu)\|\omega\|^6_{\dot{H}^{m-1}} .\end{aligned}$$

In the last estiamte, Young's inequality (9) is used for $a = \left(\frac{2\nu}{9}\right)^{\frac{3}{4}}\|\omega\|^{\frac{3}{2}}_{\dot{H}^{m+1}}$ and $b = C(m)\left(\frac{9}{2\nu}\right)^{\frac{3}{4}}\|\omega\|^{\frac{3}{2}}_{\dot{H}^{m-1}}$ with $p = \frac{4}{3}$ and $q = 4$. Similarly, the second and the third terms are evaluated as follows.

$$\begin{aligned}|-\langle \partial_x^m(u_x\omega), \partial_x^m\omega\rangle_{L^2}| &\leqq C\|u_x\omega\|_{\dot{H}^m}\|\omega\|_{\dot{H}^m} \leqq C(m)\|u_x\|_{\dot{H}^m}\|\omega\|^2_{\dot{H}^m} \leqq C(m)\|\omega\|^3_{\dot{H}^m} \\ &\leqq C(m)\|\omega\|^{\frac{3}{2}}_{\dot{H}^{m-1}}\|\omega\|^{\frac{3}{2}}_{\dot{H}^{m+1}} \leqq \frac{\nu}{6}\|\omega\|^2_{\dot{H}^{m+1}} + C(m,\nu)\|\omega\|^6_{\dot{H}^{m-1}} , \\ |\langle \partial_x^m f, \partial_x^m\omega\rangle_{L^2}| &\leqq C\|f\|_{\dot{H}^m}\|\omega\|_{\dot{H}^m} \leqq C\|f\|_{X^m_T}\|\omega\|^{\frac{1}{2}}_{\dot{H}^{m-1}}\|\omega\|^{\frac{1}{2}}_{\dot{H}^{m+1}} \\ &\leqq \frac{\nu}{6}\|\omega\|^2_{\dot{H}^{m+1}} + C(\nu)\|f\|^{\frac{4}{3}}_{X^m_T}\|\omega\|^{\frac{2}{3}}_{\dot{H}^{m-1}} \\ &\leqq \frac{\nu}{6}\|\omega\|^2_{\dot{H}^{m+1}} + C(\nu)\|\omega\|^2_{\dot{H}^{m-1}} + C(\nu)\|f\|^2_{X^m_T} .\end{aligned}$$

Substituting these estimates in the equation (5), we have

$$\frac{d}{dt}\|\omega(t)\|^2_{\dot{H}^m}+\nu\|\omega(t)\|^2_{\dot{H}^{m+1}}\leqq C(m,\nu)(\|\omega(t)\|^6_{\dot{H}^{m-1}}+\|\omega(t)\|^2_{\dot{H}^{m-1}}+\|f\|^2_{X_T^m}). \tag{6}$$

We show the estimate (4) by induction with respect to m. For $m=1$, Lemma 2.4 yields

$$\|\omega(t)\|_{L^2}\leqq C(T)(\|f\|_{X_T^1}+\|\omega_0\|_{\dot{H}^1}).$$

Substituting this estimate into (6) and using the inequality $(a+b)^p\leqq 2^{p-1}(a^p+b^p)$ for $a,b\geqq 0$ and $p\geqq 1$, we have

$$\frac{d}{dt}\|\omega(t)\|^2_{\dot{H}^1}\leqq\frac{d}{dt}\|\omega(t)\|^2_{\dot{H}^1}+\nu\|\omega(t)\|^2_{\dot{H}^2}\leqq C(m,\nu,T)(\|f\|^6_{X_T^1}+\|\omega_0\|^6_{\dot{H}^1}+\|f\|^2_{X_T^1}+\|\omega_0\|^2_{\dot{H}^1}).$$

Integrating it with respect to t yields

$$\|\omega(t)\|^2_{\dot{H}^1}\leqq C(m,\nu,T)(\|f\|^6_{X_T^1}+\|\omega_0\|^6_{\dot{H}^1}+\|f\|^2_{X_T^1}+\|\omega_0\|^2_{\dot{H}^1}).$$

This indicates that (4) holds true for $m=1$ with $P_1(x)=Q_1(x)=x^3+x$.

Suppose now that the estimate (4) is valid for $m-1$ with the polynomials P_{m-1} and Q_{m-1} of degree $3(m-1)$ having non-negative coefficients:

$$\|\omega(t)\|^2_{\dot{H}^{m-1}}\leqq C(m,\nu,T)(P_{m-1}(\|\omega_0\|^2_{\dot{H}^{m-1}})+Q_{m-1}(\|f\|^2_{X_T^{m-1}})).$$

Substituting this estimate into (6), we have

$$\begin{aligned}\frac{d}{dt}\|\omega(t)\|^2_{\dot{H}^m}&\leqq\frac{d}{dt}\|\omega(t)\|^2_{\dot{H}^m}+\nu\|\omega(t)\|^2_{\dot{H}^{m+1}}\\&\leqq C(m,\nu)(P_{m-1}(\|\omega_0\|^2_{\dot{H}^{m-1}})^3+Q_{m-1}(\|f\|^2_{X_T^{m-1}})^3\\&\qquad+P_{m-1}(\|\omega_0\|^2_{\dot{H}^{m-1}})+Q_{m-1}(\|f\|^2_{X_T^{m-1}})+\|f\|^2_{X_T^m})\\&\leqq C(m,\nu)(P_{m-1}(\|\omega_0\|^2_{\dot{H}^m})^3+Q_{m-1}(\|f\|^2_{X_T^m})^3\\&\qquad+P_{m-1}(\|\omega_0\|^2_{\dot{H}^m})+Q_{m-1}(\|f\|^2_{X_T^m})+\|f\|^2_{X_T^m}).\end{aligned}$$

Integrating this inequality with respect to t and taking $\sup_{0\leqq t\leqq T}$, we finally have

$$\|\omega\|^2_{X_T^m}\leqq C(m,\nu,T)(P_m(\|\omega_0\|^2_{\dot{H}^m})+Q_m(\|f\|^2_{X_T^m})),$$

where $P_m(x)=P_{m-1}(x)^3+P_{m-1}(x)+x$ and $Q_m(x)=Q_{m-1}(x)^3+Q_{m-1}(x)+x$ are the polynomials of degree $3m$ and their coefficients are all non-negative. □

With this estimate, we finally show the existence of the global mild solution. Note that the asymptotically stable solution obtained by Matsumoto-Sakajo[14] and the steady solution by Jeong-Kim[9] are examples of global mild solutions.

Theorem 2.6. *Let $a=-2$, $\nu>0$ and $m\in\mathbb{N}$. Suppose the forcing functon $f\in X_\infty^m$ and the initial data $\omega_0\in\dot{H}^m$. Then there exists a unique mild solution $\omega\in X_\infty^m$ to the equation (1) globally in time. Moreover, for any $T>0$, the solution satisfies the following estimate.*

$$\|\omega\|_{X_T^m}\leqq C(m,\nu)(P_m(\|\omega_0\|^2_{\dot{H}^m})+Q_m(\|f\|^2_{X_T^m})),$$

where P_m and Q_m are polynomials of degree $3m$ with non-negative coefficients.

Proof. Let us note that $f \in X^m_\infty$ implies $f|_{[0,T]} \in X^m_T$ for any $T > 0$. For fixed $N \in \mathbb{N}$, let us define the projection operator $P_N \colon \dot{H}^m \to \dot{H}^m$ by $P_N\phi \coloneqq \sum_{0<|n|\leqq N} \widehat{\phi}(n)e^{inx}$. For any $m_* > m$, if $\phi \in \dot{H}^m$, then $P_N\phi \in H^{m_*}$, and $\lim_{N\to\infty} P_N\phi = \phi$ in $\dot{H}^m$. Since the sequence of continuous functions $\{\|P_Nf(t) - f(t)\|^2_{\dot{H}^m}\}_{N\in\mathbb{N}}$ on $[0,T]$ for $f \in X^m_T$ is monotonically decreasing and it vanishes pointwise as $N \to \infty$, we have the convergence $\lim_{N\to\infty} P_Nf = f$ uniformly in X^m_T by Dini's theorem.

To extend the existence time beyond T^*, we need to show that the a priori estimate (4) for the classical solution is satisfied for the mild solution up to T^*. So we construct a classical solution that approximates the mild solution for $t < T^*$ as follows. For $m_* \in \mathbb{N}$, Theorem 2.1 assures that there exists a time $T_N(< T)$ such that the gCLMG equation (1) has a unique mild solution, say $\omega^N \in X^{m_*}_{T_N}$, for the initial data $P_N\omega_0 \in \dot{H}^{m_*}$ and the forcing function $P_Nf \in X^{m_*}_T$. When we choose $m_* = m + 4$, the mild solution becomes the classical solution. By Lemma 2.5, the solution satisfies the following estimate.

$$\left\|\omega^N\right\|^2_{X^m_{T_N}} \leqq C(m,\nu,T)(P_m(\|\omega_0\|^2_{\dot{H}^m}) + Q_m(\|f\|^2_{X^m_T})),$$

where $P_m(x)$ and $Q_m(x)$ are polynomials of degree $3m$ with non-negative coefficients that are independent on N. Note that we here use $\|P_N\omega_0\|_{\dot{H}^m} \leqq \|\omega_0\|_{\dot{H}^m}$ and $\|P_Nf\|_{X^m_T} \leqq \|f\|_{X^m_T}$. With this estimate, we can extend the existence time of the local mild solution ω^N in $\dot{H}^m$ for a certain fixed time and, by Theorem 2.3, $\omega^N(t) \in \dot{H}^{m_*}$. Repeating this step, we can finally extend the strong solution in $\dot{H}^{m_*}$ up to T, i.e., $\omega^N \in X^{m_*}_T$. Hence, for any $N \in \mathbb{N}$, we have

$$\left\|\omega^N\right\|^2_{X^m_T} \leqq C(m,\nu,T)(P_m(\|\omega_0\|^2_{\dot{H}^m}) + Q_m(\|f\|^2_{X^m_T})). \tag{7}$$

In the meantime, let $\omega \in X^m_{T_*}$ with $T_*(< T)$ be the local mild solution for the initial data $\omega_0 \in \dot{H}^m$ and the forcing function $f \in X^m_T$. It follows from the inequality (7) and the estimate (3) in Theorem 2.2 that we have

$$\begin{aligned}\left\|\omega - \omega^N\right\|_{X^m_{T_*}} &\leqq C(m,\nu,T_*,\|\omega\|_{X^m_{T_*}}, \left\|\omega^N\right\|_{X^m_{T_*}})(\|\omega_0 - P_N\omega_0\|_{\dot{H}^m} + \|f - P_Nf\|_{X^m_T})\\ &\leqq C(m,\nu,T_*,\|\omega\|_{X^m_{T_*}}, \|\omega_0\|_{\dot{H}^m}, \|f\|_{X^m_T})(\|\omega_0 - P_N\omega_0\|_{\dot{H}^m} + \|f - P_Nf\|_{X^m_T}).\end{aligned}$$

Hence, as $N \to \infty$, we obtain $\lim_{N\to\infty} \left\|\omega - \omega^N\right\|_{X^m_{T_*}} = 0$ and the estimate (7) converges as follows.

$$\|\omega\|^2_{X^m_{T_*}} \leqq C(m,\nu,T)(P_m(\|\omega_0\|^2_{\dot{H}^m}) + Q_m(\|f\|^2_{X^m_T})).$$

This estimate allows us to extend the existence time from T_* to T. Thus, for any $0 < T < \infty$, there exist the unique mild solution $\widetilde{\omega} \in X^m_T$ to the equation (1) for the initial data $\omega_0 \in \dot{H}^m$ and the forcing function $f|_{[0,T]} \in X^m_T$. Hence, for any time $t \geqq 0$, by choosing $T \geqq t$, we can define the global mild solution $\omega \in X^m_\infty$ at this time by this mild solution $\widetilde{\omega} \in X^m_T$. This finishes the proof. □

Now we switch the deterministic forcing function f and the initial data ω_0 to random variables and investigate the property of the solution to the equation (1) as a random partial differential equation. Specifically, we show that a unique global mild solution to the equation (1) is determined every time random variables in the probability space are provided.

Theorem 2.7. *Let $a = -2$, $\nu > 0$ and $m \in \mathbb{N}$. For a given probability space $(\Omega, \mathcal{F}, P)$, we introduce random variables $f\colon \Omega \to X^m_\infty$ and $\omega_0\colon \Omega \to \dot{H}^m$ satisfying $f \in \cap_{p=1}^\infty L^p(\Omega; X^m_T)$ and $\omega_0 \in \cap_{p=1}^\infty L^p(\Omega; \dot{H}^m)$ for any $T > 0$. Then there exists a stochastic process $\omega\colon \Omega \to X^m_\infty$ uniequely such that $\omega|_{[0,T]} \in L^2(\Omega; X^m_T)$ for $0 < T < \infty$, and for any $\eta \in \Omega$, $\omega^\eta = \omega(\eta) \in X^m_\infty$ is a mild solution to the equation* (1).

Proof. For $t \geqq 0$, we define the mapping $\mathcal{M}^\infty_t\colon \dot{H}^m \times X^m_\infty \to \dot{H}^m$ by $(\omega_0, f) \mapsto \omega(t)$, where $\omega(t)$ is the unique mild solution at time t for the initial data ω_0 and the forcing function f. This mapping is well-defined owing to Theorem 2.6.

On the other hand, for any $T > 0$, the mapping $\mathcal{M}^T_t$ from the pair of the initial condition and the forcing function $(\omega_0, f|_{[0,T]}) \in (\dot{H}^m \times X^m_T)$ to the unique mild solution $\omega(t) \in \dot{H}^m$ is continuous by Theorem 2.2, and so is $\mathcal{M}^\infty_t$. Hence, when we define $\omega^\eta(t) \coloneqq \mathcal{M}^\infty_t(\omega_0(\eta), f(\eta))$ for $\eta \in \Omega$, $\omega\colon \Omega \to X^m_\infty$ becomes a stochastic process.

Let us recall that the mild solution satisfies the estimate

$$\|\omega^\eta\|^2_{X^m_T} \leqq C(m, \nu, T)(P_m(\|\omega_0(\eta)\|^2_{\dot{H}^m}) + Q_m(\|f(\eta)\|^2_{X^m_T}))$$

for $0 < T < \infty$, and P_m and Q_m are the polynomials of degree $3m$. We thus have the expectation

$$\mathbb{E}[\|\omega^\eta\|^2_{X^m_T}] \leqq C(m, \nu, T)\mathbb{E}[(P_m(\|\omega_0(\eta)\|^2_{\dot{H}^m})] + \mathbb{E}[Q_m(\|f(\eta)\|^2_{X^m_T})]) < \infty,$$

since $\mathbb{E}[\|f(\eta)\|^{6m}_{X^m_T}] < \infty$ and $\mathbb{E}[\|\omega_0(\eta)\|^{6m}_{\dot{H}_m}] < \infty$ for $f \in \cap_{p=1}^\infty L^p(\Omega; X^m_T)$ and $\omega_0 \in \cap_{p=1}^\infty L^p(\Omega; \dot{H}^m)$. Consequently, we have $\omega \in L^2(\Omega; X^m_T)$. This finishes the proof. □

As we see in the proof, it is sufficient to assume $f \in \cap_{p=1}^{6m} L^p(\Omega; X^m_T)$ and $\omega_0 \in \cap_{p=1}^{6m} L^p(\Omega; \dot{H}^m)$ to show the theorem.

3. Statistical properties of solutions

We consider the time evolution of the distribution of global mild solutions to the gCLMG equation (1) when the initial data ω_0 is deterministic and the forcing function f is a time-independent random variable defined on a probability space. By approximating the distribution numerically using the Galerkin approximation of the function space and the probability space, we compute the statistical quantities of the distribution such as the average, the spectral laws of the energy and the enstrophy spectra, and the structure functions.

3.1. Numerical method

Let $P_{N,M}\colon L^2(\Omega;\dot{H}^m) \to L^2(\Omega;\dot{H}^m)$ denote the projection onto its $2N\times(M+1)$-dimensional subspace. We write $P_{N,M}f \coloneqq f^{N,M}$ for $f\in L^2(\Omega;\dot{H}^m)$. Then the Galerkin approximation of the equation (1),

$$\frac{d}{dt}\omega^{N,M} - P_{N,M}(u_x^{N,M}\omega^{N,M} - au^{N,M}\omega_x^{N,M}) = \nu\omega_{xx}^{N,M} + f^{N,M}, \quad u_x^{N,M} = \mathcal{H}(\omega^{N,M}), \quad (8)$$

for the inital data $\omega^{N,M}(0)=\omega_0^{N,M}$ gives rise to $2N\times(M+1)$-dimensional ordinary differential equations for the expansion coefficients of $\omega^{N,M}$.

The projection $P_{N,M}$ is specified as follows. For the discretization of the function space $H^m(\mathbb{S}^1)$, we use the standard dealiased pseudo-spectral method. On the other hand, the random variables in the probability space $L^2(\Omega;\dot{H}^m)$ are discretized with the generalized Polynomial Chaos (gPC). For $\mathbb{R}^d$-valued random variable $Z\colon\Omega\to\mathbb{R}^d$ and measurable functions $\widetilde{f}\colon\mathbb{R}^d\to\dot{H}^m$, we introduce $f(\eta)=\widetilde{f}(Z(\eta))$ for $\eta\in\Omega$. Let $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),P^Z)$ be the space of the pushforward measure of Z. See [1] for the definition of the pushforward measure. By $\mathbb{E}_{P^Z}$, we express the expectation with respect to P^Z. Then it follows $\mathbb{E}[\|f\|_{\dot{H}^m}^2] = \mathbb{E}_{P^Z}[\|\widetilde{f}\|_{\dot{H}^m}^2]$ that we have the existence of the mild solution $\omega(t)$ belonging to $L^2_{P^Z}(\mathbb{R}^d;\dot{H}^m)$ for any $t\geqq 0$ by using $\widetilde{f}$ on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),P^Z)$ instead of f on $(\Omega,\mathcal{F},P)$ in Theorem 2.7.

Owing to $L^2_{P^Z}(\mathbb{R}^d;\dot{H}^m)\cong L^2_{P^Z}(\mathbb{R}^d)\otimes\dot{H}^m$, its orthogonal basis of the Bochner space $L^2_{P^Z}(\mathbb{R}^d;\dot{H}^m)$ is the product of the orthonormal basis of $\dot{H}^m$, i.e., $e_n(x)=\mathrm{e}^{inx}/2\pi$, $n\in\mathbb{N}\setminus\{0\}$, and that of $L^2_{P^Z}(\mathbb{R}^d)$. The gPC uses a system of orthogonal polynomials, say $\{\Phi_\alpha(Z)\}_{\alpha=0}^\infty$, associated with the distribution of the random variable Z. That is to say, any function $f\in L^2(\Omega;\dot{H}^m)$ is represented by $f(x,\eta)=\widetilde{f}(x,Z(\eta))=\sum_{\alpha=0}^\infty \widehat{f}(x,\alpha)\Phi_\alpha(Z(\eta))$ with $\widehat{f}(x,\alpha)=\mathbb{E}_{P^Z}[\widetilde{f}(x,\cdot)\Phi_\alpha(\cdot)]/\mathbb{E}_{P^Z}[\Phi_\alpha^2]$ and it is approximated numerically by truncation. For instance, the orthogonal basis of $L^2_{P^Z}(\mathbb{R}^d)$ for the uniform distribution consists of Legendre polynomials. The mathematical theory and background behind the numerical method is provided in [20]. With this basis, the projection $P_{N,M}\colon L^2(\Omega;\dot{H}^m)\to L^2(\Omega;\dot{H}^m)$ of the function $\omega(t,x,\eta)\in L^2_{P^Z}(\Omega;\dot{H}^m)$ is expressed by

$$\omega^{N,M}(t,x,\eta)=\sum_{\alpha=0}^{M}\sum_{|n|\leqq N}\widehat{\omega}(t,n,\alpha)e^{inx}\Phi_\alpha(Z(\eta)),$$

where

$$\widehat{\omega}(t,n,\alpha)\coloneqq\frac{\mathbb{E}_{P^Z}[\langle\widetilde{\omega}(t,\cdot,\cdot),e_n(\cdot)\rangle_{L^2}\,\Phi_\alpha(\cdot)]}{\mathbb{E}_{P^Z}[\Phi_\alpha^2]}.$$

Note that the expectation $\mathbb{E}_{P^Z}$ is numerically approximated by the Gauss-Legendre quadrature rule. Then the equation (8) is rewritten to the following equations for the coefficients $\omega^{N,M}(t,x,\eta)$ for $|n| \leqq N$ and $0 \leqq \gamma \leqq M$.

$$\begin{aligned}\frac{d}{dt}\widehat{\omega}(t,n,\gamma) =& -\nu n^2 \widehat{\omega}(t,n,\gamma) + \widehat{f}(n,\gamma)\\ & - a\sum_{\alpha=0}^{M}\sum_{\beta=0}^{M} \frac{\mathbb{E}_{P^Z}[\Phi_\alpha\Phi_\beta\Phi_\gamma]}{\mathbb{E}_{P^Z}[\Phi_\alpha^2]\mathbb{E}_{P^Z}[\Phi_\beta^2]\mathbb{E}_{P^Z}[\Phi_\gamma^2]} \left\langle \mathbb{E}_{P^Z}[u\Phi_\alpha]\mathbb{E}_{P^Z}[\omega_x\Phi_\beta], e_m\right\rangle_{L^2}\\ & + \sum_{\alpha=0}^{M}\sum_{\beta=0}^{M} \frac{\mathbb{E}_{P^Z}[\Phi_\alpha\Phi_\beta\Phi_\gamma]}{\mathbb{E}_{P^Z}[\Phi_\alpha^2]\mathbb{E}_{P^Z}[\Phi_\beta^2]\mathbb{E}_{P^Z}[\Phi_\gamma^2]} \left\langle \mathbb{E}_{P^Z}[u_x\Phi_\alpha]\mathbb{E}_{P^Z}[\omega\Phi_\beta], e_m\right\rangle_{L^2}.\end{aligned}$$

We solve these ODEs numerically using the fourth-order Runge-Kutta method with the temporal step size Δt.

The gPC approximation yields the evolution of the distribution of the global mild solutions to the equation (1) for a given distribution of forcing functions on the probability space with a single numerical computation. Hence, it is unnecessary to compute the time evolutions of many sample solutions. For instance, when the vorticity $\omega \in L^2(\Omega, \dot{H}^m)$ is approximated by

$$\omega(t,x,\eta) \approx \sum_{\alpha=0}^{M}\sum_{n=-N}^{N} \widehat{\omega}(t,n,\alpha)e^{inx}\Phi_\alpha(Z(\eta)),$$

its average is simply obtained by the coefficient of $\Phi_0(Z(\eta))$.

$$\mathbb{E}[\omega](t,x) := \mathbb{E}_{P^Z}[\omega(t,x,\cdot)] \approx \sum_{n=-N}^{N} \widehat{\omega}(t,n,0)\mathrm{e}^{inx}\mathbb{E}_{P^Z}[\Phi_0].$$

Let us remark that it is the ensemble average of the distribution consisting of the mild solutions in $\dot{H}^m$ at time t. It is different from the ensemble average of the snapshots (or the long-time average) of solutions to the gCLMG equation (1) subject to a stochastic forcing in [14].

The average of the enstrophy spectra $\mathbb{E}[\|\omega\|_{L^2}^2](t,k)$ at time t is computed by

$$\mathbb{E}[\|\omega\|_{L^2}^2](t,k) := \frac{1}{2}\sum_{|\ell|=k,k+1} \mathbb{E}[|\widehat{\omega}(t,\ell,\cdot)|^2] \approx \frac{1}{2}\sum_{\alpha=0}^{M}\sum_{|\ell|=k,k+1} |\widehat{\omega}(t,\ell,\alpha)|^2\,\mathbb{E}_{P^Z}[\Phi_\alpha^2]$$

owing to the orthogonaliry of the basis functions $\{\Phi_\alpha\}_{\alpha=0}^{\infty}$. In addition, with

$$\begin{aligned}V(t,\ell) &:= \mathbb{E}[|\widehat{\omega}(t,\ell,\cdot)|^4] - \mathbb{E}[|\widehat{\omega}(t,\ell,\cdot)|^2]^2\\ &\approx \sum_{\substack{\alpha_1,\alpha_2,\\ \alpha_3,\alpha_4=0}}^{M} \widehat{\omega}(t,\ell,\alpha_1)\overline{\widehat{\omega}(t,\ell,\alpha_2)}\widehat{\omega}(t,\ell,\alpha_3)\overline{\widehat{\omega}(t,\ell,\alpha_4)}\mathbb{E}_{P^Z}[\Phi_{\alpha_1}\Phi_{\alpha_2}\Phi_{\alpha_3}\Phi_{\alpha_4}] - \mathbb{E}[|\widehat{\omega}(t,\ell,\cdot)|^2]^2,\end{aligned}$$

the standard deviation of the spectra, $\sigma[\|\omega\|_{L^2}^2](t,k)$, is given by

$$\sigma[\|\omega\|_{L^2}^2](t,k) = \frac{1}{2}\sum_{|\ell|=k,k+1}\sqrt{V(t,\ell)}.$$

The average of the energy spectra $\mathbb{E}[\|u\|^2_{L^2}](t,k)$ and the standard deviation $\sigma[\|u\|^2_{L^2}](t,k)$ are computed in the same way.

Another important statistical quantity characterizing turbulent flows is the p-th order structure function,

$$S_p[u](r) \coloneqq \langle (u(t,x+r) - u(t,x))^p \rangle.$$

Although the structure function depends on t, x, and r by definition, it is regarded as a function of the distance r only in the study of turbulence under the assumption that the turbulent flows are isotropic, homogeneous, and statistically steady. According to Kolmogorov's turbulence theory, the structure function follows the scaling law $S_p[u](r) \simeq \varepsilon^{\frac{p}{3}} |r|^{\frac{p}{3}}$. We also note that $S_2[u](r)$ is known to be relevant to the 5/3 power-law of the energy spectra. For the stochastic process $\omega(t,x,\eta) \in L^2(\Omega; \dot{H}^m)$, we introduce the local structure function $\mathcal{S}_p[\omega](t,x,r)$ by

$$\mathcal{S}_p[\omega](t,x,r) \coloneqq \mathbb{E}[|\omega(t,x+r,\cdot) - \omega(t,x,\cdot)|^p] \approx \mathbb{E}_{PZ}\left[\left|\sum_{\alpha=0}^{M} (\widetilde{\omega}(t,x+r,\alpha) - \widetilde{\omega}(t,x,\alpha))\Phi_\alpha\right|^p\right],$$

in which

$$\widetilde{\omega}(t,x,\alpha) \coloneqq \frac{\mathbb{E}_{PZ}[\omega(t,x,\cdot)\Phi_\alpha(\cdot)]}{\mathbb{E}_{PZ}[\Phi_\alpha^2]} = \sum_{n=-N}^{N} \widehat{\omega}(t,n,\alpha)\mathrm{e}^{inx}.$$

To be specific, the local structure functions of the vorticity for $p = 2, 4$ are explicitly given by

$$\mathcal{S}_2[\omega](t,x,r) \approx \sum_{\alpha_1,\alpha_2=0}^{M} \Delta\widetilde{\omega}(t,x,r,\alpha_1)\Delta\widetilde{\omega}(t,x,r,\alpha_2)\mathbb{E}_{PZ}[\Phi_{\alpha_1}\Phi_{\alpha_2}],$$

$$\mathcal{S}_4[\omega](t,x,r) \approx \sum_{\substack{\alpha_1,\alpha_2,\\ \alpha_3,\alpha_4=0}}^{M} \Delta\widetilde{\omega}(t,x,r,\alpha_1)\Delta\widetilde{\omega}(t,x,r,\alpha_2)\Delta\widetilde{\omega}(t,x,r,\alpha_3)\Delta\widetilde{\omega}(t,x,r,\alpha_4)\mathbb{E}_{PZ}[\Phi_{\alpha_1}\Phi_{\alpha_2}\Phi_{\alpha_3}\Phi_{\alpha_4}],$$

where $\Delta\widetilde{\omega}(t,x,r,m) = \widetilde{\omega}(t,x+r,m) - \widetilde{\omega}(t,x,m)$.

3.2. Numerical results

We fix the parameter $a = -2$ and the initial data $\omega_0 = 0$. The time-independent forcing function is specified by

$$f^\eta(t,x) = 0.01 \times (2Z(\eta) - 1)\sin x,$$

where $Z(\eta)$ follows the uniform distribution on $[0,1]$. Then the orthogonal bases of the gPC expansion are Legendre polynomials. Table 1 is the list of the numerical parameters used here.

Figure 1(a) is the evolution of the average of the vorticity distribution $\mathbb{E}[\omega](t,x)$ for $\nu = 1.0 \times 10^{-3}$, showing that it tends to be a stationary state. It is mathematically

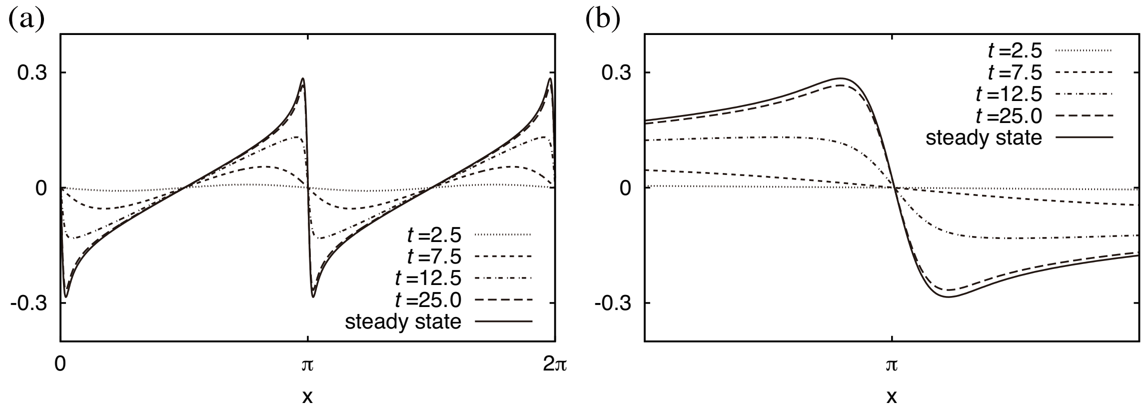

Figure 1. (a) The evolution of the average $\mathbb{E}[\omega](t,x)$ for $\nu = 1.0 \times 10^{-3}$. (b) A close-up plot of the solution in the neighborhood of the sharp peak around $x = \pi$. The steady state means the numerical solution at $t = 237.5 (\equiv T_s^\nu)$.

ν	N	M	Δt
1.0×10^{-2}	2^{10}	10	2.5×10^{-2}
1.0×10^{-3}	2^{11}	10	2.5×10^{-3}
1.0×10^{-4}	2^{12}	10	2.5×10^{-3}
5.0×10^{-5}	2^{12}	10	2.5×10^{-3}

Table 1. Numerical parapmeters

stated that there exists a function $\omega_\infty \in \dot{H}^m$ such that $\lim_{t\to\infty} \mathbb{E}[\mathcal{M}_t^\infty(0, f^\eta)] = \omega_\infty$, indicating the existence of a steady distribution of mild solutions. While the average of the steady vorticity distribution has sharp peaks around $x = 0$ and π, it remains smooth as shown in Figure 1(b). A similar pattern with sharp peaks appears in the randomly moving pulse, generating turbulent flow with the enstrophy cascade [14]. It is also similar to the asymptotically stable stationary solution to the equation (1) with the deterministic forcing function, $f(x) = 0.01 \sin x$. In what follows, to compute statistical quantities for a given ν, we use the numerical solution at $t = T_s^\nu$ when the vorticity distribution almost reaches the steady state up to numerical tolerance. For instance, when $\nu = 1.0 \times 10^{-3}$, $T_s^\nu = 237.5$ as we see in Figure 1(a,b).

The averages $\mathbb{E}[\omega]$, $\mathbb{E}[\omega_x]$, $\mathbb{E}[u]$ and $\mathbb{E}[u_x]$ of the steady distribution for $\nu = 1.0 \times 10^{-2}$, 1.0×10^{-3}, 1.0×10^{-4} are shown in Figure 2. The peaks of the average vorticity $\mathbb{E}[\omega](T_s^\nu, x)$ in Figure 2(a) get singular as the viscosity coefficient ν decreases. Figure 2(b) shows that the average of the derivative of the vorticity $\mathbb{E}[\omega_x](T_s^\nu, x)$ grows rapidly as ν decreases. The velocity average $\mathbb{E}[u](T_s^\nu, x)$ in Figure 2(c) looks smooth, but it tends to form sharp fronts in the neighborhood of $x = 0, \pi$ as ν decreases. The formation of the sharp fronts in the velocity profile is also confirmed evidently by the average of the derivative of the velocity $\mathbb{E}[u_x](T_s^\nu, x)$ in Figure 2(d), which has sharp spines at the front locations.

Figure 3 shows the averages of the energy spectra $\mathbb{E}[\|u\|_{L^2}^2](T_s^\nu, k)$ and the enstrophy

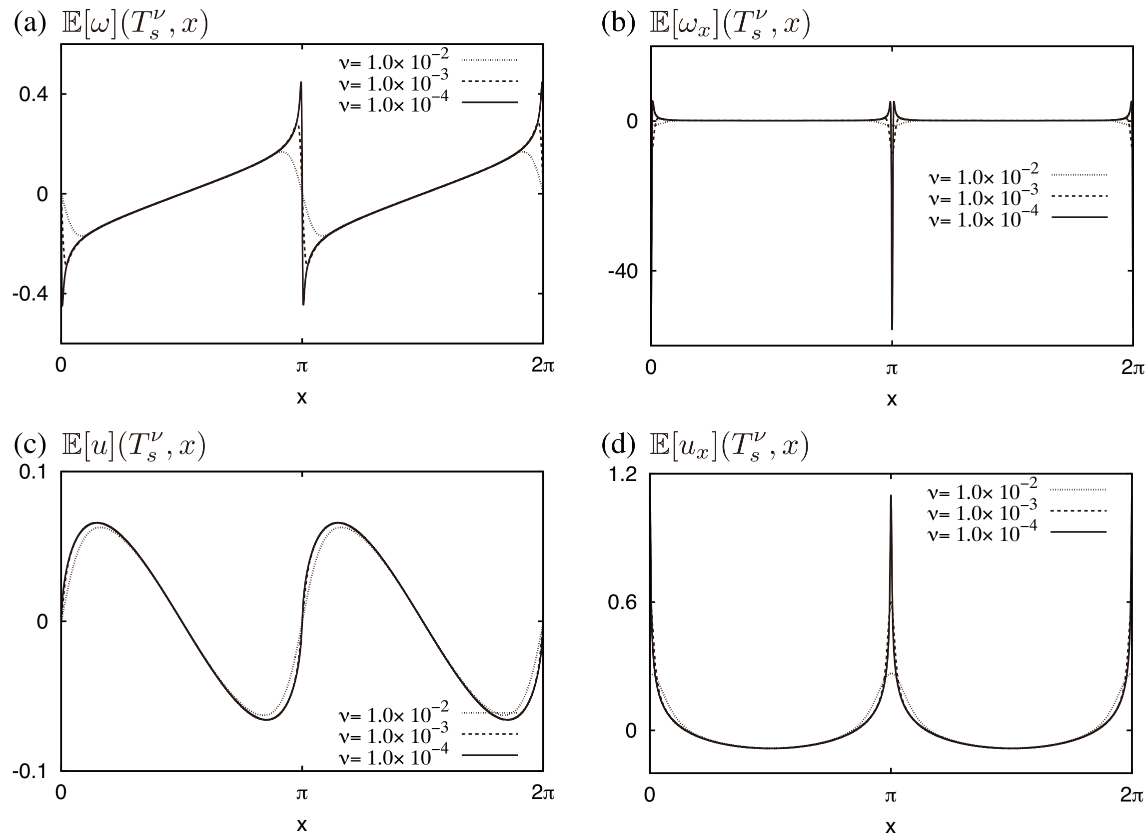

Figure 2. The averages of the steady distribution of solutions for $\nu = 1.0 \times 10^{-2}$, 1.0×10^{-3} and 1.0×10^{-4}. The numerical parameters are shown in Table 1. (a) The average of the vorticity $\mathbb{E}[\omega](T_s^\nu, x)$. (b) The average of the derivative of the vorticity $\mathbb{E}[\omega_x](T_s^\nu, x)$. (c) The average of the velocity $\mathbb{E}[u](T_s^\nu, x)$. (d) The average of the derivative of the velocity $\mathbb{E}[u_x](T_s^\nu, x)$.

spectra $\mathbb{E}[\|\omega\|_{L^2}^2](T_s^\nu, k)$ of the steady distribution for various viscous coefficients ν. For the spectra of $\nu = 5.0 \times 10^{-5}$, we show error bars $\pm\sigma[\|\omega\|_{L^2}^2](T_s^\nu, k)$ and $\pm\sigma[\|u\|_{L^2}^2](T_s^\nu, k)$ as thin gray regions. Both of them indicate that the inertial ranges are formed and expand as ν decreases. The dimensional analysis suggests that the decay rate of the energy spectra in the inertial range follows $\langle|\widehat{u}(k)|^2\rangle \simeq k^{-3}$. However, Figure 3(a) shows that it lies between k^{-4} to k^{-3}. On the other hand, the enstrophy spectra in the inertial range are expected to be $\langle|\widehat{\omega}(k)|^2\rangle \simeq k^{-1}$ owing to $\widehat{u}(k') = \widehat{\omega}(k')\,|k'|^{-1}$ by definition. Figure 3(b) again indicates a deviation from the dimensional analysis. The deviation of the decay rates has been reported for the numerical studies of the gCLMG equation (1) as a stochastic partial differential equation [14, 15]. Hence, the averages of the steady distribution of mild solutions reproduce qualitatively the same energy and enstrophy cascades in the numerical study of turbulent flows in [14].

We confirm the scaling law of the structure functions $S_p[\omega]$ and $S_p[u]$ for the solutions $\omega(t, x, \eta)$ and $u(t, x, \eta)$ to the gCLMG equation (1). The dimensional analysis of the equation in [15] has shown that $S_p[\omega](r) \simeq r^p$, $S_p[u](r) \simeq r^p$ for short ranges

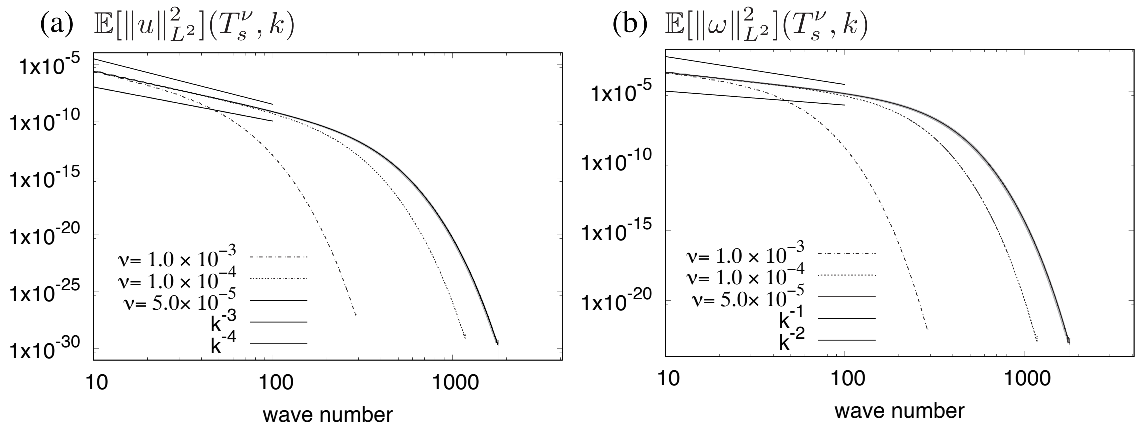

Figure 3. (a) The average of the energy spectra $\mathbb{E}[\|u\|^2_{L^2}](T^\nu_s, k)$ for the steady distribution. The inertial range extends as ν decreases. The decay rate of the spectra in the inertial range lies between k^{-3} and k^{-4}. (b) The average of the enstrophy spectra $\mathbb{E}[\|\omega\|^2_{L^2}](T^\nu_s, k)$, whose decay rate lies in the range of k^{-1} and k^{-2}. In both spectra, the decay rates slightly deviate from the expected rates $\langle \widehat{\omega}(k) \rangle \simeq k^{-1}$ and $\langle \widehat{u}(k) \rangle \simeq k^{-3}$ by the dimensional analysis. We show the spectra with error bars for $\nu = 5.0 \times 10^{-5}$.

$r \ll 1$, and $S_p[\omega](r) \simeq r^{(3-p)/3}$, $S_p[u](r) \simeq r^{(3+2p)/3}$ for long ranges $r \approx 1$. Since the distribution of the solutions tends to be stationary, we may assume the steadiness after a long-time evolution. However, we can't suppose spatial homogeneity, since the average of the steady distribution has sharp peaks. On the other hand, numerical computation of the gCLMG equation with a stochastic forcing [14] has shown that the turbulent flow is generated by the pulse of a similar profile to ω_∞ with peaks, and it wanders randomly according to the uniform distribution. Hence, to compute the structure function, we use the numerical solution at $t = T^\nu_s$ and take the ensemble average of the local structure function $\mathcal{S}_p(T^\nu_s, x, r)$ concerning x by sampling N points from the uniform distribution on $[0, 2\pi]$. That is to say, the p-th order structure function of the vorticity is given by

$$S_p[\omega](r) \coloneqq \mathbb{E}_x[\mathcal{S}_p[\omega](T^\nu_s, \cdot, r)] = \int_0^{2\pi} \mathcal{S}_p[\omega](T^\nu_s, x, r) dx \approx \frac{2\pi}{N} \sum_{n=0}^{N-1} \mathcal{S}_p[\omega](T^\nu_s, x_n, r),$$

where $x_n = \frac{2\pi}{N} n$ for $n = 0, \dots, N-1$. The p-th order structure function of the velocity is similarly approxiamted by $S_p[u](r) \approx \frac{2\pi}{N} \sum_{n=0}^{N-1} \mathcal{S}_p[u](T^\nu_s, x_n, r)$.

In Figure 4, we plot the p-th structure functions $S_p[\omega](r)$ and $S_p[u](r)$ for $p = 2, 4$. It indicates that $S_p[\omega](r) \simeq r^p$ and $S_p[u](r) \simeq r^p$ for short distances, $r \ll 1$ agree with the dimensional analysis. For long distances $r \approx 1$, we estimate $S_2[\omega](r) \sim r^{0.54}$, $S_2[u](r) \sim r^{1.83}$, $S_4[\omega] \sim r^{0.05}$ and $S_4[u] \sim r^{3.10}$ by the least square fit. The scaling laws deviate from those suggested by the dimensional analysis, showing strong intermittency.

The power-law behaviors of the structure functions with intermittency yield the information on the profile of the steady vorticity distribution in the zero viscous limits according to an exact result on the intermittency obtained by Frisch [6], which is stated

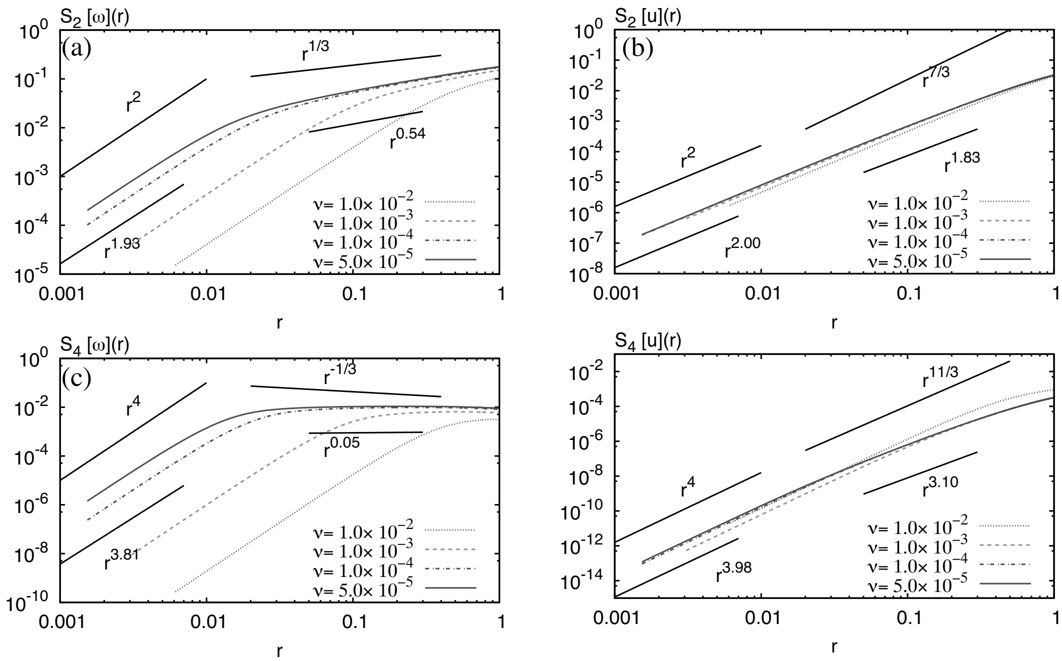

Figure 4. The p-th order structure functions $S_p[\omega](r)$ and $S_p[u](r)$ for $p = 2,\, 4$. We plot the scaling laws suggested by the dimensional analysis as well as those obtained by the least square fit. For short distances $r \ll 1$, $S_p[\omega](r) \simeq r^p$ and $S_p[u](r) \simeq r^p$, which agree with the dimensional analysis. On the other hand, for large distances $r \approx 1$, the scaling laws $S_2[\omega](r) \sim r^{0.54}$, $S_4[\omega](r) \sim r^{0.05}$, $S_2[u](r) \sim r^{1,83}$ and $S_4[u](r) \sim r^{3.10}$ deviate from the scaling laws $S_p[\omega](r) \sim r^{(3-p)/3}$ and $S_p[u](r) \sim r^{(3+2p)/3}$ owing to the dimensional analysis.

as follows. Suppose that the structure function of even order for the flow velocity v is subject to the power-law of exponent ζ_{2p} over the inertial range, namely $S_{2p}[v](r) \sim r^{\zeta_{2p}}$, and that the inertial range extends with $\nu \to 0$ as a power-law. We further assume that the two consecutive exponents satisfies $\zeta_{2p} > \zeta_{2p+2}$ for a certain $p \in \mathbb{N}$. Then the maximum velocity diverges as the viscous coefficient ν tends to zero. Let us note that, in the proof of this result, no assumption has been made on governing equations of the turbulent flows and the same argument applies to structure functions of the vorticity ω in the present study. As we have observed in Figure 4, the structure functions of the vorticity ω have the power-law behaviors over the inertial ranges, i.e., $S_2[\omega](r) \sim r^{\zeta_2}$ and $S_4[\omega](r) \sim r^{\zeta_4}$ with $\zeta_2 = 0.54 > \zeta_4 = 0.05$ and the inertial range extends as the viscous coeffcient ν vanishes, which satisfies the assumptions. Accordingly, we conclude that the maximum of the steady vorticity distribution ω diverges as $\nu \to 0$. In other words, the steady vorticity distribution tends to be a singular pulse with diverging peaks in the zero viscous limits.

4. Summary and future directions

We have established the global existence of a unique mild solution $\omega(t)$ to the gCLMG equation (1) subject to external forcing. When we regard the initial data $\omega_0^\eta(x)$ and the forcing function $f^\eta(t,x)$ as random variables sampled from a probability space, a stochastic process is defined by a continuous mapping from the pair of the random variables $(\omega_0^\eta(x), f^\eta(t,x))$ to the mild solution. The distribution of the mild solution is numerically approximated by the Galerkin approximation with the pseudo-spectral method and the gPC. We find that the distribution tends to be a steady state as $t \to \infty$. We compute some statistical quantities associated with the steady distribution and make comparisons with those of turbulent flow generated by a randomly moving pulse in [14]. The average of the steady vorticity distribution has sharp peaks at $x = 0, \pi$, which is a similar pulse pattern in this turbulent flow. The scaling laws of the energy and the enstrophy spectra also coincide with those of the pulse turbulence. Hence, the steady distribution reproduces the statistical properties of the pulse turbulence. Furthermore, we obtain the scaling laws of the p-th order structure functions of the steady distribution with $p = 2$ and $p = 4$. Both of them deviate from the scaling law expected by the dimensional analysis, which shows strong intermittency. The advantage of the gPC approximation is that such higher-order smooth structure functions can be calculated, whereas it is difficult to compute these functions due to the noisy random behavior of solutions to the gCLMG equation with a stochastic forcing as reported in [15].

We finally mention some future directions. It is mathematically important to show the existence of the steady distribution as an ω-limit solution to the gGCLM equation (1) with random external forcing. In this paper, we numerically found the special steady distribution reproducing the statistical laws of the pulse turbulence for the zero initial data and the random forcing function with uniformly distributed amplitude. We investigate how the steady distribution changes when we make the initial data a random variable. We could also consider another probability space for random forcing such as the Gaussian distribution. From the viewpoint of non-equilibrium statistical physics, it is also interesting to observe the higher-order structure function $S_p[\omega]$ for $p > 4$, since the dimensional analysis suggests a negative rate for larger p.

Appendix

The following inequalities are used in this paper.

Lemma 4.1 (Young's inequality)**.** *Let $a, b \geqq 0$. For any $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds.*

$$ab \leqq \frac{a^p}{p} + \frac{b^q}{q}. \tag{9}$$

Lemma 4.2 (Sobolev interpolation inequality)**.** *Let $0 \leqq m_1 \leqq m_2$. For $\theta \in [0,1]$, we set $m = (1-\theta)m_1 + \theta m_2$. Then if $f \in \dot{H}^{m_1} \cap \dot{H}^{m_2}$, then $f \in \dot{H}^m$. In addition, the* *following inequality holds.*

$$\|f\|_{\dot{H}^m} \leqq \|f\|_{\dot{H}^{m_1}}^{1-\theta} \|f\|_{\dot{H}^{m_2}}^{\theta}. \tag{10}$$

Lemma 4.3. *For $f, g \in \dot{H}^m$ with $m > \frac{1}{2}$, there exists a constant $C(m)$ such that*

$$\|fg\|_{\dot{H}^m} \leqq C(m) \|f\|_{\dot{H}^m} \|g\|_{\dot{H}^m}. \tag{11}$$

Proof. Suppose f and g belong to the Schwartz space $\mathcal{S}$ with $\widehat{f}(0) = \widehat{g}(0) = 0$. Then we have

$$\|fg\|_{\dot{H}^m} = \||n|^m \mathcal{F}(fg)\|_{\ell^2} = \left\||n|^m \widehat{f} * \widehat{g}\right\|_{\ell^2}.$$

Owing to $|x+y|^m \leqq 2^{m-1}(|x|^m + |y|^m)$ and Young's convolution inequality, we have

$$\begin{aligned}\left\||n|^m \widehat{f} * \widehat{g}\right\|_{\ell^2} &\lesssim \left\|\sum_{k\in\mathbb{Z}} |n-k|^m \widehat{f}(n-k)\widehat{g}(k)\right\|_{\ell^2} + \left\|\sum_{k\in\mathbb{Z}} \widehat{f}(n-k)|k|^m\widehat{g}(k)\right\|_{\ell^2} \\ &\leqq \left\||n|^m \widehat{f}\right\|_{\ell^2} \|\widehat{g}\|_{\ell^1} + \left\|\widehat{f}\right\|_{\ell^1} \||n|^m \widehat{g}\|_{\ell^2} = \|f\|_{\dot{H}^m} \|\widehat{g}\|_{\ell^1} + \left\|\widehat{f}\right\|_{\ell^1} \|g\|_{\dot{H}^m}.\end{aligned}$$

Moreover, owing to $\widehat{f}(0) = 0$ and $m > \frac{1}{2}$, we have

$$\left\|\widehat{f}\right\|_{\ell^1} = \left\||n|^{-m}|n|^m \widehat{f}\right\|_{\ell^1} \leqq \|f\|_{\dot{H}^m} \left\||n|^{-m}\right\|_{\ell^2} \leqq C(m) \|f\|_{\dot{H}^m}.$$

Hence, we have $\|fg\|_{\dot{H}^m} \leqq C(m) \|f\|_{\dot{H}^m} \|g\|_{\dot{H}^m}$ for $f, g \in \mathcal{S}$ with $\widehat{f}(0) = \widehat{g}(0) = 0$. Since $\mathcal{S}$ is dense in $\dot{H}^m$, the same inequality holds for $\dot{H}^m$ by continuity. □

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