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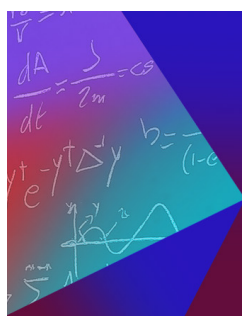


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ABSTRACT

We obtain exact analytic expressions of real tensor eigenvalue/vector distributions of real symmetric order-three tensors with Gaussian distributions for $N \leq 8$. This is achieved by explicitly computing the partition function of a zero-dimensional boson–fermion system with four interactions. The distributions are expressed by combinations of polynomial, exponential, and error functions as results of feasible complicated bosonic integrals that appear after fermionic integrations. By extrapolating the expressions and also using a previous result, we guess a large- N expression. The expressions are compared with Monte Carlo simulations, and precise agreement and good agreement are obtained with the exact and the large- N expressions, respectively. Understanding the feasibility of the integration is left for future study, which would provide a general- N analytic formula.

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I. INTRODUCTION

Eigenvalue distributions are important dynamical quantities in studies of matrix models. They model the energy eigenvalue distributions of complex dynamical systems.¹ They provide a major technique in solving matrix models.² Their topological properties differentiate phases of matrix models (for more details, see, for instance, Ref. 3), providing insights into the dynamics of gauge theories.^{4,5}

Recently, tensor models^{6–9} attract much attention in various contexts.¹⁰ While it is important to develop efficient techniques of computing eigenvalues/vectors for certain tensors in various practical applications of tensors,¹¹ it is also interesting to study their distributions for ensembles of tensors because tensors are dynamical in tensor models.

While there are already some interesting results,^{12–17} eigenvalue distributions in tensor models still remain largely unexplored. In our previous studies,^{16,17} the problem was rewritten as computations of partition functions of zero-dimensional fermion systems. In Ref. 16, an exact formula of signed distributions of real eigenvalue/vector distributions for real symmetric order-three tensors with Gaussian distributions was obtained, where each eigenvalue/vector contributed to the distribution by ± 1 , depending on the sign of a Hessian matrix associated with each eigenvalue/vector. In Ref. 17, real eigenvalue/vector distributions for the same ensemble of tensors were studied. In this study, however, only an approximate large- N expression (N denotes the dimension of the index vector space of the tensor) was obtained by truncating a Schwinger–Dyson equation, though some closely related exact results were also obtained. While the functional form of the large- N expression, which was a Gaussian, agreed well with Monte Carlo simulations, the overall factor did not, and an improvement was expected.

In this paper, we rewrite the real eigenvalue/vector problem as a computation of a partition function of a boson–fermion system with four interactions, instead of the fermion systems in the previous studies.^{16,17} We exactly compute the real eigenvalue/vector distributions for real symmetric order-three tensors with Gaussian distributions for $N \leq 8$ and extrapolate the expressions to guess a large- N expression, including the overall factor, which was not correctly obtained by the approximation of the previous paper.¹⁷ We compare the exact expressions for small N with Monte Carlo simulations and obtain precise agreement. We also compare the large- N expression and obtain good agreement.

Computing the real eigenvalue/vector distributions is essentially the same as the computation of complexity in the p -spin spherical model of spin glasses^{18,19} (see Appendix D). By using a rotational symmetry, the problem can be mapped to a random matrix model, which can subsequently be solved by the standard matrix model methods.^{12,13,20} In fact, the results of Sec. VI can be obtained from those in Ref. 20, as explained in Appendix D. However, we expect that the field theoretical method employed in this paper may provide new insights and new applications in future analysis of the tensor eigenvalue/vector problems, which are still largely unexplored.

This paper is organized as follows: In Sec. II, we define what we compute, the real eigenvalue/vector distributions for real symmetric order-three tensors with Gaussian distributions. In Sec. III, we rewrite the problem as a computation of a partition function of a zero-dimensional boson–fermion system with four interactions. In Sec. IV, we determine the general form, which is obtained by integrating out the fermions in the partition function. In Sec. V, we perform the remaining bosonic integrals but one. In Sec. VI, we explicitly perform the last bosonic integral for each case of $N \leq 8$ and obtain the explicit exact expressions of the distributions. Interestingly, the apparently complicated bosonic integrals are feasible, and the final results have simple expressions. In Sec. VII, we perform some Monte Carlo simulations, and precise agreement with the exact expressions is obtained. In Sec. VIII, we perform an extrapolation of the exact results for small N to guess a large- N expression. We compare it with Monte Carlo simulations, and good agreement is obtained. Section IX is devoted to a summary and future prospects.

II. REAL EIGENVECTOR/VALUE DISTRIBUTIONS OF TENSORS

In this paper, we restrict ourselves to the real symmetric order-three tensors, i.e., $C_{abc} \in \mathbb{R}$ ($a, b, c = 1, 2, \dots, N$), satisfying $C_{abc} = C_{bac} = C_{bca}$, as the simplest case. As for eigenvalues/vectors of tensors, there are a few similar but slightly different definitions in the literature.^{21–23} In this paper, we employ the definition that the real eigenvectors v of a given C are the non-zero solutions to

$$C_{abc}v_bv_c = v_a, \quad (v \neq 0, v \in \mathbb{R}^N), \quad (1)$$

where the repeated indices are assumed to be summed over, as will be assumed in the rest of this paper. Then, the distribution of the eigenvectors for a given C is given by

$$\begin{aligned} \rho(v, C) &= \sum_{i=1}^{n_C} \delta^N(v - v^i) \\ &= |\det M| \prod_{a=1}^N \delta(v_a - C_{abc}v_bv_c) \end{aligned} \quad (2)$$

for $v \neq 0$ under the volume measure $d^N v = \prod_{a=1}^N dv_a$, where v^i ($i = 1, 2, \dots, n_C$) are all the solutions to (1), $|\cdot|$ denotes the absolute value, and

$$M_{ab} = \frac{\partial}{\partial v_a} (v_b - C_{bcd}v_cv_d) = \delta_{ab} - 2C_{abc}v_c. \quad (3)$$

Here, the absolute value of the determinant, $|\det M|$, is the Jacobian associated with the change of the arguments of the δ -functions performed in (2).

When the tensor C has a Gaussian distribution, the real eigenvector distribution is given by

$$\rho(v) = A^{-1} \int_{\mathbb{R}^{\#C}} dC e^{-\alpha C^2} \rho(v, C), \quad (4)$$

where $dC = \prod_{a \leq b \leq c=1}^N dC_{abc}$, $A = \int_{\mathbb{R}^{\#C}} dC e^{-\alpha C^2}$, $C^2 = C_{abc}C_{abc}$, $\alpha > 0$, and $\#C = N(N+1)(N+2)/6$, which is the total number of the independent components of C .

It is worth commenting on the eigenvalue distribution corresponding to the real eigenvector distribution above. An eigenvalue ζ accompanied with a real eigenvector v (a Z -eigenvalue in the terminology of Ref. 21) is defined by

$$C_{abc}w_bw_c = \zeta w_a \quad (|w| = 1, w \in \mathbb{R}^N), \quad (5)$$

where $|w| = \sqrt{w_a w_a}$. Comparing with (1), we obtain the relation

$$\zeta = \frac{1}{|v|}. \quad (6)$$

Therefore, by using (6) and the fact that $\rho(v)$ is actually a function of $|v|$ because of the rotational symmetry of the distribution, we obtain the real eigenvalue distribution,

$$\rho_{\text{eig}}(\zeta) = \rho(1/\zeta)S_{N-1}\zeta^{-N-1}, \tag{7}$$

where $1/\zeta$ in the argument abusively represents an arbitrary vector of size $1/\zeta$ and $S_{N-1} = 2\pi^{N/2}/\Gamma[N/2]$, the surface volume of a unit sphere in an N -dimensional space.

Here, we would like to stress that we are specifically considering real eigenvalues/vectors only. Unlike real symmetric matrices, real symmetric tensors can also have complex eigenvalues/vectors by allowing complex solutions to the equation in (1). To include such complex solutions in our analysis, the expressions in this section need to appropriately be modified. Note also that the connection to the p -spin spherical model explained in Appendix D is lost in this case. Therefore, the end results for the complex case may have non-trivial differences from the real case.

III. A ZERO-DIMENSIONAL BOSON-FERMION SYSTEM

In this section, we will rewrite (4) with (2) as a partition function of a zero-dimensional boson–fermion system with four interactions. An immediate obstacle in doing this is the presence of an absolute value in (2), which is not an analytic function. To rewrite it in an analytic form, we take

$$|\det M| = \lim_{\epsilon \rightarrow +0} \frac{\det(M^2 + \epsilon I)}{\sqrt{\det(M^2 + \epsilon I)}}, \tag{8}$$

where $I_{ab} = \delta_{ab}$ is an identity matrix, and the parameter ϵ is a positive small regularization parameter, which assures the convergence of the integrals below.

Determinant factors, such as (8), can be managed inside partition functions of zero-dimensional field theoretical systems. This technique is common in supersymmetric approaches to disorder averaging in statistical physics (see, for instance, Ref. 24 and references therein). The numerator $\det(M^2 + \epsilon I)$ can be rewritten as $\det(M^2 + \epsilon I) = \int d\tilde{\psi}d\psi e^{\tilde{\psi}_a(M^2)_{ab}\psi_b + \epsilon\tilde{\psi}_a\psi_a}$ by introducing a fermion pair, $\tilde{\psi}$ and ψ .²⁵ However, the exponent of this expression contains C in a quadratic manner [see (3)] and is difficult to handle when we perform an integration over C as will be done below. Therefore, as was done previously in Ref. 17, we introduce another fermion pair, $\tilde{\varphi}, \varphi$, to rewrite the exponent in a form linear in C ,

$$\det(M^2 + \epsilon I) = (-1)^N \int d\tilde{\psi}d\psi d\tilde{\varphi}d\varphi e^{-\tilde{\varphi}\varphi - \tilde{\psi}M\varphi - \tilde{\varphi}M\psi + \epsilon\tilde{\psi}\psi}, \tag{9}$$

where the contracted indices are suppressed for brevity, $\tilde{\varphi}\varphi = \tilde{\varphi}_a\varphi_a$, $\tilde{\psi}M\varphi = \tilde{\psi}_a M_{ab}\varphi_b$, and so on. The equality can be shown by noting that $\tilde{\varphi}\varphi + \tilde{\psi}M\varphi + \tilde{\varphi}M\psi = (\tilde{\varphi} + \tilde{\psi}M)(\varphi + M\psi) - \tilde{\psi}M^2\psi$. In a similar manner, we obtain

$$\frac{1}{\sqrt{\det(M^2 + \epsilon I)}} = \pi^{-N} \int_{\mathbb{R}^{2N}} d\phi d\sigma e^{-\sigma^2 - 2i\sigma M\phi - \epsilon\phi^2}, \tag{10}$$

where σ is a new bosonic variable and $\sigma^2 = \sigma_a\sigma_a$, and so on.

By using (8)–(10) and a well-known formula, $\int_{\mathbb{R}} dx e^{ipx} = 2\pi\delta(p)$, (4) with (2) can be rewritten as

$$\rho(v) = \lim_{\epsilon \rightarrow +0} A^{-1} (2\pi)^{-N} \pi^{-N} (-1)^N \int dC d\lambda d\phi d\sigma d\tilde{\psi}d\psi d\tilde{\varphi}d\varphi e^{S_1}, \tag{11}$$

where

$$S_1 = -\alpha C^2 + i\lambda_a(v_a - C_{abc}v_bv_c) - \sigma^2 - 2i\sigma M\phi - \epsilon\phi^2 - \tilde{\varphi}\varphi - \tilde{\psi}M\varphi - \tilde{\varphi}M\psi + \epsilon\tilde{\psi}\psi, \tag{12}$$

with a new bosonic variable λ_a ($a = 1, 2, \dots, N$) to rewrite the δ -functions.

Next, let us perform the integrations over C and λ in the expression (11). Let us first consider C . The terms containing C in (12) are

$$S_C = -\alpha C^2 - iC_{abc}\lambda_a v_b v_c + 4iC_{abc}v_a \sigma_b \phi_c + 2C_{abc}v_a \tilde{\psi}_b \varphi_c + 2C_{abc}v_a \tilde{\varphi}_b \psi_c. \tag{13}$$

Then, we obtain

$$\int_{\mathbb{R}^{6C}} dC e^{S_C} = A e^{\delta S_C}, \tag{14}$$

where

$$\delta S_C = \frac{1}{\alpha} \left(\frac{1}{6} \sum_s \left(-\frac{i}{2} \lambda_{s_a} v_{s_b} v_{s_c} + 2i v_{s_a} \sigma_{s_b} \phi_{s_c} + v_{s_a} \tilde{\psi}_{s_b} \varphi_{s_c} + v_{s_a} \tilde{\varphi}_{s_b} \psi_{s_c} \right) \right)^2, \tag{15}$$

where the summation is over all the permutations of a, b, c , which is necessary because C is a symmetric tensor. By explicitly expanding (15), we obtain

$$\delta S_C = -\frac{|v|^4}{12\alpha} \lambda_a B_{ab} \lambda_b - i\lambda_a (D_a + \tilde{D}_a) + E_1 + E_2 + E_3, \quad (16)$$

where

$$\begin{aligned} B_{ab} &= \delta_{ab} + 2\hat{v}_a \hat{v}_b = I_{\perp ab} + 3I_{\parallel ab}, \\ D_a &= \frac{|v|^3}{3\alpha} (\tilde{\psi}_{\parallel} \varphi_{\parallel} + \tilde{\varphi}_{\parallel} \psi_{\parallel}) \hat{v}_a + \frac{|v|^3}{3\alpha} (\tilde{\psi}_a \varphi_{\parallel} + \tilde{\psi}_{\parallel} \varphi_a + \tilde{\varphi}_a \psi_{\parallel} + \tilde{\varphi}_{\parallel} \psi_a), \\ \tilde{D}_a &= \frac{2i}{\alpha} v_b v_c \cdot \frac{1}{6} \sum_s v_{s_a} \sigma_{s_b} \phi_{s_c}, \\ E_1 &= \frac{1}{\alpha} \left(\frac{1}{6} \sum_s (\tilde{\psi}_{s_a} \varphi_{s_b} v_{s_c} + \tilde{\varphi}_{s_a} \psi_{s_b} v_{s_c}) \right)^2, \\ E_2 &= -\frac{4}{\alpha} \left(\frac{1}{6} \sum_s v_{s_a} \sigma_{s_b} \phi_{s_c} \right)^2, \\ E_3 &= \frac{4i}{\alpha} (v_a \tilde{\psi}_b \varphi_c + v_a \tilde{\varphi}_b \psi_c) \cdot \frac{1}{6} \sum_s v_{s_a} \sigma_{s_b} \phi_{s_c}. \end{aligned} \quad (17)$$

Here, $\hat{v} = v/|v|$, $\tilde{\psi}_{\parallel} = \tilde{\psi}_a \hat{v}_a$, etc., and I_{\parallel} and I_{\perp} are, respectively, the projection matrices to the parallel and transverse subspaces to \hat{v} .

Next, let us perform the integration over λ . Picking up the terms containing λ (with no C) in (12) and (16), we obtain

$$S_{\lambda} = -\frac{|v|^4}{12\alpha} \lambda_a B_{ab} \lambda_b + i\lambda_a (v_a - D_a - \tilde{D}_a). \quad (18)$$

Considering that B is a sum of the projection matrices as in (17), we obtain

$$\int_{\mathbb{R}^N} d\lambda e^{S_{\lambda}} = |v|^{-2N} (12\pi\alpha)^{\frac{N}{2}} (\det B)^{-\frac{1}{2}} e^{\delta S_{\lambda}}, \quad (19)$$

where $\det B = 3$ from (17), and

$$\begin{aligned} \delta S_{\lambda} &= -3\alpha |v|^{-4} (v - D - \tilde{D})_a B_{ab}^{-1} (v - D - \tilde{D})_b \\ &= -\alpha |v|^{-2} + 2\alpha |v|^{-3} D_{\parallel} + 2\alpha |v|^{-3} \tilde{D}_{\parallel} \\ &\quad - 3\alpha |v|^{-4} \left(D_{\perp} \cdot D_{\perp} + \frac{1}{3} D_{\parallel}^2 + 2D_{\perp} \cdot \tilde{D}_{\perp} + \tilde{D}_{\perp}^2 + \frac{2}{3} D_{\parallel} \tilde{D}_{\parallel} + \frac{1}{3} \tilde{D}_{\parallel}^2 \right). \end{aligned} \quad (20)$$

Here, we have used $B^{-1} = I_{\perp} + \frac{1}{3} I_{\parallel}$, $D_{\perp} \cdot \tilde{D}_{\perp} = D_{\perp a} \tilde{D}_{\perp a}$, and so on, where X_{\perp} ($X = D, \tilde{D}$) denotes vector X projected to the transverse subspace to \hat{v} . From (17), the projections of D and \tilde{D} are more explicitly given by

$$\begin{aligned} D_{\parallel} &= \frac{|v|^3}{\alpha} (\tilde{\psi}_{\parallel} \varphi_{\parallel} + \tilde{\varphi}_{\parallel} \psi_{\parallel}), \\ D_{\perp} &= \frac{|v|^3}{3\alpha} (\tilde{\psi}_{\perp} \varphi_{\parallel} + \tilde{\psi}_{\parallel} \varphi_{\perp} + \tilde{\varphi}_{\perp} \psi_{\parallel} + \tilde{\varphi}_{\parallel} \psi_{\perp}), \\ \tilde{D}_{\parallel} &= \frac{2i|v|^3}{\alpha} \sigma_{\parallel} \phi_{\parallel}, \\ \tilde{D}_{\perp} &= \frac{2i|v|^3}{3\alpha} (\sigma_{\parallel} \phi_{\perp} + \sigma_{\perp} \phi_{\parallel}). \end{aligned} \quad (21)$$

The first term in the second line of (20) is a constant contribution to the potential. The second and the third terms in the same line are corrections to the kinetic terms of the parallel components of the fermions and the bosons, respectively. The other terms describe the four interaction terms among the bosons and the fermions. Collecting the results above, we obtain the following intermediate expression:

$$\rho(v) = \lim_{\epsilon \rightarrow +0} 3^{\frac{N-1}{2}} \pi^{-\frac{3N}{2}} \alpha^{\frac{N}{2}} |v|^{-2N} e^{-\frac{\alpha}{|v|^2}} (-1)^N \int d\phi d\sigma d\tilde{\psi} d\psi d\tilde{\varphi} d\varphi e^{\tilde{K}_F + \tilde{K}_B + \tilde{V}_F + \tilde{V}_B + \tilde{V}_{BF}}, \quad (22)$$

where

$$\begin{aligned}\tilde{K}_F &= -\tilde{\varphi}_\perp \cdot \varphi_\perp - \tilde{\psi}_\perp \cdot \varphi_\perp - \tilde{\varphi}_\perp \cdot \psi_\perp + \epsilon \tilde{\psi}_\perp \cdot \psi_\perp - \tilde{\varphi}_\parallel \varphi_\parallel + \tilde{\psi}_\parallel \varphi_\parallel + \tilde{\varphi}_\parallel \psi_\parallel + \epsilon \tilde{\psi}_\parallel \psi_\parallel, \\ \tilde{K}_B &= -\sigma_\perp^2 - 2i\sigma_\perp \cdot \phi_\perp - \epsilon \phi_\perp^2 - \sigma_\parallel^2 + 2i\sigma_\parallel \phi_\parallel - \epsilon \phi_\parallel^2, \\ \tilde{V}_F &= E_1 - 3\alpha|v|^{-4} \left(D_\perp \cdot D_\perp + \frac{1}{3} D_\parallel^2 \right), \\ \tilde{V}_B &= E_2 - 3\alpha|v|^{-4} \left(\tilde{D}_\perp^2 + \frac{1}{3} \tilde{D}_\parallel^2 \right), \\ \tilde{V}_{BF} &= E_3 - 3\alpha|v|^{-4} \left(2D_\perp \cdot \tilde{D}_\perp + \frac{2}{3} D_\parallel \tilde{D}_\parallel \right).\end{aligned}\tag{23}$$

Here, \tilde{K}_F, \tilde{V}_F contain only the fermions, \tilde{K}_B, \tilde{V}_B contain only the bosons, and \tilde{V}_{BF} is a mixture. Note that the quadratic terms of the parallel components in \tilde{K}_F and \tilde{K}_B have been corrected by the aforementioned terms from the second line of (20).

Now, we want to compute $\tilde{V}_B, \tilde{V}_F, \tilde{V}_{BF}$ more explicitly. The computation of \tilde{V}_F in (23) is essentially the same as the derivation of the four Fermi interaction terms in Ref. 17 for $R = 1$. By noting that two of the terms in Ref. 17 do not appear, because $\tilde{\psi}_\perp \cdot \tilde{\psi}_\perp = \tilde{\varphi}_\perp \cdot \tilde{\varphi}_\perp = 0$ (for $R = 1$), we obtain

$$\tilde{V}_F = -\frac{|v|^2}{6\alpha} \left((\tilde{\psi}_\perp \cdot \varphi_\perp)^2 + (\tilde{\varphi}_\perp \cdot \psi_\perp)^2 + 2\tilde{\psi}_\perp \cdot \tilde{\varphi}_\perp \varphi_\perp \cdot \psi_\perp + 2\tilde{\psi}_\perp \cdot \psi_\perp \tilde{\varphi}_\perp \cdot \varphi_\perp \right).\tag{24}$$

What seems surprising in this expression is that the parallel components of the fermions cancel out from the interactions after all.

As for \tilde{V}_B and \tilde{V}_{BF} in (23), by using the results of the computations of E_2 and E_3 in Appendix A, we obtain

$$\begin{aligned}\tilde{V}_B &= -\frac{2|v|^2}{3\alpha} \left(\sigma_\perp^2 \phi_\perp^2 + (\sigma_\perp \cdot \phi_\perp)^2 \right), \\ \tilde{V}_{BF} &= \frac{2i|v|^2}{3\alpha} \left(\tilde{\psi}_\perp \cdot \sigma_\perp \varphi_\perp \cdot \phi_\perp + \tilde{\varphi}_\perp \cdot \sigma_\perp \psi_\perp \cdot \phi_\perp + \tilde{\psi}_\perp \cdot \phi_\perp \varphi_\perp \cdot \sigma_\perp + \tilde{\varphi}_\perp \cdot \phi_\perp \psi_\perp \cdot \sigma_\perp \right).\end{aligned}\tag{25}$$

We again find the surprising fact that no parallel components appear in \tilde{V}_B and \tilde{V}_{BF} .

Because the parallel components only exist in \tilde{K}_B, \tilde{K}_F and do not interact, these can trivially be integrated out that generates the overall factors of π and -1 for the bosons and fermions, respectively. Therefore, we finally obtain

$$\rho(v) = \lim_{\epsilon \rightarrow +0} 3^{\frac{N-1}{2}} \pi^{-\frac{3N}{2}+1} \alpha^{\frac{N}{2}} |v|^{-2N} e^{-\frac{\alpha}{|v|^2}} (-1)^{N-1} \int d\phi_\perp d\sigma_\perp d\tilde{\psi}_\perp d\psi_\perp d\tilde{\varphi}_\perp d\varphi_\perp e^{\tilde{K}_F + \tilde{K}_B + \tilde{V}_F + \tilde{V}_B + \tilde{V}_{BF}},\tag{26}$$

where

$$\begin{aligned}\tilde{K}_F^\perp &= -\tilde{\varphi}_\perp \cdot \varphi_\perp - \tilde{\psi}_\perp \cdot \varphi_\perp - \tilde{\varphi}_\perp \cdot \psi_\perp + \epsilon \tilde{\psi}_\perp \cdot \psi_\perp, \\ \tilde{K}_B^\perp &= -\sigma_\perp^2 - 2i\sigma_\perp \cdot \phi_\perp - \epsilon \phi_\perp^2,\end{aligned}\tag{27}$$

and $\tilde{V}_F, \tilde{V}_B, \tilde{V}_{BF}$ are given in (24) and (25).

IV. INTEGRATIONS OVER FERMIONS

In expression (26), the variables projected to the transverse directions, i.e., $\tilde{\psi}_\perp$, etc., just represent $N - 1$ -dimensional degrees of freedom with no other restrictions. Therefore, we can simply regard them as $N - 1$ -dimensional variables from the beginning. The external variable, the vector v , only appears in the coupling constants of the four interactions through $|v|$. Therefore, we can simply write

$$\rho(v) = \lim_{\epsilon \rightarrow +0} 3^{\frac{N-1}{2}} \pi^{-\frac{3N}{2}+1} \alpha^{\frac{N}{2}} |v|^{-2N} e^{-\frac{\alpha}{|v|^2}} (-1)^{N-1} \int_{N^*} d\phi d\sigma d\tilde{\psi} d\psi d\tilde{\varphi} d\varphi e^{K_F + K_B + V_F + V_B + V_{BF}},\tag{28}$$

where all the variables are $N - 1$ -dimensional and

$$\begin{aligned}K_F &= -\tilde{\varphi} \cdot \varphi - \tilde{\psi} \cdot \varphi - \tilde{\varphi} \cdot \psi + \epsilon \tilde{\psi} \cdot \psi, \\ K_B &= -\sigma^2 - 2i\sigma \cdot \phi - \epsilon \phi^2, \\ V_F &= -\frac{|v|^2}{6\alpha} \left((\tilde{\psi} \cdot \varphi)^2 + (\tilde{\varphi} \cdot \psi)^2 + 2\tilde{\psi} \cdot \tilde{\varphi} \varphi \cdot \psi + 2\tilde{\psi} \cdot \psi \tilde{\varphi} \cdot \varphi \right), \\ V_B &= -\frac{2|v|^2}{3\alpha} \left(\sigma^2 \phi^2 + (\sigma \cdot \phi)^2 \right), \\ V_{BF} &= \frac{2i|v|^2}{3\alpha} \left(\tilde{\psi} \cdot \sigma \varphi \cdot \phi + \tilde{\varphi} \cdot \sigma \psi \cdot \phi + \tilde{\psi} \cdot \phi \varphi \cdot \sigma + \tilde{\varphi} \cdot \phi \psi \cdot \sigma \right).\end{aligned}\tag{29}$$

Note that, for simplicity, we are abusively using the same notations of the variables as those in Sec. III with a different dimension, and to indicate this difference, the symbol N^* is attached to the integral symbol in (28).

To compute $\rho(v)$ explicitly, we first perform the fermionic integrations. The integrand in (28) can be expanded in the fermions. A useful property in this expansion is that the expansion in V_{BF} stops at the fourth order,

$$e^{V_{BF}} = \sum_{n=0}^4 \frac{1}{n!} (V_{BF})^n. \quad (30)$$

This can be proven as follows: The fermions in V_{BF} are projected to ϕ or σ . Therefore, V_{BF} contains only eight independent fermions in total, i.e., $\bar{\psi} \cdot \phi, \bar{\psi} \cdot \sigma, \psi \cdot \phi, \psi \cdot \sigma, \bar{\varphi} \cdot \phi, \bar{\varphi} \cdot \sigma, \varphi \cdot \phi, \varphi \cdot \sigma$. Since each term of V_{BF} contains two of these fermions and products of more than eight of these fermions vanish, we obtain

$$(V_{BF})^n = 0 \text{ for } n > 4. \quad (31)$$

We also note that when $\sigma = \pm\phi$, only four of these fermions are independent. Therefore,

$$(V_{BF})^n = 0 \text{ for } n > 2, \text{ when } \sigma = \pm\phi. \quad (32)$$

Now, we want to determine the functional forms of the fermionic integrals of each summand in (30),

$$\beta_n = \frac{(-1)^{N-1}}{n!} \int_{N^*} d\bar{\psi} d\psi d\bar{\varphi} d\varphi (V_{BF})^n e^{K_F+V_F} \quad (n \leq 4). \quad (33)$$

From the form of V_{BF} and the $O(N-1)$ symmetry, β_n should be a polynomial function of $\sigma^2, \phi^2, \sigma \cdot \phi$, and its order in σ, ϕ should be $2n$. In addition, each term of the polynomial function should contain equal numbers of σ and ϕ , and the polynomial function should also be invariant under the interchange $\sigma \leftrightarrow \phi$ as a whole.

From the above considerations, we uniquely obtain for $n = 1$,

$$\beta_1 = a_1 \sigma \cdot \phi. \quad (34)$$

Here, a_1 is a function of ϵ and $|v|^2/\alpha$, and all a_i below are also so.

For $n = 2$, we have two possibilities,

$$\beta_2 = a_2 (\sigma \cdot \phi)^2 + a_3 \sigma^2 \phi^2. \quad (35)$$

As for $n = 3$, β_3 must vanish for $\sigma = \pm\phi$ due to (32). Considering also the other conditions mentioned above, we uniquely obtain

$$\beta_3 = a_4 (\sigma^2 \phi^2 \sigma \cdot \phi - (\sigma \cdot \phi)^3). \quad (36)$$

As for $n = 4$, we need not only (32) but also the following property: Taking the derivatives of $(V_{BF})^4$ with respect to σ, ϕ three times or less must vanish when $\sigma = \pm\phi$. This can be proven as follows. Obviously, $(V_{BF})^4$ is proportional to the product of the eight projected fermions, $\bar{\psi} \cdot \phi, \bar{\psi} \cdot \sigma, \dots$. After taking the derivatives of this three times or less with respect to σ, ϕ , each term contains at least five of the eight projected fermions. Therefore, all these terms vanish for $\sigma = \pm\phi$ because of the same reason for (32). Now, using this property, we can uniquely determine

$$\beta_4 = a_5 (\sigma^2 \phi^2 - (\sigma \cdot \phi)^2). \quad (37)$$

Collecting all the results above, we conclude that the fermionic integrations of the partition function has the following general form:

$$\int_{N^*} d\bar{\psi} d\psi d\bar{\varphi} d\varphi e^{K_F+V_F+V_{BF}} = a_0 + a_1 \sigma \cdot \phi + a_2 (\sigma \cdot \phi)^2 + a_3 \sigma^2 \phi^2 + a_4 (\sigma^2 \phi^2 \sigma \cdot \phi - (\sigma \cdot \phi)^3) + a_5 (\sigma^2 \phi^2 - (\sigma \cdot \phi)^2). \quad (38)$$

Here, a_i are generally functions of ϵ and $|v|^2/\alpha$, but ϵ in them turn out to simply disappear in the $\epsilon \rightarrow +0$ limit in (28) without any essential roles. Therefore, in the following computations, we ignore it by just putting $\epsilon = 0$ in a_i .

V. INTEGRATIONS OVER BOSONS

Assuming the form (38) of the fermionic integrations, we perform the bosonic integrations but one in this section.

Let us first generalize the bosonic kinetic term by introducing new parameters ϵ_i ($i = 1, 2, 3$),

$$K_B^\epsilon = -\epsilon_1 \sigma^2 - 2i\epsilon_2 \sigma \cdot \phi - \epsilon_3 \phi^2, \tag{39}$$

where $\epsilon_1, \epsilon_3 > 0$ are assumed for the convergence of the bosonic integration. The original kinetic term in (29) corresponds to $\epsilon_1 = \epsilon_2 = 1, \epsilon_3 = \epsilon$. Using this new kinetic term, (28), and (38), $\rho(v)$ can be expressed as

$$\rho(v) = 3^{\frac{N-1}{2}} \pi^{-\frac{3N}{2}+1} \alpha^{\frac{N}{2}} |v|^{-2N} e^{-\frac{\alpha}{|v|^2}} G_N, \tag{40}$$

where

$$G_N = (a_0 + a_1 D_1 + a_2 D_2^2 + a_3 D_1 + a_4 (D_1 D_2 - D_2^3) + a_5 (D_1 - D_2^2)^2) \int_{N^*} d\sigma d\phi e^{K_B^\epsilon + V_B} \Big|_{\substack{\epsilon_1 = \epsilon_2 = 1 \\ \epsilon_3 = +0}}, \tag{41}$$

with D_1, D_2 being the following partial derivative operators:

$$\begin{aligned} D_1 &= \frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_3}, \\ D_2 &= -\frac{1}{2i} \frac{\partial}{\partial \epsilon_2}. \end{aligned} \tag{42}$$

The bosonic integration in (41) does not seem to be fully integrable, but it has a simpler expression. Since σ appears at most quadratically in $K_B^\epsilon + V_B$, the σ integration can be performed,

$$\int_{N^*} d\sigma e^{K_B^\epsilon + V_B} = \pi^{\frac{N-1}{2}} (\epsilon_1 + 8z\phi^2)^{-\frac{1}{2}} (\epsilon_1 + 4z\phi^2)^{\frac{N-2}{2}} \exp\left(-\frac{(\epsilon_2^2 + \epsilon_3(\epsilon_1 + 8z\phi^2))\phi^2}{\epsilon_1 + 8z\phi^2}\right), \tag{43}$$

where for brevity we have introduced

$$z = \frac{|v|^2}{6\alpha}. \tag{44}$$

(43) does not depend on the angular directions of ϕ , and therefore, the integration over these produces the spherical volume, $2\pi^{(N-1)/2} |\phi|^{N-2} / \Gamma[(N-1)/2]$. Then, after applying the derivative operators in (41) and performing a replacement of variable, $|\phi| = \sqrt{x}/\sqrt{1-8zx}$, we obtain

$$\begin{aligned} G_N &= \frac{\pi^{N-1}}{4\Gamma[\frac{N-1}{2}]} \int_0^{\frac{1}{8z}} dx e^{-x} x^{\frac{N-3}{2}} (1-4zx)^{-\frac{N+2}{2}} \\ &\quad \cdot (4a_0 + 2(-2ia_1 + a_2 + (N-1)a_3)x \\ &\quad + (8iza_1 - (3+4z)(a_2 + a_3) - i(N-2)a_4)x^2 + 8z(a_2 + a_3)x^3), \end{aligned} \tag{45}$$

where we have used

$$64z^2 a_0 - 8iza_1 + (4z-1)a_2 + (-1-4z+8(N-1)z)a_3 - i(N-2)a_4 + N(N-2)a_5 = 0 \tag{46}$$

to delete a_5 . As we will see in Sec. VI, (46) holds for all the cases we consider (namely, $N \leq 8$). In fact, this relation is essentially important because otherwise the integrand has an extra factor $1/(1-8zx)$, and the feasibility of the integration over x which will be performed in Sec. VI would become unclear.

VI. ANALYTIC EXPRESSIONS

In this section, we explicitly compute the integration (45) for each case of $N \leq 8$ to obtain exact analytic expressions. What is surprising is that the seemingly difficult integrations can be done explicitly. The apparent reason is that the integrand of (45) turns out to be a sum of a total derivative and a simple integrable function.

A. $N = 1$

This case is trivial, since the integration in (28) can just be ignored. We obtain

$$\rho_{N=1}(v) = \pi^{-\frac{1}{2}} \alpha^{\frac{1}{2}} |v|^{-2} e^{-\frac{\alpha}{|v|^2}}. \tag{47}$$

Since $v = 1/C$ for $N = 1$ from (1), $\rho_{N=1}(v)dv$ is indeed equivalent to $\alpha^{1/2} \pi^{-1/2} e^{-\alpha C^2} dC$, which is the Gaussian distribution of C .

B. $N = 2$

In this case, since the variables in (28) are all one-dimensional, we can ignore the indices. Then, we immediately obtain

$$\begin{aligned} V_F^{N=2} &= -4z \bar{\psi} \psi \bar{\phi} \phi, \\ V_B^{N=2} &= -8z \sigma^2 \phi^2, \\ V_{BF}^{N=2} &= 8iz \sigma \phi (\bar{\psi} \phi + \bar{\phi} \psi). \end{aligned} \tag{48}$$

Hence, by explicitly expanding the integrand, we obtain

$$\int_{N^*} d\bar{\psi} d\psi d\bar{\phi} d\phi e^{K_F^{N=2} + V_F^{N=2} + V_{BF}^{N=2}} = 1 + 4z - 16iz\sigma\phi - 64z^2\sigma^2\phi^2. \tag{49}$$

This determines

$$\begin{aligned} a_0 &= 1 + 4z, \\ a_1 &= -16iz, \\ a_2 &= -64z^2, \\ \text{Others} &= 0 \end{aligned} \tag{50}$$

by comparing with (38). This indeed satisfies (46). Putting (50) into (45), we obtain

$$\begin{aligned} G_{N=2} &= \sqrt{\pi} \int_0^{\frac{1}{8z}} dx e^{-x} x^{-\frac{1}{2}} (1 + 4z - 8zx) \\ &= \sqrt{\pi} \left((1 + 4z) \gamma \left[\frac{1}{2}, \frac{1}{8z} \right] - 8z \gamma \left[\frac{3}{2}, \frac{1}{8z} \right] \right) \\ &= \sqrt{\pi} \left(\gamma \left[\frac{1}{2}, \frac{1}{8z} \right] + \sqrt{8z} e^{-\frac{1}{8z}} \right), \end{aligned} \tag{51}$$

where the lower incomplete gamma function $\gamma[\cdot, \cdot]$ is defined by

$$\gamma[a, y] = \int_0^y dt t^{a-1} e^{-t}, \tag{52}$$

and we have used its property,

$$\gamma[a + 1, y] = a\gamma[a, y] - y^a e^{-y}. \tag{53}$$

The lower incomplete gamma function with index 1/2 in (51) is related to the error function by

$$\gamma \left[\frac{1}{2}, y \right] = \sqrt{\pi} \operatorname{erf}[\sqrt{y}]. \tag{54}$$

Therefore, from (44) and (51), $G_{N=2}$ is represented by a combination of polynomial, exponential, and error functions of $|v|$. This is common to the other cases of N shown below.

C. $N = 3$

The fermionic integration of (38) seems too complicated to perform by hand. Rather, we use a Mathematica package for Grassmann variables.²⁶ The result is

$$\begin{aligned} a_0 &= 1 + 4z + 28z^2, \\ a_1 &= -16i(z + 2z^2), \\ a_2 &= -32(3z^2 + 2z^3), \\ a_3 &= -32(-z^2 + 6z^3), \\ a_4 &= -256iz^3, \\ a_5 &= 256z^4. \end{aligned} \tag{55}$$

This indeed satisfies (46). Then, by putting this into (45), we obtain

$$G_{N=3} = \pi^2 \int_0^{\frac{1}{8z}} dx \frac{e^{-x}}{(1-4zx)^{\frac{5}{2}}} (1+4z+28z^2-16z(1+3z+14z^2)x + 16z^2(5+16z+16z^2)x^2 - 128z^3(1+4z)x^3). \tag{56}$$

A surprising fact is that the integrand in (56) is actually a total derivative, and we therefore obtain

$$G_{N=3} = -\pi^2 \int_0^{\frac{1}{8z}} dx \frac{d}{dx} \left(\frac{e^{-x}}{(1-4zx)^{\frac{3}{2}}} (1-2z-4z(3+4z)x+32z^2(1+4z)x^2) \right) = \pi^2 (1-2z+4\sqrt{2}ze^{-\frac{1}{8z}}). \tag{57}$$

D. Larger odd N

The strategy taken in Sec. VI C can be generalized for larger odd N in the following manner. We first compute a_i ($i = 0, 1, \dots, 5$), which are listed for $N = 5, 7$ in Appendix B, by using the aforementioned Mathematica package. They indeed satisfy (46). Then, by putting them into (45), we obtain G_N . Similarly to the case of $N = 3$ in Sec. VI C, what we find is that the integrand of G_N is a total derivative of the following form for $N = 5, 7$:

$$G_{N:\text{odd}} = \frac{\pi^{N-1}}{4\Gamma[\frac{N-1}{2}]} \int_0^{\frac{1}{8z}} dx \frac{d}{dx} \frac{e^{-x} \sum_{n=0}^{\frac{N+1}{2}} b_n x^n}{(1-4zx)^{\frac{N}{2}}}, \tag{58}$$

where b_n ($n = 0, 1, \dots, (N+1)/2$) are some polynomial functions of z . The explicit forms of b_n are given in Appendix B. Therefore, we obtain

$$G_{N:\text{odd}} = \frac{\pi^{N-1}}{4\Gamma[\frac{N-1}{2}]} \left(2^{\frac{N}{2}} e^{-\frac{1}{8z}} \sum_{n=0}^{\frac{N+1}{2}} \frac{b_n}{(8z)^n} - b_0 \right). \tag{59}$$

By using b_n in Appendix B, the explicit expressions of G_N for $N = 5, 7$ are given by

$$G_{N=5} = \pi^4 \left(1 - 12z + 12z^2 + \sqrt{2}e^{-\frac{1}{8z}} (1 + 12z + 12z^2) \right),$$

$$G_{N=7} = \pi^6 \left(1 - 30z + 180z^2 - 120z^3 + \frac{\sqrt{2}e^{-\frac{1}{8z}}}{8z} (1 + 8z + 120z^2 - 480z^3 + 2640z^4) \right). \tag{60}$$

Let us finally comment on our equipment. The computations were done on a machine that had a Xeon W2295 (3.0 GHz, 18 cores), 128 GB DDR4 memory, and Ubuntu 20 as OS. The computation of a_i quickly takes longer time as N becomes larger. $G_{N=8}$, which appears in Subsection VI E, was the largest feasible case, while we failed to obtain $G_{N=9}$ seemingly because of a memory shortage.

E. Larger even N

The difference from the odd case of Sec. VI D is that the integrand of G_N is a sum of a total derivative and a simple integrable term,

$$G_N = \frac{\pi^{N-1}}{4\Gamma[\frac{N-1}{2}]} \int_0^{\frac{1}{8z}} dx \left(c_0 x^{-\frac{1}{2}} e^{-x} + \frac{d}{dx} \frac{x^{\frac{1}{2}} e^{-x} \sum_{n=0}^{\frac{N}{2}} b_n x^n}{(1-4zx)^{\frac{N}{2}}} \right). \tag{61}$$

Then, by doing the integration, we obtain

$$G_N = \frac{\pi^{N-1}}{4\Gamma[\frac{N-1}{2}]} \left(c_0 \gamma \left[\frac{1}{2}, \frac{1}{8z} \right] + 2^{\frac{N}{2}} e^{-\frac{1}{8z}} \sum_{n=0}^{\frac{N}{2}} \frac{b_n}{(8z)^{n+\frac{1}{2}}} \right). \tag{62}$$

The lists of a_i, b_i, c_0 for $N = 4, 6, 8$ are given in Appendix C. a_i indeed satisfy (46). By putting the values of b_i, c_0 , we obtain the explicit forms of G_N as

$$\begin{aligned} G_{N=4} &= \pi^{\frac{5}{2}} \left(6\sqrt{2}e^{-\frac{1}{8z}} \sqrt{z}(1+2z) + (1-6z) \gamma\left[\frac{1}{2}, \frac{1}{8z}\right] \right), \\ G_{N=6} &= \pi^{\frac{9}{2}} \left(\frac{2\sqrt{2}e^{-\frac{1}{8z}}(1+15z+180z^3)}{3\sqrt{z}} + (1-20z+60z^2) \gamma\left[\frac{1}{2}, \frac{1}{8z}\right] \right), \\ G_{N=8} &= \pi^{\frac{13}{2}} \left(\frac{\sqrt{2}e^{-\frac{1}{8z}}(1+210z^2-2100z^3+12600z^4+25200z^5)}{15z^{\frac{3}{2}}} \right. \\ &\quad \left. + (1-42z+420z^2-840z^3) \gamma\left[\frac{1}{2}, \frac{1}{8z}\right] \right). \end{aligned} \tag{63}$$

VII. COMPARISON WITH MONTE CARLO SIMULATIONS

In this section, we compare the results in Sec. VI with Monte Carlo simulations. The procedure is basically the same as that used in Refs. 16 and 17. To make this paper self-contained, however, we review the method below.

The eigenvector equation (1) is a system of polynomial equations and it can be solved by an appropriate polynomial equation solver, unless N is too large. We use Mathematica 13 for this purpose. It gives generally complex solutions to Eq. (1), and we pick up only real ones, since we are only counting real eigenvectors (or Z-eigenvalues). Whether this method covers all the real solutions or not can be checked by whether the number of generally complex solutions obtained for each C by a polynomial equation solver agrees with the known number $2^N - 1$.²³ (In fact, for large N , Mathematica 13 seems to miss a few solutions for some C . We have not pursued the reason for that, but the missing portion is $\sim 10^{-4}$ even for our largest case of $N = 16$, and is statistically irrelevant in the present study.)

With the above method of solving the eigenvector equation (1), our procedure of Monte Carlo simulation is given as follows:

- Randomly generate a real symmetric tensor C with components, $C_{ijk} = \sigma/\sqrt{d(i, j, k)}$ ($1 \leq i \leq j \leq k \leq N$), where σ has a normal distribution with mean value zero and standard deviation one, and $d(i, j, k)$ is a degeneracy factor,

$$d(i, j, k) = \begin{cases} 1, & i = j = k \\ 3, & i = j \neq k, i \neq j = k, k = i \neq j. \\ 6, & i \neq k \neq j \neq i \end{cases} \tag{64}$$

This random generation of C corresponds to $\alpha = 1/2$ in (4) since

$$C^2 = C_{abc}C_{abc} = \sum_{i \leq j \leq k=1}^N d(i, j, k) C_{ijk}^2 \tag{65}$$

due to C being a symmetric tensor.

- Compute the real eigenvectors of a generated C by the aforementioned method.
- Store each size $|v|$ of all the real eigenvectors.
- Repeat the above processes.

By this procedure, we obtain a sequence of sizes of real eigenvectors, $|v|_i$ ($i = 1, 2, \dots, L$). Then, the size distribution of real eigenvectors can be obtained by

$$\rho_{\text{size}}^{\text{MC}}((k+1/2)\delta v) = \frac{1}{\delta v N_C} \sum_{i=1}^L \theta(k\delta v < |v|_i \leq (k+1)\delta v), \tag{66}$$

where δv is a bin size, N_C is the total number of randomly generated C , $k = 0, 1, 2, \dots$, and $\theta(\cdot)$ is a support function that takes 1 if the inequality of the argument is satisfied, but zero otherwise. For the comparison with the analytical results obtained in Sec. VI, (66) should be compared with the size distribution,

$$\rho_{\text{size}}(|v|) = \rho(v) S_{N-1} |v|^{N-1}, \tag{67}$$

where $S_{N-1} = 2\pi^{N/2}/\Gamma[N/2]$ denotes the surface volume of a unit sphere in an N -dimensional space and the vector v in the argument of $\rho(\cdot)$ is an arbitrary vector of size $|v|$ due to the rotational symmetry.

Since a Z-eigenvalue is related with the size of an eigenvector by relation (6), the real (or Z-) eigenvalue distribution is given by

$$\rho_{\text{eig}}^{\text{MC}}((k+1/2)\delta\zeta) = \frac{1}{\delta\zeta N_C} \sum_{i=1}^L \theta(k\delta\zeta < 1/|v|_i \leq (k+1)\delta\zeta), \tag{68}$$

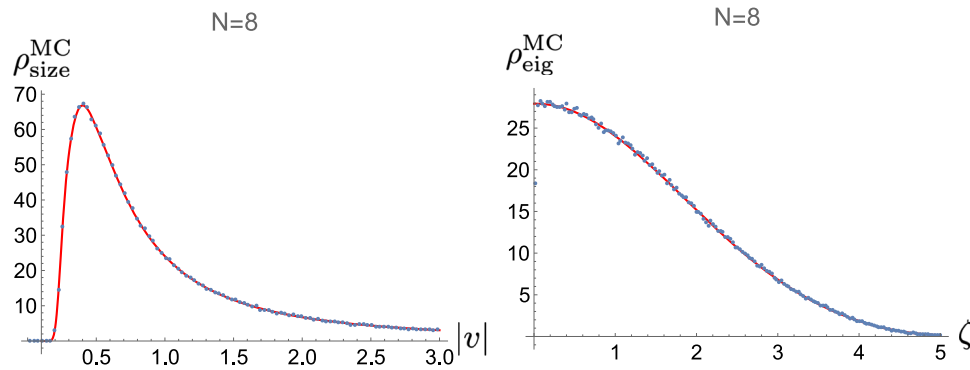


FIG. 1. The results of the Monte Carlo simulation for $N = 8$, $N_C = 10\,000$ are compared with the analytic expressions. Left: the eigenvector size distribution. The Monte Carlo result (66) with $\delta v = 0.03$ (dots) is compared with (67) using (40) and $G_{N=8}$ in (63) (solid line). Right: the eigenvalue distribution. The Monte Carlo result (68) with $\delta\zeta = 0.03$ (dots) is compared with the analytic expression through (7).

where $\delta\zeta$ is a bin size. This quantity should be compared with (7).

In Fig. 1, the Monte Carlo results are compared with the analytic expressions for $N = 8$. They agree precisely. Similar precise agreement has been obtained for all $N \leq 8$.

VIII. EXTRAPOLATION TO GENERAL N

In this section, we point out a few patterns that exist in the expressions of the distributions derived for small N in Sec. VI and guess an extrapolation to general N . A motivation for doing this is to improve the large- N expression previously obtained by an approximation using a Schwinger–Dyson equation.¹⁷ The issue of the previous result was that while the functional form agreed well with the numerical simulation for large- N , the overall factor did not. In this section, using the extrapolation, we will guess the overall factor for general N and find good agreement with Monte Carlo simulations.

After a thought, one notices that G_N for even N (namely, $N = 2, 4, 6, 8$) in Sec. VI can be expressed by the following general form:

$$G_{N:\text{even}} = \pi^{N-\frac{3}{2}} z^{\frac{N-1}{2}} H_{N-1} \left[\frac{1}{2\sqrt{z}} \right] \gamma \left[\frac{1}{2}, \frac{1}{8z} \right] + \sqrt{2} \pi^{N-\frac{3}{2}} \frac{N!}{\frac{N}{2}!} z^{\frac{N-1}{2}} e^{-\frac{1}{8z}} \left(1 + \frac{d_1}{z} + \frac{d_2}{z^2} + \dots + \frac{d_{N-3}}{z^{N-3}} \right), \quad (69)$$

where $H_n[\cdot]$ are Hermite polynomials, d_i are some coefficients generally depending on N , and specifically,

$$d_1 = \frac{1 + (-1)^{\frac{N}{2}}}{4}. \quad (70)$$

As for d_i ($i \geq 2$), we could not find reasonably simple functions of N .

Let us discuss the real eigenvalue distribution for large- N , assuming (69) with (70). Because of relations (6) and (44) and the fact that the major part of the distribution is around $\zeta \sim 0$ as in Fig. 1, we are interested in a $1/z$ expansion of (69) with (70). By explicitly doing this, one obtains

$$G_N \sim \sqrt{2} \pi^{N-\frac{3}{2}} \frac{\Gamma[N+1]}{\Gamma[\frac{N}{2}+1]} z^{\frac{N-1}{2}} \left(1 + \frac{1}{8z} + \dots \right), \quad (71)$$

where the factorials are replaced by Gamma functions to also be applicable for odd N below. By combining with the previous result in Ref. 17 that the large- N eigenvalue distribution is given by a Gaussian function of $|v|$, one could assume that the expansion in $1/z$ of (71) comes from the expression

$$G_N \sim \sqrt{2} \pi^{N-\frac{3}{2}} \frac{\Gamma[N+1]}{\Gamma[\frac{N}{2}+1]} z^{\frac{N-1}{2}} e^{-\frac{1}{8z}}. \quad (72)$$

Putting this into (7) using (6), (40), and (44), we obtain

$$\rho_{\text{eig}}(\zeta) \sim 2^{-\frac{N}{2}+2} \alpha^{\frac{1}{2}} \pi^{-\frac{1}{2}} \frac{\Gamma[N+1]}{\Gamma[\frac{N}{2}+1] \Gamma[\frac{N}{2}]} e^{-\frac{\alpha}{4}\zeta^2}, \quad (73)$$

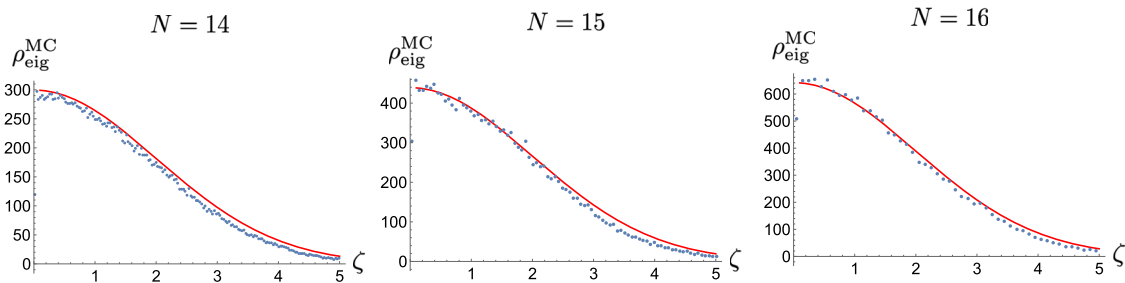


FIG. 2. The guessed large- N formula (73) (solid line) is compared with the Monte Carlo simulation (68) (dots). Left: $N = 14$, $\delta\zeta = 0.03$, $N_C = 400$. Middle: $N = 15$, $\delta\zeta = 0.06$, $N_C = 100$. Right: $N = 16$, $\delta\zeta = 0.1$, $N_C = 72$.

which indeed is a Gaussian distribution. While the coefficient $\alpha/4$ in the exponent indeed agrees with the previous result in Ref. 17, the overall factor is different. By integrating over ζ , one can obtain the mean total number of real eigenvalues in large- N as

$$\text{Mean number of real eigenvalues} \sim \frac{2^{-\frac{N}{2}+2}\Gamma[N+1]}{\Gamma[\frac{N}{2}+1]\Gamma[\frac{N}{2}]}. \quad (74)$$

In the large- N limit, this expression indeed agrees with the result given in Refs. 13 and 20 (namely, Eq. (1.2) in Ref. 13).

In Fig. 2, we compare the large- N expression (73) with the Monte Carlo simulations for $N = 14, 15, 16$. [With our machine power (see the last part of Sec. VI D), it took one full day for the computation of the $N = 16$ case.] They agree well, and the agreement seems to become slightly better, as N becomes larger.

In the above discussion, we have not used the odd N cases. The reason is that we could not find simple expressions valid across all the odd N cases obtained in Sec. VI. We merely notice

$$G_{N:\text{odd}} = \pi^{N-1} z^{\frac{N-1}{2}} H_{N-1}\left(\frac{1}{2\sqrt{z}}\right) + d'_0 e^{-\frac{1}{8z}} \left(1 + \frac{d'_1}{z} + \dots + \frac{d'_{N-3}}{z^{N-3}}\right), \quad (75)$$

where we do not have any simple expressions for d'_i .

Rather, we find that the large- z behavior of $G_{N:\text{odd}}$ approaches that of $G_{N:\text{even}}$, as N becomes larger. To see this, let us introduce the parameters of the $1/z$ expansion as

$$G_N \sim e_0 z^{\frac{N-1}{2}} \left(1 + \frac{e_1}{z} + \dots\right). \quad (76)$$

As in (71), for even N , we have

$$e_0 = \frac{\sqrt{2\pi}^{N-\frac{3}{2}}\Gamma[N+1]}{\Gamma[\frac{N}{2}+1]}, \quad e_1 = \frac{1}{8}. \quad (77)$$

In Fig. 3, we plot e_0, e_1 for each N , which are extracted by performing the $1/z$ expansions of the explicit expressions in Sec. VI. Indeed, e_0, e_1 for odd N seem to approach those for even N , as N becomes larger. This implies that the large- N formula (73) can commonly be used for both even and odd N .

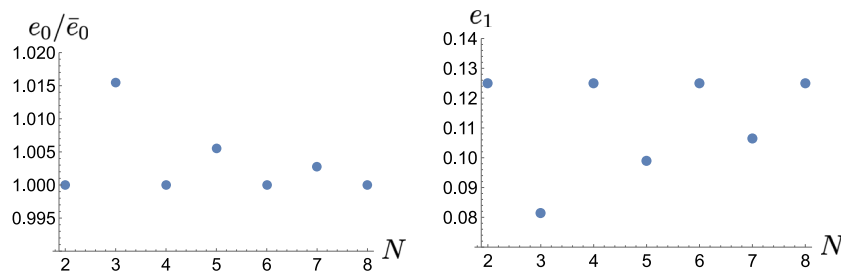


FIG. 3. The coefficients e_0, e_1 of the expansion (76) are plotted against N . $\bar{e}_0 = \sqrt{2\pi}^{N-3/2}\Gamma[N]/\Gamma[N/2+1]$ [see (71)]. The values of e_0, e_1 for odd N seem to approach those for even N , as N becomes larger.

IX. SUMMARY AND FUTURE PROSPECTS

In this paper, we have explicitly computed the real eigenvalue/vector distributions for real symmetric order-three tensors with Gaussian distributions for $N \leq 8$. This has been achieved by rewriting the problem as the computation of a partition function of a zero-dimensional boson fermion system with four interactions. We have extrapolated the exact expressions for $N \leq 8$ to guess a large- N expression. Monte Carlo simulations have been performed, and precise agreement and good agreement have been obtained for the exact small- N expressions and the large- N expression, respectively.

It seems surprising that the complicated integrations of the partition function can be performed exactly with the final simple results expressed by polynomial, exponential, and error functions. It is suspected that this is true for any N , but deeper understanding of this aspect needed for such generalization is left for future study. Considering the final simple forms, it would be possible that there exist much easier ways to compute the distributions than what has been done in this paper.

There would exist various directions to pursue by a similar strategy of this paper. One obvious direction is the generalization of tensors to those with different orders and non-Gaussian distributions. It would also be interesting to pursue the possibility of topological changes of the distributions in tensor models as in matrix models. In fact, some tensor systems seemingly suggest such possibilities.^{27,28} It would also be possible to compute correlations among eigenvalues/vectors and also to incorporate complex eigenvalues/vectors.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Naoki Sasakura: Conceptualization (lead).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: COMPUTATIONS OF E_2 AND E_3

From (17),

$$E_2 = -\frac{4}{\alpha} A_1, \quad (\text{A1})$$

with

$$A_1 = \left(\frac{1}{6} \sum_s v_{s_a} \sigma_{s_b} \phi_{s_c} \right)^2. \quad (\text{A2})$$

By explicitly writing down the sum and noting that one of the symmetrization is not necessary, we obtain

$$\begin{aligned} A_1 &= \frac{1}{6} v_a \sigma_b \phi_c (v_a \sigma_b \phi_c + v_b \sigma_c \phi_a + v_c \sigma_a \phi_b + v_b \sigma_a \phi_c + v_a \sigma_c \phi_b + v_c \sigma_b \phi_a) \\ &= \frac{1}{6} (v^2 \sigma^2 \phi^2 + v \cdot \sigma \sigma \cdot \phi \phi \cdot v + v \cdot \phi \sigma \cdot v \phi \cdot \sigma + v \cdot \sigma \sigma \cdot v \phi^2 + v^2 \sigma \cdot \phi \phi \cdot \sigma + v \cdot \phi \sigma^2 \phi \cdot v) \\ &= \frac{|v|^2}{6} (\sigma^2 \phi^2 + 2\phi_{\parallel} \sigma_{\parallel} \sigma \cdot \phi + \sigma_{\parallel}^2 \phi^2 + \phi_{\parallel}^2 \sigma^2 + (\sigma \cdot \phi)^2) \\ &= \frac{|v|^2}{6} (\sigma_{\perp}^2 \phi_{\perp}^2 + (\phi_{\perp} \cdot \sigma_{\perp})^2 + 2\sigma_{\parallel}^2 \phi_{\perp}^2 + 2\sigma_{\perp}^2 \phi_{\parallel}^2 + 4\sigma_{\parallel} \phi_{\parallel} \phi_{\perp} \cdot \sigma_{\perp} + 6\sigma_{\parallel}^2 \phi_{\parallel}^2), \end{aligned} \quad (\text{A3})$$

where we have used $v \cdot \phi = |v| \phi_{\parallel}$, etc., and $\phi \cdot \sigma = \phi_{\parallel} \sigma_{\parallel} + \phi_{\perp} \cdot \sigma_{\perp}$, etc.

Let us next compute

$$E_3 = \frac{4i}{\alpha} A_2, \tag{A4}$$

where

$$A_2 = (v_a \bar{\psi}_b \varphi_c + v_a \bar{\varphi}_b \psi_c) \frac{1}{6} \sum_s v_{s_a} \sigma_{s_b} \phi_{s_c} \tag{A5}$$

from (17). By explicitly expanding the sum, we obtain

$$\begin{aligned} A_2 &= v_a \bar{\psi}_b \varphi_c (v_a \sigma_b \phi_c + v_b \sigma_c \phi_a + v_c \sigma_a \phi_b + v_b \sigma_a \phi_c + v_a \sigma_c \phi_b + v_c \sigma_b \phi_a) + (\psi \leftrightarrow \varphi) \\ &= \frac{|v|^2}{6} (\bar{\psi}_\perp \cdot \sigma_\perp \varphi_\perp \cdot \phi_\perp + \bar{\psi}_\perp \cdot \phi_\perp \varphi_\perp \cdot \sigma_\perp + \bar{\varphi}_\perp \cdot \sigma_\perp \psi_\perp \cdot \phi_\perp + \bar{\varphi}_\perp \cdot \phi_\perp \psi_\perp \cdot \sigma_\perp \\ &\quad + 2\bar{\psi}_\parallel \sigma_\parallel \varphi_\perp \cdot \phi_\perp + 2\phi_\parallel \bar{\psi}_\parallel \varphi_\perp \cdot \sigma_\perp + 2\varphi_\parallel \phi_\parallel \bar{\psi}_\perp \cdot \sigma_\perp + 2\varphi_\parallel \sigma_\parallel \bar{\psi}_\perp \cdot \phi_\perp \\ &\quad + 2\bar{\varphi}_\parallel \sigma_\parallel \psi_\perp \cdot \phi_\perp + 2\phi_\parallel \bar{\varphi}_\parallel \psi_\perp \cdot \sigma_\perp + 2\psi_\parallel \phi_\parallel \bar{\varphi}_\perp \cdot \sigma_\perp + 2\psi_\parallel \sigma_\parallel \bar{\varphi}_\perp \cdot \phi_\perp \\ &\quad + 6\bar{\psi}_\parallel \varphi_\parallel \phi_\parallel \sigma_\parallel + 6\bar{\varphi}_\parallel \psi_\parallel \phi_\parallel \sigma_\parallel), \end{aligned} \tag{A6}$$

where we have taken similar computational steps as those for A_1 .

APPENDIX B: LISTS OF a_i AND b_i FOR $N = 5, 7$

For $N = 5$,

$$\begin{aligned} a_0 &= 1 - 8z + 120z^2 + 480z^3 + 2640z^4, \\ a_1 &= -16i(z - 6z^2 + 60z^3 + 120z^4), \\ a_2 &= -32(3z^2 - 10z^3 + 100z^4 + 200z^5), \\ a_3 &= -32(-z^2 + 10z^3 + 20z^4 + 280z^5), \\ a_4 &= -256i(z^3 + 20z^5), \\ a_5 &= 256(z^4 + 4z^5 + 28z^6) \end{aligned} \tag{B1}$$

and

$$\begin{aligned} b_0 &= -4(1 - 12z + 12z^2), \\ b_1 &= 4(-1 + 22z - 132z^2 + 120z^3), \\ b_2 &= 48(z - 8z^2 + 60z^3 + 80z^4), \\ b_3 &= -128(z^2 + 60z^4 + 240z^5). \end{aligned} \tag{B2}$$

For $N = 7$,

$$\begin{aligned} a_0 &= 1 - 36z + 660z^2 - 3360z^3 + 25\,200z^4 + 100\,800z^5 + 504\,000z^6, \\ a_1 &= -16i(z - 30z^2 + 440z^3 - 1680z^4 + 8400z^5 + 16\,800z^6), \\ a_2 &= -32(3z^2 - 70z^3 + 840z^4 - 1680z^5 + 11\,760z^6 + 30\,240z^7), \\ a_3 &= -32(-z^2 + 30z^3 - 280z^4 + 1680z^5 + 5040z^6 + 36\,960z^7), \\ a_4 &= -256i(z^3 - 16z^4 + 168z^5 + 1680z^7), \\ a_5 &= 256(z^4 - 8z^5 + 120z^6 + 480z^7 + 2640z^8) \end{aligned} \tag{B3}$$

and

$$\begin{aligned} b_0 &= 8(-1 + 30z - 180z^2 + 120z^3), \\ b_1 &= -8(1 - 44z + 600z^2 - 2640z^3 + 1680z^4), \\ b_2 &= 4(-1 + 58z - 1160z^2 + 9360z^3 - 28\,560z^4 + 16\,800z^5), \\ b_3 &= 16(3z - 100z^2 + 1480z^3 - 6720z^4 + 25\,200z^5 + 33\,600z^6), \\ b_4 &= -128(z^2 - 20z^3 + 280z^4 + 8400z^6 + 33\,600z^7). \end{aligned} \tag{B4}$$

APPENDIX C: LISTS OF a_i, b_j, c_0 FOR $N = 4, 6, 8$

For $N = 4$,

$$\begin{aligned} a_0 &= 1 + 60z^2 + 240z^3, \\ a_1 &= -16i(z + 20z^3), \\ a_2 &= -32(3z^2 + 2z^3 + 24z^4), \\ a_3 &= -32(-z^2 + 6z^3 + 32z^4), \\ a_4 &= -256i(z^3 + 2z^4), \\ a_5 &= 256(z^4 + 4z^5) \end{aligned} \tag{C1}$$

and

$$\begin{aligned} c_0 &= -2(-1 + 6z), \\ b_0 &= 4(-1 + 6z), \\ b_1 &= 16(3z - 2z^2 + 40z^3), \\ b_2 &= -128(z^2 + 4z^3 + 28z^4). \end{aligned} \tag{C2}$$

For $N = 6$,

$$\begin{aligned} a_0 &= 1 - 20z + 280z^2 + 8400z^4 + 33\,600z^5, \\ a_1 &= -16i(z - 16z^2 + 168z^3 + 1680z^5), \\ a_2 &= -32(3z^2 - 34z^3 + 300z^4 + 360z^5 + 2400z^6), \\ a_3 &= -32(-z^2 + 18z^3 - 60z^4 + 600z^5 + 2880z^6), \\ a_4 &= -256i(z^3 - 6z^4 + 60z^5 + 120z^6), \\ a_5 &= 256(z^4 + 60z^6 + 240z^7) \end{aligned} \tag{C3}$$

and

$$\begin{aligned} c_0 &= 3(1 - 20z + 60z^2), \\ b_0 &= -6(1 - 20z + 60z^2), \\ b_1 &= 4(-1 + 18z)(1 - 20z + 60z^2), \\ b_2 &= 16(3z - 56z^2 + 540z^3 - 240z^4 + 3360z^5), \\ b_3 &= -128(z^2 - 8z^3 + 120z^4 + 480z^5 + 2640z^6). \end{aligned} \tag{C4}$$

For $N = 8$,

$$\begin{aligned} a_0 &= 1 - 56z + 1428z^2 - 15\,120z^3 + 105\,840z^4 + 2\,116\,800z^6 + 8\,467\,200z^7, \\ a_1 &= -16i(z - 48z^2 + 1020z^3 - 8640z^4 + 45\,360z^5 + 302\,400z^7), \\ a_2 &= -32(3z^2 - 118z^3 + 2080z^4 - 12\,880z^5 + 58\,800z^6 + 84\,000z^7 + 470\,400z^8), \\ a_3 &= -32(-z^2 + 46z^3 - 760z^4 + 6160z^5 - 8400z^6 + 117\,600z^7 + 537\,600z^8), \\ a_4 &= -256i(z^3 - 30z^4 + 440z^5 - 1680z^6 + 8400z^7 + 16\,800z^8), \\ a_5 &= 256(z^4 - 20z^5 + 280z^6 + 8400z^8 + 33\,600z^9) \end{aligned} \tag{C5}$$

and

$$\begin{aligned} c_0 &= -(15/2)(-1 + 42z - 420z^2 + 840z^3), \\ b_0 &= 15(-1 + 42z - 420z^2 + 840z^3), \\ b_1 &= -10(-1 + 24z)(-1 + 42z - 420z^2 + 840z^3), \\ b_2 &= 4(1 - 40z + 360z^2)(-1 + 42z - 420z^2 + 840z^3), \\ b_3 &= 48(z - 52z^2 + 1140z^3 - 10\,360z^4 + 50\,400z^5 - 16\,800z^6 + 201\,600z^7), \\ b_4 &= -128(z^2 - 36z^3 + 660z^4 - 3360z^5 + 25\,200z^6 + 100\,800z^7 + 504\,000z^8). \end{aligned} \tag{C6}$$

APPENDIX D: COMPLEXITY OF THE SPHERICAL p -SPIN MODEL

Computing the distributions of the real tensor eigenvalues/vectors is the same as the computation of complexity (see, for example, Ref. 29 for a review) of the spherical p -spin model.^{18,19} In this appendix, we limit ourselves to $p = 3$ corresponding to order-three tensors.

The Hamiltonian of the spherical p -spin model (with $p = 3$) is given by

$$H = -\frac{1}{N} J_{abc} \sigma_a \sigma_b \sigma_c, \quad (\text{D1})$$

with the continuous spin variable $\sigma \in \mathbb{R}^N$, which has a constraint

$$\sigma_a \sigma_a = N. \quad (\text{D2})$$

Here, the real symmetric tensor J is assumed to have the normal Gaussian distribution and can be identified with C with $\alpha = 1/2$ in our notation. To match (D2) with our notation, we introduce the change of variable,

$$\sigma_a = \sqrt{N} w_a, \quad (\text{D3})$$

and then, we have

$$H = -\sqrt{N} C_{abc} w_a w_b w_c \text{ with } w_a w_a = 1. \quad (\text{D4})$$

The problem of computing complexity of the p -spin spherical model is to count the number of local minima (and stationary points) of the above Hamiltonian. By using the method of Lagrange multiplier, counting the stationary points is equivalent to solve

$$C_{abc} w_b w_c = -\sqrt{N} u w_a, \quad (\text{D5})$$

where $u \in \mathbb{R}$ and $w_a w_a = 1$ ($w \in \mathbb{R}^N$). The value of u is the energy of the stationary point and can be identified with the same variable employed in Ref. 20. Comparing with the Z -eigenvalue equation (5), we have the relation

$$\zeta = -\sqrt{N} u. \quad (\text{D6})$$

By using (D6), it can be checked that the expressions of the mean distributions of the numbers of the critical points given for general N in Sec. 7.2 of Ref. 20 agree with our results in Sec. VI.

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