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Maps preserving triple transition pseudo-probabilities

By

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Abstract

Let e and v be minimal tripotents in a JBW*-triple M. We introduce the notion of triple transition pseudo-probability from e to v as the complex number $TTP(e,v) = \varphi_v(e)$, where φ_v is the unique extreme point of the closed unit ball of M_* at which v attains its norm. In the case of two minimal projections in a von Neumann algebra, this correspond to the usual transition probability. We prove that every bijective transformation Φ preserving triple transition pseudo-probabilities between the lattices of tripotents of two atomic JBW*-triples M and N admits an extension to a bijective (complex) linear mapping between the socles of these JBW*-triples. If we additionally assume that Φ preserves orthogonality, then Φ can be extended to a surjective (complex-)linear (isometric) triple isomorphism from M onto N. In case that M and N are two spin factors or two type 1 Cartan factors we show, via techniques and results on preservers, that every bijection preserving triple transition pseudo-probabilities between the lattices of tripotents of M and N automatically preserves orthogonality, and hence admits an extension to a triple isomorphism from M onto N.

§ 1. Introduction

The available mathematical models for quantum mechanic make use of complex Hilbert spaces to define the states of a quantum system. Given a complex Hilbert space H, the normal state space of $S(H)$ is identified, via trace duality, with those

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positive norm-one elements (states) in the predual of the von Neumann algebra, $B(H)$, of all bounded linear operators on H. Each observable is associated with a self-adjoint operator $A \in B(H)$, and its expected value on the the system in state p is $A(p) = \operatorname{tr}(Ap)$, where $tr(.)$ stands for the usual trace on $B(H)$. The elements in $S(H)$ are called the normal states of a quantum system associated to the Hilbert space H. The extreme points of $S(H)$, as a convex set inside the closed unit ball of $B(H)_*$, are called pure states, and they can be also identified with rank-one projections on H. The set of all rank-one projections on H will be denoted by $\mathcal{P}_1(H)$, while $\mathcal{P}(H)$ or $\mathcal{P}(B(H))$ will stand for the set of all (orthogonal) projections on H.

If two pure states are represented by the minimal projections $p = \xi\otimes\xi$ and $q = \eta\otimes\eta$, with ξ and η in the unit sphere of H, according to Born's rule, the *transition probability* from p to q is defined as

$$TP(p,q) = \operatorname{tr}(pq) = \operatorname{tr}(pq^*) = \operatorname{tr}(qp^*) = |\langle \xi, \eta\rangle|^2.$$

Here, and along this note, for ξ in another complex Hilbert space K and $\eta \in H$, the symbol $\xi \otimes \eta$ will stand for the operator from H to K defined by $\xi \otimes \eta(\zeta) := \langle \zeta, \eta\rangle \xi$.

A bijective map $\Phi : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ is called a symmetry transformation or a Wigner symmetry if it preserves the transition probability between minimal projections, that is,

$$TP(\Phi(p), \Phi(q)) = \operatorname{tr}(\Phi(p)\Phi(q)) = \operatorname{tr}(pq) = TP(p,q), \text{ for all } (p,q \in \mathcal{P}_1(H)).$$

A linear (respectively, conjugate-linear) mapping $u : H \to H$ is called a unitary (respectively, an anti-unitary) if $uu^* = u^*u = 1$. The celebrated Wigner's theorem admits the following statement:

Theorem 1.1. (Wigner theorem, [41], [36, page 12]) *Let H be a complex Hilbert space. A bijective mapping $\Phi : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ is a symmetry transformation if and only if there is an either unitary or anti-unitary operator u on H, unique up to multiplication by a unitary scalar, such that $\Phi(p) = upu^*$ for all $p \in \mathcal{P}_1(H)$. Furthermore, the real linear (actually complex-linear or conjugate-linear) mapping $T : B(H) \to B(H)$, $T(x) = uxu^*$ is a *-automorphism whose restriction to $\mathcal{P}_1(H)$ coincides with Φ.*

It is known (see, for example, [10, §4 and 6]) that for a complex Hilbert space H with $\dim(H) \geq 3$, the following mathematical models employed in the Hilbert space formulation of quantum mechanics are equivalent:

$(M.1)$ The set $\mathbf{P}$ of pure states on H (which algebraically corresponds to the set $\mathcal{P}_1(H)$) whose automorphisms are the bijections preserving transition probabilities.

$(M.2)$ The orthomodular lattice $\mathbf{L}$ of closed subspaces of H, or equivalently, the lattice of all projections in $B(H)$, where the automorphisms are the bijections preserving orthogonality and order.

The equivalence of these two models implies that if $\dim(H) \geq 3$, every bijection $\Phi : \mathcal{P}(B(H)) \to \mathcal{P}(B(H))$ preserving the partial ordering and orthogonality in both directions is given by a real linear *-automorphism on $B(H)$ determined either by a unitary or by an anti-unitary operator on H (cf. [10, §2.3 and Proposition 4.9]).

The lattice of projections in $B(H)$ is a subset of the strictly bigger lattice of partial isometries in $B(H)$. We recall that an element e in $B(H)$ is a partial isometry if ee^* (equivalently, e^*e) is a projection. Partial isometries are also called tripotents since an element e is a partial isometry if and only if $ee^*e = e$. Let the symbol $\mathcal{PI}(H) = \mathcal{U}(B(H))$ stand for the set of all partial isometries on H. We shall write $\mathcal{PI}_1(H) = \mathcal{U}_{min}(B(H))$ for the set of all rank–1 or minimal partial isometries on H. We say that $e, v \in \mathcal{U}(B(H))$ are orthogonal if and only if $\{ee^*, vv^*\}$ and $\{e^*e, v^*v\}$ are two sets of orthogonal projections. The standard partial ordering on $\mathcal{U}(B(H))$ is defined in the following terms: $e \leq u$ if $u - e$ is a partial isometry orthogonal to e.

L. Molnár seems to be the first author in considering a Wigner type theorem for bijections on the lattice of partial isometries of $B(H)$ preserving the partial order and orthogonality in both directions.

Theorem 1.2. [37, Theorem 1] *Let H be a complex Hilbert space with $\dim(H) \geq 3$. Suppose that $\Phi : \mathcal{U}(B(H)) \to \mathcal{U}(B(H))$ is a bijective transformation which preserves the partial ordering and the orthogonality between partial isometries in both directions. If Φ is continuous (in the operator norm) at a single element of $\mathcal{U}(B(H))$ different from 0, then Φ extends to a real-linear triple isomorphism.*

During the mini-symposium *"Research on preserver problems on Banach algebras and related topics"* held at RIMS (Research Institute for Mathematical Sciences), Kyoto University on October 25–27, 2021, the author of this note presented the following generalization of the previous theorem to the case of atomic JBW*-triples (i.e. JB*-triples which are ℓ_∞-sums of Cartan factors).

Theorem 1.3. [18, Theorem 6.1] *Let $M = \bigoplus_{i\in I}^{\ell_\infty} C_i$ and $N = \bigoplus_{j\in J}^{\ell_\infty} \tilde{C}_j$ be atomic JBW*-triples, where C_i and C_j are Cartan factors with rank ≥ 2. Suppose that $\Phi : \mathcal{U}(M) \to \mathcal{U}(N)$ is a bijective transformation which preserves the partial ordering in both directions and orthogonality between tripotents. We shall additionally assume that Φ is continuous at a tripotent $u = (u_i)_i$ in M with $u_i \neq 0$ for all i (or we shall simply assume that $\Phi|_{\mathbb{T}u}$ is continuous at a tripotent $(u_i)_i$ in M with $u_i \neq 0$ for all i). Then* *there exists a real linear triple isomorphism $T : M \to N$ such that $T(w) = \Phi(w)$ for all $w \in \mathcal{U}(M)$.*

It should be remarked that the hypothesis concerning the ranks of the Cartan factors in the previous theorem cannot be relaxed (cf. [18, Remark 3.6]). Anyway, the validity of the result for rank-2 Cartan factors is undoubtedly an advantage.

Back to the essence of Wigner theorem expressed in Theorem 1.1, we find the following contribution by L. Molnár.

Theorem 1.4. [37, Theorem 2] *Let $\Phi : \mathcal{U}_{min}(B(H)) \to \mathcal{U}_{min}(B(H))$ be a bijective mapping satisfying*

$$tr(\Phi(e)^*\Phi(v)) = tr(e^*v), \text{ for all } e, v \in \mathcal{U}_{min}(B(H)). \tag{1.1}$$

Then Φ extends to a surjective complex-linear isometry. Moreover, one of the following statements holds:

(*a*) *there exist unitaries u, w on H such that $\Phi(e) = uew$ $(e \in \mathcal{U}_{min}(B(H)))$;*

(*b*) *there exist anti-unitaries u, w on H such that $\Phi(e) = ue^*w$ $(e \in \mathcal{U}_{min}(B(H)))$.*

Let us observe that for each minimal partial isometry e in $B(H)$, the functional $\varphi_e(x) = \mathrm{tr}(e^*x)$ is the unique extreme point of the closed unit ball of $B(H)_*$, the predual of $B(H)$, at which e attains its norm. A similar property holds in the wider setting of JBW*-triples (see subsection 1.1 for details and definitions). Namely, for each minimal tripotent e in a JBW*-triple, M, there exists a unique pure atom (i.e. an extreme point of the closed unit ball of M_*) φ_e at which e attains its norm and the corresponding Peirce-2 projection writes in the form $P_2(e)(x) = \varphi_e(x)e$ for all $x \in M$ (cf. [19, Proposition 4]). The mapping

$$\mathcal{U}_{min}(M) \to \partial_e(\mathcal{B}_{M_*}), \quad e \mapsto \varphi_e$$

is a bijection from the set of minimal tripotents in M onto the set of pure atoms of M. Given two minimal tripotents e and v in a JBW*-triple M, we define the *triple transition pseudo-probability* from e to v as the complex number given by $TTP(e, v) = \varphi_v(e)$. So, the hypothesis (1.1) in Theorem 1.4 is equivalent to say that Φ preserves triple transition pseudo-probabilities. In the case of $B(H)$, the triple transition pseudo-probability between two minimal projections is precisely the usual transition probability. We shall show that this pseudo-probability is symmetric in the sense that $TTP(e, v) = \overline{TTP(v, e)}$, for every couple of minimal tripotents $e, v \in M$.

We shall also see below that the triple transition pseudo-probability between any two minimal projections p and q in a von Neumann algebra W is zero if and only if p

and q are orthogonal (i.e. $pq = 0$). The same equivalence does not necessarily hold when projections are replaced with tripotents or partial isometries, for example, the partial isometries $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are not orthogonal in $M_2(\mathbb{C})$, but $TTP(e, v) = 0$. This is a theoretical handicap for the triple transition pseudo-probability. However, despite Theorem 1.3 above does not hold for rank-one JB*-triples (cf. [18, Remark 3.6]), every (non-necessarily surjective) mapping between the lattices of tripotents of two rank-one JB*-triples preserving triple transition pseudo-probabilities always admits an extension to a linear and isometric triple homomorphism between the corresponding JB*-triples (see Proposition 2.2). This will be obtained by an application of a theorem of Ding on the extension of isometries on the unit sphere of a Hilbert space [13].

In Theorem 2.3 we establish that if M and N are atomic JBW*-triples and $\Phi : \mathcal{U}_{min}(M) \to \mathcal{U}_{min}(N)$ is a bijective transformation preserving triple transition pseudo-probabilities between the sets of minimal tripotents, then there exists a bijective (complex) linear mapping T_0 from the socle of M onto the socle of N whose restriction to $\mathcal{U}_{min}(M)$ is Φ, where the socle of a JB*-triple is the subspace linearly generated by its minimal tripotents. If we additionally assume that Φ preserves orthogonality, then we prove the existence of a surjective (complex-)linear (isometric) triple isomorphism from M onto N extending the mapping Φ (cf. Corollary 2.5).

Due to the just commented result, the natural question is whether every bijection between the sets of minimal tripotents in two atomic JBW*-triples preserving triple transition pseudo-probabilities must automatically preserve orthogonality among them. The rest of the paper is devoted to present a couple of positive answers to this problem in the case of spin and type 1 Cartan factors.

Section 3 is devoted to study bijections preserving triple transition pseudo-probabilities between the sets of minimal tripotents in two spin factors. We shall show that any such bijection preserves orthogonality, and hence admits an extension to a triple isomorphism between the spin factors (see Theorem 3.2). The proof is based on an remarkable results on preservers, due to J. Chmieliński, asserting that a non-vanishing mapping between two inner product spaces is linear and preserves orthogonality in the Euclidean sense if and only if it is a positive multiple of a linear isometry [11, Theorem 1].

In section 4 we also establish a positive answer to the problem stated above in the case of a bijection between the sets of minimal tripotents in two type 1 Cartan factors (see Theorem 4.4). On this occasion, our arguments run closer to those given by Molnár in the proof of Theorem 1.4. For this purpose we shall establish a variant of several results previously explored by M. Marcus, B.N. Moyls [35], R. Westwick [40] and M. Omladič and P. Šemrl [38]. We concretely prove in Theorem 4.3 that for each linear bijection $\Phi : soc(B(H_1, K_1)) \to soc(B(H_2, K_2))$ preserving rank-one operators in both directions, where H_1, H_2, K_1 and K_2 are complex Hilbert spaces with dimensions ≥ 2, one of the next statements holds:

(a) either there are bijective linear mappings $u : K_1 \to K_2$, and $v : H_1 \to H_2$ such that $\Phi(\xi \otimes \eta) = u(\xi \otimes \eta)v = u(\xi) \otimes v(\eta)$ $(\xi \in K_1, \eta \in H_1)$;

(b) or there are bijective conjugate-linear mappings $u : H_1 \to K_2$, $v : K_1 \to H_2$ such that $\Phi(\xi \otimes \eta) = u(\xi \otimes \eta)^* v = u(\eta \otimes \xi)v = u(\eta) \otimes v(\xi)$ $(\xi \in K_1,\ \eta \in H_1)$.

Let us finish this introduction with a kind of announcement or statement of intentions, it would be desirable to find a positive argument to prove that every bijection between the sets of minimal tripotents in two atomic JBW*-triple automatically preserves orthogonality. Perhaps a more general point of view could provide a better understanding. At the present moment it seems a open problem. Some other additional questions also arise after this first study on triple transition pseudo-probabilities.

§ 1.1. Definitions and terminology

The model which motivated the study of C*-algebras is the space $B(H)$, of all bounded linear operators on a complex Hilbert space H. Left and right weak* closed ideals of $B(H)$ are precisely subspaces of the form $B(H)p$ and $pB(H)$, respectively, where p is a projection in $B(H)$. These ideals are identified with subspaces of operators of the form $B(p(H), H)$ and $B(H, p(H))$. However, given two complex Hilbert spaces H and K (where we can always assume that K is a closed subspace of H), the Banach space $B(H, K)$, of all bounded linear operators from H to K, is not, in general, a C*-subalgebra of some $B(H)$. Despite of this handicap, $B(H, K)$ is stable under products of the form

$$\{x, y, z\} = \frac{1}{2}\left(xy^*z + zy^*x\right) \quad (x, y, z \in B(H, K)). \tag{1.2}$$

Closed (complex) subspaces of $B(H, K)$ which are closed for the triple product defined in (1.2) were called *J*-algebras* by L. Harris in [27, 28]. J*-algebras include, in particular, all C*-algebras, all JC*-algebras, all complex Hilbert spaces, and all ternary algebras of operators. Harris also proved that the open unit ball of every J*-algebra enjoys the interesting holomorphic property of being a bounded symmetric domain (see [27, Corollary 2]). In [7], R. Braun, W. Kaup and H. Upmeier extended Harris' result by showing that the open unit ball of every (unital) JB*-algebra satisfies the same property.

If the holomorphic-property "being a bounded symmetric domain" is employed to classify the open unit balls of complex Banach spaces, the definitive result is due to W.

Kaup, who in his own words *"introduced the concept of JB*-triple and showed that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple, and in this way, the category of all bounded symmetric domains with base point is equivalent to the category of JB*-triples"* (see [31]).

A complex Banach space E is called a *JB*-triple* if it admits a continuous triple product $\{\cdot,\cdot,\cdot\} : E\times E\times E \to E$, which is symmetric and bilinear in the first and third variables, conjugate-linear in the middle one, and satisfies the following axioms:

(a) (Jordan identity)

$$L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$$

for a, b, x, y in E, where $L(a, b)$ is the operator on E given by $x \mapsto \{a, b, x\}$;

(b) $L(a, a)$ is a hermitian operator with non-negative spectrum for all $a \in E$;

(c) $\|\{a, a, a\}\| = \|a\|^3$ for every $a \in E$.

The first examples of JB*-triples include C*-algebras and $B(H, K)$ spaces with respect to the triple product given in (1.2), the latter are known as *Cartan factors of type 1*.

There are six different types of Cartan factors, the first one has been introduced in the previous paragraph. In order to define the next two types, let j be a conjugation (i.e. a conjugate-linear isometry or period 2) on a complex Hilbert space H. We consider a linear involution on $B(H)$ defined by $x \mapsto x^t := jx^*j$. *Cartan factors of type 2 and 3* are the JB*-subtriples of $B(H)$ of all t-skew-symmetric and t-symmetric operators, respectively.

A *Cartan factor of type 4*, also called a *spin factor*, is a complex Hilbert space M provided with a conjugation $x \mapsto \overline{x}$, where the triple product and the norm are defined by

$$\{x, y, z\} = \langle x, y\rangle z + \langle z, y\rangle x - \langle x, \overline{z}\rangle \overline{y}, \tag{1.3}$$

and

$$\|x\|^2 = \langle x, x\rangle + \sqrt{\langle x, x\rangle^2 - |\langle x, \overline{x}\rangle|^2}, \tag{1.4}$$

respectively (cf. [17, Chapter 3]). The *Cartan factors of types 5 and 6* (also called *exceptional* Cartan factors) are spaces of matrices over the eight dimensional complex algebra of Cayley numbers; the type 6 consists of all 3×3 self-adjoint matrices and has a natural Jordan algebra structure, and the type 5 is the subtriple consisting of all 1×2 matrices (see [32, 27, 29] and the recent references [22, §6.3 and 6.4], [23, §3] for more details).

An element e in a JB*-triple E is called a *tripotent* if $\{e, e, e\} = e$. When a C*-algebra is regarded as a JB*-triple with the triple product in (1.2), tripotents and partial isometries correspond to the same elements. If we fix a tripotent e in E, we can find a decomposition of the space in terms of the eigenspaces of the operator $L(e, e)$ which is expressed as follows:

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e), \tag{1.5}$$

where $E_k(e) := \{x \in E : L(e, e)x = \frac{k}{2}x\}$ is a subtriple of E called the *Peirce-k subspace* ($k = 0, 1, 2$). *Peirce-k projection* is the name given to the natural projection of E onto $E_k(e)$, and it is usually denoted by $P_k(e)$. The Peirce-2 subspace $E_2(e)$ is a unital JB*-algebra with respect to the product and involution given by $x \circ_e y = \{x, e, y\}$ and $x^{*_e} = \{e, x, e\}$, respectively.

A tripotent e in E is called *algebraically minimal* (respectively, *complete* or *algebraically maximal*) if $E_2(e) = \mathbb{C}e \neq \{0\}$ (respectively, $E_0(e) = \{0\}$). We shall say that e is a *unitary tripotent* if $E_2(e) = E$. The symbols $\mathcal{U}(E)$, $\mathcal{U}_{min}(E)$, and $\mathcal{U}_{max}(E)$ will stand for the sets of all tripotents, minimal tripotents, and complete tripotents in E, respectively.

A JB*-triple might contain no non-trivial tripotents, that is the case of the JB*-triple $C_0[0, 1]$ of all complex-valued continuous functions on $[0, 1]$ vanishing at 0. However, in a JB*-triple E the extreme points of its closed unit ball are precisely the complete tripotents in E (cf. [7, Lemma 4.1], [33, Proposition 3.5] or [14, Corollary 4.8]). Thus, every JB*-triple which is also a dual Banach space contains an abundant set of tripotents. JB*-triples which are additionally dual Banach spaces are called *JBW*-triples*. Each JBW*-triple admits a unique (isometric) predual and its triple product is separately weak* continuous (cf. [2]).

A JBW*-triple is called *atomic* if it coincides with the w*-closure of the linear span of its minimal tripotents. A very natural example is given by $B(H)$, where each minimal tripotent is of the form $\xi \otimes \eta$ with ξ, η in the unit sphere of H. Every Cartan factor is an atomic JBW*-triple. Cartan factors are enough to exhaust all possible cases since every atomic JBW*-triple is an ℓ_∞-sum of Cartan factors (cf. [20, Proposition 2 and Theorem E]).

The notion of orthogonality between tripotents is an important concept in the theory of JB*-triples. Suppose e and v are two tripotents in a JB*-triple E. According to the standard notation (see, for example [34, 3]) we say that e is *orthogonal* to u ($e \perp u$ in short) if $\{e, e, u\} = 0$. It is known that $e \perp u$ if and only if $\{u, u, e\} = 0$ (and the latter is equivalent to any of the next statements: $L(e, u) = 0$; $L(u, e) = 0$; $e \in E_0(u)$; $u \in E_0(e)$ cf. [34, Lemma 3.9]). It is worth to remark that two projections p and q in a C*-algebra A, regarded as a JB*-triple, are orthogonal if and only if $pq = 0$

(that is, they are orthogonal in the usual sense).

We can also speak about orthogonality for pairs of general elements in a JB*-triple E. We shall say that x and y in E are *orthogonal* ($x \perp y$ in short) if $L(x,y) = 0$ (equivalently $L(y,x) = 0$, compare [9, Lemma 1.1] for several reformulations). Any two orthogonal elements a and b in JB*-triple E are M-orthogonal in a strict geometric sense, that is, $\|a+b\| = \max\{\|a\|, \|b\|\}$ (see [19, Lemma 1.3(a)]).

Building upon the relation "being orthogonal" we can define a canonical order "$\leq$" on tripotents in E given by $e \leq u$ if and only if $u - e$ is a tripotent and $u - e \perp e$. This partial ordering is precisely the order consider by L. Molnár in Theorem 1.2, and it provides an important tool in JB*-triples (see, for example, the recent papers [25, 26, 22, 23, 21, 24] where it plays an important role). The partial order in $\mathcal{U}(E)$ enjoys several interesting properties; for example, $e \leq u$ if and only if e is a projection in the JB*-algebras $E_2(e)$ (cf. [3, Lemma 3.2] or [19, Corollary 1.7] or [22, Proposition 2.4]). In particular, if e and p are tripotents (i.e. partial isometries) in a C*-algebra A regarded as a JB*-triple with the triple product in (1.2) and p is a projection, the condition $e \leq p$ implies that e is a projection in A with $e \leq p$ in the usual order on projections (i.e. $pe = e$).

A non-zero tripotent e in E is called (*order*) *minimal* (respectively, (*order*) *maximal*) if $0 \neq u \leq e$ for a tripotent u in E implies that $u = e$ (respectively, $e \leq u$ for a tripotent u in E implies that $u = e$). Clearly, every algebraically minimal tripotent is (order) minimal but the reciprocal implication does not necessarily hold, for example, the unit element in $C[0,1]$ is order minimal but not algebraically minimal. In the C*-algebra $C_0[0,1]$, of all continuous functions on $[0,1]$ vanishing at 0, the zero tripotent is order maximal but it is not algebraically maximal. In the setting of JBW*-triples these pathologies do not happen, that is, in a JBW*-triple order and algebraic maximal (respectively, minimal) tripotents coincide (cf. [14, Corollary 4.8] and [3, Lemma 4.7]).

A triple homomorphism between JB*-triples E and F is a linear map $T : E \to F$ such that $T\{a,b,c\} = \{T(a), T(b), T(c)\}$ for all $a,b,c \in E$. Every triple homomorphism between JB*-triples is continuous [1, Lemma 1]. A triple isomorphism is a bijective triple homomorphism. Clearly, the inclusion $T(\mathcal{U}(E)) \subseteq \mathcal{U}(F)$ holds for each triple homomorphism T, while the equality $T(\mathcal{U}(E)) = \mathcal{U}(F)$ is true for every triple isomorphism T. Every injective triple homomorphism is an isometry (see [1, Lemma 1]). Actually a deep result in the theory of JB*-triples, established by W. Kaup in [31, Proposition 5.5], proves that a linear bijection between JB*-triples is a triple isomorphism if and only if it is an isometry. Therefore, each triple isomorphism $T : E \to F$ induces a surjective isometry $T|_{\mathcal{U}(E)} : \mathcal{U}(E) \to \mathcal{U}(F)$ which preserves orthogonality and partial order in both directions. Similar arguments prove that the mappings $T|_{\mathcal{U}_{min}(E)} : \mathcal{U}_{min}(E) \to \mathcal{U}_{min}(F)$ and $T|_{\mathcal{U}_{max}(E)} : \mathcal{U}_{max}(E) \to \mathcal{U}_{max}(F)$ are surjective isometries.

Along this note, the unit sphere of each normed space X will be denoted by S_{X}, and we shall write $\mathbb{T}$ for $S_{\mathbb{C}}$.

§ 2. Maps preserving triple transition pseudo-probabilities between minimal tripotents

As we recalled at the introduction, the *transition probability* between two minimal projections $p = \xi \otimes \xi$ and $q = \eta \otimes \eta$ in $B(H)$ is given by $\operatorname{tr}(pq) = \operatorname{tr}(pq^*) = \operatorname{tr}(qp^*) = |\langle \xi, \eta\rangle|^2$. Let us observe that each minimal projection $p = \xi \otimes \xi$ in $B(H)$ is bi-univocally associated with a pure normal state $\varphi_p \in B(H)_*$ (i.e. an extreme point of the normal state space) at which p attains its norm. Clearly φ_p is identified with the pure normal state given by $\varphi(a) = (\xi \otimes \xi)(a) := \langle a(\xi), \xi\rangle = \operatorname{tr}(ap)$ $(a \in B(H))$. Thus, the transition probability between p and q is given by the identity

$$\operatorname{tr}(pq) = |\langle \xi, \eta\rangle|^2 = |\varphi_p(q)|^2 = |\varphi_q(p)|^2. \tag{2.1}$$

For each minimal partial isometry $e = \xi \otimes \eta$ in $B(H)$, with ξ, η unitary vectors in H, there exists a unique extreme point φ_e of the closed unit ball of $C_1(H) = B(H)_*$ such that $\varphi_e(e) = 1$. Actually φ_e is defined by $\varphi_e(x) := \langle x(\xi), \eta\rangle = \operatorname{tr}(e^* x)$ $(x \in B(H))$. Motivated by the identity in (2.1), for each couple e, v of minimal partial isometries in $B(H)$, we define the *triple transition pseudo-probability* between e and v as the scalar $\varphi_e(v)$ –this is not a real probability, since it actually takes complex values. The question is whether we can extend this definition to the wider setting of Cartan factors and atomic JBW*-triples.

The lacking of a positive cone in general JB*-triples induced us to replace the lattice of projections in $B(H)$ by the poset of tripotents in a Cartan factor or in an atomic JBW*-triple in our recent study on bijections preserving the partial ordering and orthogonality between the poset of two atomic JBW*-triples in [18]. Here we introduce the triple transition pseudo-probability between two minimal tripotents in an atomic JBW*-triple. To understand well the definition we need to recall some geometric properties of JBW*-triples. Following [19], the extreme points of the closed unit ball, $\mathcal{B}_{M_*}$, of the predual, M_*, of a JBW*-triple M are called *atoms* or *pure atoms*. We recall that the extreme points of the convex set of all positive functionals with norm ≤ 1 in the predual of a von Neumann algebra are called *pure states*. The symbol $\partial_e(\mathcal{B}_{M_*})$ will stand for the set of all pure atoms of M.

By [19, Proposition 4], for each minimal tripotent e in a JBW*-triple M there exists a unique pure atom φ_e satisfying $P_2(e)(x) = \varphi_e(x) e$ for all $x \in M$. Furthermore, the mapping

$$\mathcal{U}_{min}(M) \to \partial_e(\mathcal{B}_{M_*}), \quad e \mapsto \varphi_e$$

is a bijection from the set of minimal tripotents in M onto the set of pure atoms of M.

We are now in a position to introduce the key notion of this note.

Definition 2.1. *Let e and v be minimal tripotents in a JBW*-triple M. We define the* triple transition pseudo-probability *from e to v as the complex number given by*

$$TTP(e,v) = \varphi_v(e). \tag{2.2}$$

Observe that every triple transition pseudo-probability lies in the closed unit ball of $\mathbb{C}$. Formally speaking, the triple transition pseudo-probability is not a probability because it can take complex values. However, it satisfies many interesting and natural properties. For example, by [19, Lemma 2.2] we have

$$TTP(v,e) = \varphi_e(v) = \overline{\varphi_v(e)} = \overline{TTP(e,v)}, \tag{2.3}$$

for every $e, v \in \mathcal{U}_{min}(M)$, which is naturally expressing the property of symmetry of the triple transition pseudo-probability.

If p and q are two minimal projections in a von Neumann algebra W, having in mind that φ_p is a norm-one functional attaining its norm at p, it follows that φ_p is a positive normal state on W, and hence $TTP(q,p) = \varphi_p(q)$ is a real number in the interval $[0,1]$ and coincides with $TTP(p,q) = \varphi_q(p)$. Therefore the new notion of triple transition pseudo-probability agrees with the usual transition probability in the case of minimal projections.

Molnár's theorem [37, Theorem 2], presented as Theorem 1.4 in the introduction, can be now restated in the following terms: Let $\Phi : \mathcal{U}_{min}(B(H)) \to \mathcal{U}_{min}(B(H))$ be a bijective mapping preserving triple transition pseudo-probabilities. Then Φ extends to a surjective complex-linear isometry. Inspired by Molnár's result, it seems natural to study the bijections preserving the triple transition pseudo-probabilities between the sets of minimal tripotents of two atomic JBW*-triples. The first unexpected conclusion appears when dealing with rank-one JB*-triples. Contrary to the serious obstacles affecting bijective preservers of partial ordering in both directions and orthogonality in the case of rank-one Cartan factors cf. [18, Remark 3.6]), preservers of triple transition pseudo-probabilities between sets of minimal tripotents have an excellent behaviour in the case of rank-one Cartan factors.

Let us first recall that a subset $\mathcal{S}$ of a JB*-triple E is called *orthogonal* if $0 \notin \mathcal{S}$ and $a \perp b$ for all $a, b \in \mathcal{S}$. The minimal cardinal number r satisfying $\operatorname{card}(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the *rank* of E. Spin factors have rank 2 and the exceptional Cartan factors of type 5 and 6 have ranks 2 and 3, respectively. A JB*-triple has finite rank if and only if it is reflexive (cf. [8, Proposition 4.5] and [12, Theorem 6] or [6, 5]). Furthermore, if E is a JB*-triple of rank-one, it must be reflexive and a rank-one Cartan factor, and moreover, it must be isometrically isomorphic to a complex Hilbert space (see the discussion in [5, §3] and [32, Table 1 in page 210]).

The rank of a tripotent e in a JB*-triple E is defined as the rank of $E_2(e)$. It is known that for each tripotent e in a Cartan factor C we have $r(e) = r(C_2(e)) = n < \infty$ if and only if it can be written as an orthogonal sum of n mutually orthogonal minimal tripotents in C (see, for example, [32, page 200]).

The rank theory plays a fundamental role in the different solutions to Tingley's problem in the case of compact C*-algebras [39] and weakly compact JB*-triples [15, 16], as well as to prove that every JBW*-triple satisfies the Mazur–Ulam property [4, 30]. In our next result we shall apply some of the techniques developed in the just quoted results.

Proposition 2.2. *Let $\Phi : \mathcal{U}_{min}(E) \to \mathcal{U}_{min}(F)$ be a transformation preserving triple transition pseudo-probabilities, that is,*

$$TTP(\Phi(u), \Phi(e)) = \varphi_{\Phi(e)}(\Phi(u)) = \varphi_e(u) = TTP(u, e), \text{ for all } e, u \in \mathcal{U}_{min}(E),$$

where E and F are two rank-one JB*-triples. Then Φ extends to a (complex-)linear isometric triple homomorphism from E to F.

Proof. As we have seen before the statement of this proposition, we can assume that E and F are two complex Hilbert spaces regarded as type 1 Cartan factors. We observe that $\mathcal{U}(E)\backslash\{0\} = \mathcal{U}_{min}(E) = S_{_E}$, the unit sphere of E, and $\mathcal{U}(F)\backslash\{0\} = \mathcal{U}_{min}(F) = S_{_F}$. Since for each $e \in S_{_E}$, φ_e is precisely the functional given by $\varphi_e(x) = \langle x, e\rangle$ $(x \in E)$, the hypothesis on Φ is equivalent to

$$\langle \Phi(u), \Phi(e)\rangle = \langle u, e\rangle, \text{ for all } e, u \in \mathcal{U}_{min}(E) = S_{_E}.$$

A simple computation shows that

$$\begin{aligned}\|\Phi(e) - \Phi(v)\|^2 &= \langle \Phi(e) - \Phi(v), \Phi(e) - \Phi(v)\rangle \\ &= \langle \Phi(e), \Phi(e)\rangle - \langle \Phi(v), \Phi(e)\rangle - \langle \Phi(e), \Phi(v)\rangle + \langle \Phi(v), \Phi(v)\rangle \\ &= \langle e, e\rangle - \langle v, e\rangle - \langle e, v\rangle + \langle v, v\rangle = \|e - v\|^2,\end{aligned}$$

for all $e, v \in S(E)$. That is $\Phi : S_{_E} \to S_{_F}$ is an isometry. Moreover, by the assumptions on Φ we also have

$$\langle -\Phi(e), \Phi(-e)\rangle = -\langle \Phi(e), \Phi(-e)\rangle = -\langle e, -e\rangle = 1,$$

which proves that $\Phi(-e) = -\Phi(e)$, for all $e \in S_{_E}$. An application of the solution to Tingley's problem for Hilbert spaces established by G.G. Ding in [13, Theorem 2.2] guarantees the existence of a real linear isometry $T : E \to F$ whose restriction to $S_{_E}$ is Φ.

We shall finally show that T is complex linear. As before, by the assumptions on Φ, for each $\lambda \in \mathbb{T}$ we also have

$$\langle \lambda\Phi(e), \Phi(\lambda e)\rangle = \lambda\langle \Phi(e), \Phi(\lambda e)\rangle = \lambda\langle e, \lambda e\rangle = 1,$$

witnessing that $\Phi(\lambda e) = \lambda\Phi(e)$, for all $e \in S_{_E}$ and $\lambda \in \mathbb{T}$. The rest is clear. $\square$

Let us note that in the previous proposition we are not assuming that Φ is injective nor surjective.

It is now time to see a handicap or a limitation of the triple transition pseudo-probability. Let p and q be two minimal projections in a von Neumann algebra W. Suppose that the transition probability between p and q, as given in (2.1), is zero, that is $\varphi_p(q) = 0$, or equivalently, $P_2(p)(q) = pqp = 0$. Since $0 = pqp = (pq)(pq)^*$, we deduce that $pq = qp = 0$, and thus $q = (1-p)q(1-q) = P_0(p)(q) \perp p$. This property does not always hold when projections are replaced with tripotents or partial isometries, for example if e and v are minimal tripotents in a Cartan factor C with $v \in C_1(e)$ we clearly have $\varphi_e(v) = 0$ but e and v are not orthogonal. A simple example can be given by $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $M_2(\mathbb{C})$. However, every (real linear) triple homomorphism between JB*-triples preserves orthogonality. Despite of this handicap, we can now get a first extension of every bijective transformation preserving triple transition probabilities between the sets of minimal tripotents of two atomic JBW*-triples to their socles.

Let us first recall some structure results for atomic JBW*-triples. Every JBW*-triple M decomposes as the orthogonal sum of two weak*-closed ideals $\mathcal{A}$ and $\mathcal{N}$, where $\mathcal{A}$ is an atomic JBW*-triple (called the *atomic part* of M) and $\mathcal{N}$ contains no minimal tripotents [19, Theorem 2]. Furthermore, M_* decomposes as the ℓ_1-sum of two norm closed subspaces $\mathcal{A}_*$–the predual of $\mathcal{A}$– and $\mathcal{N}_*$–the predual of $\mathcal{N}$– satisfying that $\mathcal{A}_*$ is the norm closure of the linear span of all pure atoms of M and the closed unit ball of $\mathcal{N}_*$ contains no extreme points [19, Theorem 1].

At this stage the reader should also get some information about elementary JB*-triples. Let C_j be a Cartan factor of type $j \in \{1, \ldots, 6\}$. The elementary JB*-triple, K_j, of type j associated with C_j is defined as follows: $K_1 = K(H_1, H_2)$; $K_i = C \cap K(H)$ when C is of type $i = 2, 3$, and $K_j = C_j$ in the remaining cases (cf. [8]). For each elementary JB*-triple of type j, its bidual space is precisely a Cartan factor of j.

The socle of a JB*-triple E, soc(E), is the (non-necessarily closed) linear subspace of E generated by all minimal tripotents in E. For example, the socle of $B(H)$ is the subspace, $\mathcal{F}(H)$, of all finite rank operators, and it is not, in general, closed. If C is a

Cartan factor of finite rank (or, more generally, a reflexive JB*-triple), every element in C can be written as a finite linear combination of mutually orthogonal minimal tripotents (see [8, Proposition 4.5 and Remark 4.6] or [5]), and thus the socle of C is the whole C–that is $\operatorname{soc}(C) = \mathcal{K}(C) = C$. For a general Cartan factor we have $\overline{\operatorname{soc}(C)}^{\|-\|} = \mathcal{K}(C)$ and $\overline{\mathcal{K}(C)}^{w^*} = C$. In an atomic JBW*-triple M, the symbol $\mathcal{K}(M)$ will stand for the c_0-sum of the elementary JB*-triples associated with the Cartan factors expressing M as an ℓ_∞-sum.

Theorem 2.3. *Let $\Phi : \mathcal{U}_{min}(M) \to \mathcal{U}_{min}(N)$ be a bijective transformation preserving triple transition pseudo-probabilities (i.e., $TTP(\Phi(v), \Phi(e)) = \varphi_{\Phi(e)}(\Phi(v)) = \varphi_e(v) = TTP(v, e)$, for all e, v in $\mathcal{U}_{min}(M)$), where M and N are atomic JBW*-triples. Then there exists a bijective (complex) linear mapping $T_0 : soc(M) \to soc(N)$ whose restriction to $\mathcal{U}_{min}(M)$ is Φ.*

Proof of Theorem 2.3. Clearly, the pure atoms of M and N are norming sets for $\mathcal{K}(M)$ and $\mathcal{K}(N)$, respectively. Let us suppose that $\displaystyle\sum_{i=1}^{m} \alpha_i e_i = \sum_{j=1}^{m} \beta_j v_j \in \operatorname{soc}(M)$, where $\alpha_i, \beta_j \in \mathbb{C}$ and $e_i, v_j \in \mathcal{U}_{min}(M)$. By the hypothesis on Φ, for each $\psi \in \partial_e(\mathcal{B}_{N_*})$, there exists $\Phi(w) = \tilde{w} \in \mathcal{U}_{min}(N)$ (and $w \in \mathcal{U}_{min}(M)$) such that $\psi = \psi_{\tilde{w}} = \psi_{\Phi(w)}$. It also follows from the hypotheses that

$$\begin{aligned}
\psi\left(\sum_{i=1}^{m} \alpha_i \Phi(e_i)\right) &= \psi_{\Phi(w)}\left(\sum_{i=1}^{m} \alpha_i \Phi(e_i)\right) = \sum_{i=1}^{m} \alpha_i \psi_{\Phi(w)}\left(\Phi(e_i)\right) \\
&= \sum_{i=1}^{m} \alpha_i \psi_w\left(e_i\right) = \psi_w\left(\sum_{i=1}^{m} \alpha_i e_i\right) = \psi_w\left(\sum_{j=1}^{m} \beta_j v_j\right) \\
&= \sum_{j=1}^{m} \beta_j \psi_w\left(v_j\right) = \sum_{j=1}^{m} \beta_j \psi_{\Phi(w)}\left(\Phi(v_j)\right) \\
&= \psi_{\Phi(w)}\left(\sum_{j=1}^{m} \beta_j \Phi(v_j)\right) = \psi\left(\sum_{j=1}^{m} \beta_j \Phi(v_j)\right).
\end{aligned}$$

The arbitrariness of $\psi \in \partial_e(\mathcal{B}_{M_*})$ together with the fact that the set of pure atoms of N separates the point of $\mathcal{K}(M)$ imply that

$$\sum_{i=1}^{m} \alpha_i \Phi(e_i) = \sum_{j=1}^{m} \beta_j \Phi(v_j).$$

Therefore, the mapping $T_0 : \operatorname{soc}(M) \to \operatorname{soc}(N)$, $T_0\Big(\sum_{i=1}^{m} \alpha_i e_i\Big) = \sum_{i=1}^{m} \alpha_i \Phi(e_i)$ is well-defined and linear. We further know that $T_0(e) = \Phi(e)$ for all $e \in \mathcal{U}_{min}(M)$.

We can similarly define a linear mapping $R_0 : \text{soc}(N) \to \text{soc}(M)$ satisfying $R_0(\Phi(e)) = e$ for all $e \in \mathcal{U}_{min}(M)$ and $R_0 = T_0^{-1}$. Therefore T_0 and R_0 are bijections. $\square$

It should be remarked that, at this stage the hypotheses of the previous Theorem 2.3 do not imply, in a simple way, that the linear mapping T_0 is continuous. Actually, if $e_1, \ldots, e_n$ are mutually orthogonal minimal tripotents in M, we have $\left\| T_0\left(\sum_{j=1}^n e_j \right) \right\| = \left\| \sum_{j=1}^n \Phi(e_j) \right\| \leq n$. We cannot get a better bound without assuming orthogonality (and hence M-orthogonality) on the minimal tripotents $\Phi(e_1), \ldots, \Phi(e_n)$. In this line we recall next a result by F.J. Herves and J.M. Isidro from [29].

Theorem 2.4. [29, Theorem in page 199] *Let E be a finite-rank JB*-triple, and let $T : E \to E$ be a linear mapping (continuity is not assumed). Then the following statements are equivalent:*

(1) *T is a triple automorphism.*

(2) *$T(\mathcal{U}_{min}(E)) = \mathcal{U}_{min}(E)$ and preserves orthogonality.*

We establish now a hybrid version of the previous two results.

Corollary 2.5. *Let $\Phi : \mathcal{U}_{min}(M) \to \mathcal{U}_{min}(N)$ be a bijective transformation preserving orthogonality and triple transition pseudo-probabilities (i.e. $TTP(\Phi(v), \Phi(e)) = \varphi_{\Phi(e)}(\Phi(v)) = \varphi_e(v) = TTP(v, e)$, for all e, v in $\mathcal{U}_{min}(M)$), where M and N are atomic JBW*-triples. Then Φ extends (uniquely) to a surjective complex-linear (isometric) triple isomorphism from M onto N.*

Proof. By Theorem 2.3 there exists a linear bijection $T_0 : soc(M) \to soc(N)$ whose restriction to $\mathcal{U}_{min}(M)$ coincides with Φ. By hypotheses, given $u, v \in \mathcal{U}_{min}(M)$ with $u \perp v$, we have $\Phi(u) \perp \Phi(v)$. Having in mind that for each x in the closed unit ball of $\text{soc}(M)$ there exists a finite family $\{e_n\}_n$ of mutually orthogonal minimal tripotents in M and $\{\lambda_n\}_n$ in $\mathbb{R}^+$ such that $x = \sum\limits_n \lambda_n e_n$ and $1 = \|x\| = \max\{\lambda_n : n\}$ (cf. [8, Remark 4.6]). It follows from the definition of T_0 that

$$T_0(x) = T_0\left(\sum_n \lambda_n e_n\right) = \sum_n \lambda_n T_0(e_n) = \sum_n \lambda_n \Phi(e_n),$$

and hence $\|T_0(x)\| = \|x\|$, because, by hypotheses, $\{\Phi(e_n)\}_n$ is a family of mutually orthogonal minimal tripotents in N. Furthermore, by the previous conclusion

$$\begin{aligned}\{T_0(x), T_0(x), T_0(x)\} &= \sum_n \lambda_n^3 \{\Phi(e_n), \Phi(e_n), \Phi(e_n)\} \\ &= \sum_n \lambda_n^3 \Phi(e_n) = \sum_n \lambda_n^3 T_0(e_n) = T_0(\{x, x, x\}),\end{aligned}$$

which shows that T_0 is a contractive triple isomorphism from $\mathrm{soc}(M)$ onto $\mathrm{soc}(N)$.

We can therefore find a continuous linear extension of T_0 to a continuous (isometric) linear triple isomorphism from $\mathcal{K}(M)$ onto $\mathcal{K}(N)$ denoted by the same symbol T_0. The bitransposed mapping $T_0^{**}: \mathcal{K}(M)^{**} = M \to \mathcal{K}(N)^{**} = N$ is a triple isomorphism whose restriction to $\mathcal{U}_{min}(M)$ coincides with Φ. This finishes the proof of the result. $\square$

It seems a natural (and important) question to ask whether a bijection preserving triple transition pseudo-probabilities between the sets of minimal tripotents in two atomic JBW*-triples also preserves orthogonality. That is, whether in Corollary 2.5 the hypothesis concerning preservation of orthogonality can be relaxed. This will be answered for spin and type 1 Cartan factors along the next sections.

§ 3. The case of spin factors

As well as the study of those maps preserving triple transition pseudo-probabilities between the sets of minimal tripotents in two rank-one JB*-triples deserved its own treatment in Proposition 2.2, the case of spin factors is also worth to study by itself.

Let us fix a spin factor M whose inner product, involution and triple product are given by $\langle\cdot,\cdot\rangle$, $x \mapsto \overline{x}$, and

$$\{a, b, c\} = \langle a, b\rangle c + \langle c, b\rangle a - \langle a, \bar{c}\rangle \bar{b},$$

respectively (cf. the definition in page 7). It is usually assumed that $\dim(M) \geq 3$; actually if $\dim(M) = 2$, the defined structure produces $\mathbb{C} \oplus^{\infty} \mathbb{C}$, which is not a factor (cf. [32, Remark 4.3]). The real subspace

$$M_{\mathbb{R}}^{-} = \{a \in M : a = \bar{a}\},$$

of all fixed points for the involution $\bar{\cdot}$ is a real Hilbert space with respect to the restricted inner product $\langle a, b\rangle = \Re\mathrm{e}\langle a, b\rangle$ $(a, b \in M_{\mathbb{R}}^{-})$, and $M = M_{\mathbb{R}}^{-} \oplus i M_{\mathbb{R}}^{-}$. We shall also make use of the real Hilbert space $\mathcal{H} = M_{\mathbb{R}}$ given by the real underlying space of M equipped with the inner product $\Re\mathrm{e}\langle .,. \rangle$. Clearly $M_{\mathbb{R}}^{-}$ is a closed subspace of $M_{\mathbb{R}}$. The symbol $\perp_2$ will denote orthogonality in the Euclidean sense.

Each triple automorphism Φ on the spin factor M is precisely described in the following form:

$$\Phi(a + ib) = \lambda(U(a) + iU(b)) \text{ for all } a, b \in M_{\mathbb{R}}^{-},$$

where $\lambda \in \mathbb{T}$ and $U : M_{\mathbb{R}}^{-} \to M_{\mathbb{R}}^{-}$ is a unitary operator (cf. [29, Theorem in page 196] or [17, Section 3.1.3]).

The set of tripotents in M has been intensively studied along the last forty years. If we exclude the zero tripotent, M only contains tripotents of rank-one (minimal) and of rank-two (maximal and unitaries). In the second case we have

$$\mathcal{U}_{max}(M) = \{\lambda a : \lambda \in \mathbb{T},\ a \in S_{M_{\mathbb{R}}^-}\},$$

while

$$\mathcal{U}_{min}(M) = \left\{\frac{a+ib}{2} : a, b \in S_{M_{\mathbb{R}}^-},\ \langle a, b\rangle = 0\right\} \tag{3.1}$$

(see, for example, [17, Section 3.1.4] or [22, Lemma 6.1], [18] or [30, Section 3]).

It is well-known, and easy to check, that for each minimal tripotent $v = \frac{a+ib}{2}$ with $a, b \in S_{M_{\mathbb{R}}^-}$, $\langle a, b\rangle = 0$ the Peirce-0 and Peirce-1 subspaces are the following:

$$M_0(v) = \mathbb{C}\overline{v}, \text{ and } M_1(v) = \{x \in M : x \perp_2 v, \overline{v}\} = \{x \in M : x \perp_2 a, b\}. \tag{3.2}$$

Another important fact to have in mind is the following: for each minimal tripotent v in a spin factor M we have

$$P_2(v)(x) = 2\langle x, v\rangle v, \text{ and hence } \varphi_v(x) = 2\langle x, v\rangle \quad (x \in M). \tag{3.3}$$

The next technical lemma is presented separately to simplify the arguments.

Lemma 3.1. *Let v and w be minimal tripotents in a spin factor M. Suppose that $M_1(v) = M_1(w)$. Then w lies in the linear span of v and $\overline{v}$. If we additionally assume that $TTP(w, v) = \varphi_v(w) = 0$, then w belongs to $\mathbb{T}\overline{v} \subset M_0(v)$.*

Proof. Let us write $v = \frac{a+ib}{2}$, $w = \frac{c+id}{2}$ with $a, b, c, d \in S_{M_{\mathbb{R}}^-}$ and $\langle a, b\rangle = 0 = \langle c, d\rangle$. Since $\{x \in M : x \perp_2 a, b\} = M_1(v) = M_1(w) = \{x \in M : x \perp_2 c, d\}$, it follows that the Euclidean orthogonal complements of the sets $\{a, b\}$ and $\{c, d\}$ in M coincide. It is straightforward to check that, under these circumstances, $c = \alpha_1 a + \alpha_2 b$ and $d = \beta_1 a + \beta_2 b$, with $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in S_{\ell_2^2(\mathbb{R})} \equiv \mathbb{T}$. Therefore

$$w = \frac{c+id}{2} = \frac{\alpha_1 a + \alpha_2 b + i(\beta_1 a + \beta_2 b)}{2} \in \operatorname{span}\{v, \overline{v}\},$$

as desired.

Finally, if $TTP(w, v) = \varphi_v(w) = 0$, the expression of the pure states given in (3.3) assures that

$$0 = \varphi_v(w) = 2\langle w, v\rangle = 2\left\langle \frac{\alpha_1 a + \alpha_2 b + i(\beta_1 a + \beta_2 b)}{2}, \frac{a+ib}{2}\right\rangle,$$

or equivalently,

$$\alpha_1 + \beta_2 = 0 = \beta_1 - \alpha_2,$$

that is, $\beta_1 + i\beta_2 = \alpha_2 - i\alpha_1 = (-i)(\alpha_1 + i\alpha_2)$, and thus

$$w = \frac{(\alpha_1 + i\alpha_2)a + (\alpha_2 - i\alpha_1)b}{2} = (\alpha_1 + i\alpha_2)\frac{a - ib}{2} \in \mathbb{T}\overline{v},$$

which finishes the proof. □

We can now prove that every bijection preserving triple transition pseudo-probabilities between the sets of minimal tripotents in two spin factors also preserves orthogonality, and consequently extends to a triple isomorphism between the spin factors.

Theorem 3.2. *Let $\Phi : \mathcal{U}_{min}(M) \to \mathcal{U}_{min}(N)$ be a bijective transformation preserving triple transition pseudo-probabilities (i.e., $TTP(\Phi(v), \Phi(e)) = \varphi_{\Phi(e)}(\Phi(v)) = \varphi_e(v) = TTP(v, e)$, for all e, v in $\mathcal{U}_{min}(M)$), where M and N are spin factors. Then there exists a (unique) triple isomorphism $T : M \to N$ whose restriction to $\mathcal{U}_{min}(M)$ is Φ.*

Proof. We shall denote by the same symbols $\langle .,. \rangle$ and $x \mapsto \overline{x}$ the inner products and the involutions on M and N. Let $T_0 : \mathrm{soc}(M) \to \mathrm{soc}(N)$ be the linear bijection whose restriction to $\mathcal{U}_{min}(M)$ is Φ given by Theorem 2.3. Since M and N are spin factors with rank-two we have $\mathrm{soc}(M) = M$ and $\mathrm{soc}(N) = N$. Furthermore, every element x in M writes in the form $x = \lambda_1 v_1 + \lambda_2 v_2$, where v_1 and v_2 are two orthogonal minimal tripotents in M, $\lambda_i \in [0, \|x\|]$. In particular, $\|T_0(x)\| \leq \lambda_1 \|\Phi(v_1)\| + \lambda_2 \|\Phi(v_2)\| \leq 2\|x\|$. Therefore T_0 is a bounded linear bijection from M onto N sending minimal tripotents to minimal tripotents and preserving triple transition pseudo-probabilities.

Let us take $a, b \in S_{M_{\mathbb{R}}^-}$ with $\langle a, b \rangle = 0$. Let us write $T_0(a) = a_1 + ia_2$ and $T_0(b) = b_1 + ib_2$, with $a_j, b_j \in N_{\mathbb{R}}^-$. Since the elements $T_0(\frac{a+ib}{2}) = \frac{(a_1-b_2)+i(b_1+a_2)}{2}$ and $T_0(\frac{a-ib}{2}) = \frac{(a_1+b_2)+i(a_2-b_1)}{2}$ are minimal tripotents we deduce that the following identities hold:

$$\begin{cases} \|a_1 - b_2\|_2^2 = \|a_2 + b_1\|_2^2 = 1, \\ \|a_1 + b_2\|_2^2 = \|a_2 - b_1\|_2^2 = 1, \\ \langle a_1 + b_2, a_2 - b_1 \rangle = 0, \\ \langle a_1 - b_2, a_2 + b_1 \rangle = 0, \end{cases},$$

equivalently,

$$\begin{cases} \|a_1\|_2^2 + \|b_2\|_2^2 = \|b_1\|_2^2 + \|a_2\|_2^2 = 1, \\ \langle a_1, b_2 \rangle = 0 = \langle a_2, b_1 \rangle, \\ \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = 0, \\ \langle a_1, b_1 \rangle - \langle a_2, b_2 \rangle = 0. \end{cases} \tag{3.4}$$

On the other hand, by (3.3), for each minimal tripotent v in a spin factor we have $\varphi_v(x) = 2\langle x, v\rangle$. Applying now that Φ (and T_0) preserves triple transition pseudo-probabilities among minimal tripotents and the explicit expression of the pure atoms given above we have

$$\begin{aligned} 0 = 2\left\langle \frac{a+ib}{2}, \frac{a-ib}{2}\right\rangle &= \varphi_{\frac{a-ib}{2}}\left(\frac{a+ib}{2}\right) = \varphi_{T_0\left(\frac{a-ib}{2}\right)}\left(T_0\left(\frac{a+ib}{2}\right)\right) \\ &= 2\left\langle T_0\left(\frac{a+ib}{2}\right), T_0\left(\frac{a-ib}{2}\right)\right\rangle \\ &= 2\left\langle \frac{(a_1-b_2)+i(b_1+a_2)}{2}, \frac{(a_1+b_2)+i(a_2-b_1)}{2}\right\rangle \end{aligned}.$$

By computing the imaginary parts in the previous equality we arrive at

$$0 = \left\langle \frac{(b_1+a_2)}{2}, \frac{(a_1+b_2)}{2}\right\rangle - \left\langle \frac{(a_1-b_2)}{2}, \frac{(a_2-b_1)}{2}\right\rangle$$

equivalently,

$$\langle a_1, b_1\rangle + \langle a_2, b_2\rangle = 0,$$

which combined with the equality in the fourth line of (3.4) gives

$$\langle a_1, b_1\rangle = \langle a_2, b_2\rangle = 0.$$

The last identity together with the conclusion in the second line of (3.4) show that

$$\begin{aligned} &\langle a_1, b_1\rangle = \langle a_2, b_2\rangle = \langle a_1, b_2\rangle = \langle a_2, b_1\rangle = 0 \\ &\Leftrightarrow a_j \perp_2 b_k,\ j,k = 1,2 \text{ in the Hilbert space } N_{\mathbb{R}}. \end{aligned}$$

We have therefore shown that

$$\boxed{a \perp_2 b \text{ in the Hilbert space } M_{\mathbb{R}}^-} \Rightarrow \boxed{T_0(a) \perp_2 T_0(b) \text{ in the Hilbert space } N_{\mathbb{R}}.} \tag{3.5}$$

Therefore $T_0|_{M_{\mathbb{R}}^-} : M_{\mathbb{R}}^- \to N_{\mathbb{R}}$ is an injective linear mapping preserving orthogonality in the Hilbert sense, that is, $\langle a, b\rangle = 0 \Rightarrow \langle T_0(a), T_0(b)\rangle = 0$. Another interesting result on preservers, proved by J. Chmieliński, assures that $T_0|_{M_{\mathbb{R}}^-}$ is a positive multiple of an isometry, that is, there exists a positive γ and a real linear isometry $U : M_{\mathbb{R}}^- \to N_{\mathbb{R}}$ such that $T_0|_{M_{\mathbb{R}}^-} = \gamma U$ (cf. [11, Theorem 1]).

We shall next prove that $\gamma = 1$. Indeed, let us fix two orthogonal elements a, b in the unit sphere of $M_{\mathbb{R}}^-$. By hypothesis, $\widehat{v} = T_0(\frac{a+ib}{2}) = \frac{\gamma U(a)+i\gamma U(b)}{2} = \gamma\frac{U(a)+iU(b)}{2}$ is a minimal tripotent in N. It is well-known that in this case

$$\left\langle \frac{U(a)+iU(b)}{2}, \frac{\overline{U(a)}-i\overline{U(b)}}{2}\right\rangle = \left\langle \gamma^{-1}\widehat{v}, \gamma^{-1}\overline{\widehat{v}}\right\rangle = \gamma^{-2}\left\langle \widehat{v}, \overline{\widehat{v}}\right\rangle = 0. \tag{3.6}$$

By applying the properties of U and the previous identity we get

$$\begin{aligned}
&\left\{\frac{U(a)+iU(b)}{2},\frac{U(a)+iU(b)}{2},\frac{U(a)+iU(b)}{2}\right\}\\
&=2\left\langle\frac{U(a)+iU(b)}{2},\frac{U(a)+iU(b)}{2}\right\rangle\frac{U(a)+iU(b)}{2}\\
&-\left\langle\frac{U(a)+iU(b)}{2},\frac{\overline{U(a)}-i\overline{U(b)}}{2}\right\rangle\frac{\overline{U(a)}-i\overline{U(b)}}{2}\quad ,\\
&=\frac{1}{2}\left(\|U(a)\|_2^2+\|U(b)\|_2^2\right)\frac{U(a)+iU(b)}{2}\\
&=\frac{1}{2}\left(\|a\|_2^2+\|b\|_2^2\right)\frac{U(a)+iU(b)}{2}=\frac{U(a)+iU(b)}{2}
\end{aligned}$$

witnessing that $\frac{U(a)+iU(b)}{2}$ is a tripotent. Since $\frac{U(a)+iU(b)}{2}=\gamma^{-1}\widehat{v}$ we obtain $\gamma=1$ as desired.

Let us go back to the identity in (3.6) to deduce that

$$\begin{aligned}
0&=\left\langle U(a)+iU(b),\overline{U(a)}-i\overline{U(b)}\right\rangle\\
&=\left\langle U(a),\overline{U(a)}\right\rangle-\left\langle U(b),\overline{U(b)}\right\rangle+i\left\langle U(a),\overline{U(b)}\right\rangle+i\left\langle U(b),\overline{U(a)}\right\rangle\\
&=\left\langle U(a),\overline{U(a)}\right\rangle-\left\langle U(b),\overline{U(b)}\right\rangle+2i\left\langle U(a),\overline{U(b)}\right\rangle.
\end{aligned}\tag{3.7}$$

Since in this argument the roles of a and b are completely symmetric, by replacing $\frac{a+ib}{2}$ with $\frac{b+ia}{2}$ we derive

$$\begin{aligned}
0&=\left\langle U(b),\overline{U(b)}\right\rangle-\left\langle U(a),\overline{U(a)}\right\rangle+2i\left\langle U(b),\overline{U(a)}\right\rangle\\
&=\left\langle U(b),\overline{U(b)}\right\rangle-\left\langle U(a),\overline{U(a)}\right\rangle+2i\left\langle U(a),\overline{U(b)}\right\rangle.
\end{aligned}\tag{3.8}$$

Now, adding (3.7) and (3.8) we get

$$\left\langle T_0(a),\overline{T_0(b)}\right\rangle=\left\langle U(a),\overline{U(b)}\right\rangle=0,\ \text{for all } a,b\in S_{M_{\mathbb{R}}^-}\ \text{with } a\perp_2 b.\tag{3.9}$$

We shall next show that for any minimal tripotent $v=\frac{a+ib}{2}$ the identities

$$N_1(T_0(\overline{v}))=T_0\left(M_1(\overline{v})\right)=T_0\left(M_1(v)\right)=N_1(T_0(v))=N_1(\overline{T_0(v)})\tag{3.10}$$

hold. The second and fourth equalities are clear because $M_1(\overline{v})=M_1(v)$ and $N_1(T_0(v))=N_1(\overline{T_0(v)})$ (cf. (3.2)). For the third one suppose that $\dim(M)=\dim(M_{\mathbb{R}}^-)\geq 4$, and observe that in this case every minimal tripotent in $M_1(v)$ is of the form $\frac{c+id}{2}$ with $c,d\in M_{\mathbb{R}}^-$ and $c,d\perp_2 a,b$, and every element in $M_1(v)$ writes as the linear combination of a minimal tripotent of this form and its orthogonal. Having in mind (3.9) and the properties of U we obtain

$$\begin{aligned}\left\{T_0(v),T_0(v),T_0\left(\frac{c+id}{2}\right)\right\} &= \left\{\frac{U(a)+iU(b)}{2},\frac{U(a)+iU(b)}{2},\frac{U(c)+iU(d)}{2}\right\}\\ &= \left\langle\frac{U(a)+iU(b)}{2},\frac{U(a)+iU(b)}{2}\right\rangle\frac{U(c)+iU(d)}{2}\\ &+\left\langle\frac{U(c)+iU(d)}{2},\frac{U(a)+iU(b)}{2}\right\rangle\frac{U(a)+iU(b)}{2}\\ &-\left\langle\frac{U(a)+iU(b)}{2},\frac{\overline{U(c)}-i\overline{U(d)}}{2}\right\rangle\frac{\overline{U(a)}-i\overline{U(b)}}{2}\\ &= \frac{1}{2}\frac{U(c)+iU(d)}{2},\end{aligned}$$

witnessing that $T_0\left(\frac{c+id}{2}\right)=\frac{U(c)+iU(d)}{2}\in N_1\left(T_0(v)\right)$, and consequently $T_0\left(M_1(v)\right)\subseteq N_1(T_0(v))$, but the equality holds by the bijectivity of T_0.

If $\dim(M)=\dim(M_{\mathbb{R}}^-)=3$, then $M_1(v)$ is one dimensional and of the form $\mathbb{C}c$ with c in the unit sphere of $M_{\mathbb{R}}^-$ and $c\perp_2 a,b$. In this case, by the properties of U and (3.9), we get

$$\begin{aligned}\{T_0(v),T_0(v),T_0(\alpha c)\} &= \left\{\frac{U(a)+iU(b)}{2},\frac{U(a)+iU(b)}{2},\alpha U(c)\right\}\\ &= \left\langle\frac{U(a)+iU(b)}{2},\frac{U(a)+iU(b)}{2}\right\rangle\alpha U(c)\\ &+\left\langle\alpha U(c),\frac{U(a)+iU(b)}{2}\right\rangle\frac{U(a)+iU(b)}{2}\\ &-\left\langle\frac{U(a)+iU(b)}{2},\overline{\alpha U(c)}\right\rangle\frac{\overline{U(a)}-i\overline{U(b)}}{2}=\frac{1}{2}\alpha U(c).\end{aligned}$$

This shows that $T_0\left(M_1(v)\right)\subseteq N_1(T_0(v))$, and consequently, $T_0\left(M_1(v)\right)=N_1(T_0(v))$. The same argument shows the validity of the first equality in (3.10).

We have therefore proved that $N_1(T_0(\overline{v}))=N_1(T_0(v))$, and since, by hypothesis, $TTP(T_0(\overline{v}),T_0(v))=TTP(\overline{v},v)=0$, Lemma 3.1 assures that $T_0(\overline{v})\in\mathbb{T}\,\overline{T_0(v)}$, and thus $T_0(\overline{v})\perp T_0(v)$ in the spin factor N. Since $v=\frac{a+ib}{2}$ is any arbitrary minimal tripotent in M and each minimal tripotent orthogonal to v lies in $\mathbb{T}\overline{v}$, we deduce that T_0 preserves orthogonality among $\mathcal{U}_{min}(M)$. Finally, Corollary 2.5 asserts that T_0 is a triple isomorphism. □

§ 4. The case of type 1 Cartan factors

This section is aimed to study those bijections preserving triple transition pseudo-probabilities between the sets of minimal tripotents of two type 1 Cartan factors. In this case we shall try to extend the arguments settled by L. Molnár in [37, Theorem 2]. For this purpose we shall focus on those linear operators between type 1 Cartan factors preserving rank-one operators. Let us begin by recalling a result by M. Marcus, B.N. Moyls [35] and R. Westwick [40].

Theorem 4.1. [35, 40] *Let $T : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ be a linear mapping sending rank-one operators to rank-one operators. Then there exist invertible matrices $u \in M_m(\mathbb{C})$ and $v \in M_n(\mathbb{C})$ such that one of the next statements holds:*

(1) *$T(a) = uav$ for all $a \in M_{m,n}(\mathbb{C})$;*

(2) *$m = n$ and $T(a) = ua^tv$ for all $a \in M_{m,n}(\mathbb{C})$; where a^t denotes the transpose of a.*

One of the key tools employed by Molnár in [37, Theorem 2] is the following consequence of a result due to M. Omladič and P. Šemrl.

Theorem 4.2. [38, Theorem 3.3] *Let H be a complex Hilbert space. Suppose $\Phi : soc(B(H)) \to soc(B(H))$ is a surjective linear transformation preserving the rank-one operators in both directions. Then:*

(a) *either there are bijective linear mappings u, v on H such that $\Phi(\xi \otimes \eta) = u(\xi) \otimes v(\eta)$ $(\xi, \eta \in H)$;*

(b) *or there are bijective conjugate-linear mappings u, v on H such that $\Phi(\xi \otimes \eta) = u(\eta) \otimes v(\xi)$ $(\xi, \eta \in H)$.*

It is known that (linear) triple automorphisms on a type 1 Cartan factor of the form $B(H, K)$ are either of the form $T(x) = uxv$ $(x \in B(H, K))$, with $u \in B(K)$ and $v \in B(H)$ unitaries, or $\dim(H)$ =$\dim(K)$ and $T(x) = ux^*v$ $(x \in B(H, K) \equiv B(H))$, with $u, v : H \to H$ anti-unitaries (cf. [32, page 199]). We establish next the tool required for our purposes. It should be noted that we have strengthened the hypotheses with respect to the mentioned result by Omladič and Šemrl, but the current statement will serve for our goals in this note.

Theorem 4.3. *Let $\Phi : soc(B(H_1, K_1)) \to soc(B(H_2, K_2))$ be a bijective linear transformation, where H_1, H_2, K_1 and K_2 are complex Hilbert spaces with dimensions ≥ 2. Suppose that Φ preserves rank-one operators in both directions. Then:*

(a) *either there are bijective linear mappings $u : K_1 \to K_2$, and $v : H_1 \to H_2$ such that $\Phi(\xi \otimes \eta) = u(\xi) \otimes v(\eta)$ $(\xi \in K_1, \eta \in H_1)$;*

(b) *or there are bijective conjugate-linear mappings $u : H_1 \to K_2$, $v : K_1 \to H_2$ such that $\Phi(\xi \otimes \eta) = u(\eta) \otimes v(\xi)$ $(\xi \in K_1,\ \eta \in H_1)$.*

Proof. We shall mimic the notation and arguments by Omladič and Šemrl in [38]. So, for each $\xi \in K_j, \eta \in H_j$ we set $L_\xi := \{\xi \otimes \eta : \eta \in H_j\}$ and $R_\eta := \{\xi \otimes \eta : \xi \in K_j\}$. It is not hard to check, as in the proof of [38, Lemma 2.1], that L_ξ and R_η are maximal among additive subgroups of rank-one operators, that is, every additive group of $soc(H_j, K_j)$ consisting of operators of rank-one is either a subgroup of an L_ξ, for a vector $\xi \in K$, or a subgroup of an R_η for $\eta \in H_j$.

Step 1. We claim that for every $\xi \in K_1$, either there is a $\widehat{\xi} \in K_2$, depending on ξ, such that $\Phi(L_\xi) = L_{\widehat{\xi}}$, or there is an $\widehat{\eta} \in H_2$, depending on ξ, such that $\Phi(L_\xi) = R_{\widehat{\eta}}$. This is clear because Φ is linear, bijective, and preserves rank-one operators in both directions, and thus it must preserve maximal additive subgroups of rank-one operators.

Step 2. If there exists $\xi_0 \in K_1 \backslash \{0\}$ such that $\Phi(L_{\xi_0}) = L_{\widehat{\xi_0}}$ for some $\widehat{\xi_0} \in K_2$ depending on ξ_0 (respectively, $\Phi(L_{\xi_0}) = R_{\widehat{\eta_0}}$ for some $\widehat{\eta_0} \in H_2$, depending on ξ_0), then for each $\xi \in K_1$, $\Phi(L_\xi) = L_{\widehat{\xi}}$ for some $\widehat{\xi} \in K_2$ depending on ξ (respectively, $\Phi(L_\xi) = R_{\widehat{\eta}}$ for some $\widehat{\eta} \in H_2$, depending on ξ). We shall only prove the first statement. Suppose we can find ξ_0 and ξ_1 in $K_1 \backslash \{0\}$ such that $\Phi(L_{\xi_0}) = L_{\widehat{\xi_0}}$ for some $\widehat{\xi_0} \in K_2$ depending on ξ_0 and $\Phi(L_{\xi_1}) = R_{\widehat{\eta_1}}$ for some $\widehat{\eta_1} \in H_2$, depending on ξ_1. Observing that $L_{\xi_0} = L_{\xi_1}$ if and only if ξ_0 and ξ_1 are linearly dependent, it follows from the assumptions that ξ_0 and ξ_1 must be linearly independent. The element $\widehat{\xi_0} \otimes \widehat{\eta_1} \in L_{\widehat{\xi_0}} \cap R_{\widehat{\eta_1}} = \Phi(L_{\xi_0}) \cap \Phi(L_{\xi_1})$, and thus we can find $\eta_0, \eta_1 \in H_1 \backslash \{0\}$ such that $\Phi(\xi_0 \otimes \eta_0) = \widehat{\xi_0} \otimes \widehat{\eta_1} = \Phi(\xi_1 \otimes \eta_1)$. The injectivity of Φ assures that $\xi_0 \otimes \eta_0 = \xi_1 \otimes \eta_1$, however this equality implies that ξ_0 and ξ_1 are linearly dependent, which is impossible.

Step 3. Let us assume, by Steps 1 and 2, that for each $\xi \in K_1$, there exists a $\widehat{\xi} \in K_2$, depending on ξ, such that $\Phi(L_\xi) = L_{\widehat{\xi}}$. Consequently, for each $\eta \in H_1$ there exists a unique $v_\xi(\eta) \in H_2$ such that

$$\Phi(\xi \otimes \eta) = \widehat{\xi} \otimes v_\xi(\eta). \tag{4.1}$$

It is easy to see that v_ξ inherits the linearity of Φ, and so $v_\xi : H_1 \to H_2$ is a well-defined linear bijection.

Fix a non-zero ξ_0 in K_1. We shall next prove the existence of a non-zero constant $\tau = \tau(\xi)$ such that $v_\xi = \tau v_{\xi_0}$ for all $\xi \in K_1$. Namely, fix $\eta \in H_1$. If $\widehat{\xi}$ and $\widehat{\xi_0}$ are linearly independent, let us find, by the hypothesis on v_{ξ_0}, and the fact that $\dim(H_1) \geq 2$, another element η_1 such that $v_{\xi_0}(\eta)$ and $v_{\xi_0}(\eta_1)$ are linearly independent. Since the elements $\xi_0 \otimes \eta + \xi \otimes \eta$, $\xi_0 \otimes \eta_1 + \xi \otimes \eta_1$ and $\xi_0 \otimes (\eta + \eta_1) + \xi \otimes (\eta + \eta_1)$ have rank-one, the same holds for their images under Φ, that is, the elements

$$\begin{cases} \widehat{\xi_0} \otimes v_{\xi_0}(\eta) + \widehat{\xi} \otimes v_{\xi}(\eta), \\ \widehat{\xi_0} \otimes v_{\xi_0}(\eta_1) + \widehat{\xi} \otimes v_{\xi}(\eta_1), \\ \widehat{\xi_0} \otimes v_{\xi_0}(\eta+\eta_1) + \widehat{\xi} \otimes v_{\xi}(\eta+\eta_1), \end{cases}$$

must be rank-one elements. It necessarily follows that each one of the sets

$$\{v_{\xi_0}(\eta), v_{\xi}(\eta)\},\ \{v_{\xi_0}(\eta_1), v_{\xi}(\eta_1)\},\ \{v_{\xi_0}(\eta+\eta_1), v_{\xi}(\eta+\eta_1)\}$$

must be linearly dependent, which implies the existence of a non-zero constant α such that $v_\xi(\eta) = \alpha v_{\xi_0}(\eta)$ and $v_\xi(\eta_1) = \alpha v_{\xi_0}(\eta_1)$ for every η and η_1 linearly independent. It follows from the hypothesis on the dimension of H_1 that $v_\xi = \alpha v_{\xi_0}$. The constant α might depend on the element ξ, so we shall denote it by $\alpha(\xi) \in \mathbb{C}$.

Suppose now that $\widehat{\xi}$ and $\widehat{\xi_0}$ are linearly dependent. Find another ξ_1 such that $\widehat{\xi_1}$ and any of $\{\widehat{\xi_0}, \widehat{\xi}\}$ are linearly independent. It follows from the above arguments that $v_{\xi_1} = \alpha(\xi_1) v_{\xi_0}$, $v_\xi = \beta(\xi) v_{\xi_1}$ and $v_{\xi_0} = \beta(\xi_0) v_{\xi_1}$ for some non-zero scalars $\beta(\xi), \beta(\xi_0)$. It follows that $\beta(\xi_0)^{-1} = \alpha(\xi_1)$ and $v_\xi = \beta(\xi) v_{\xi_1} = \beta(\xi)\alpha(\xi_1) v_{\xi_0} = \beta(\xi)\beta(\xi_0)^{-1} v_{\xi_0}$. That is, v_ξ is a scalar multiple of v_{ξ_0}.

We have therefore shown the existence of a mapping $\tau : K_1 \to \mathbb{C}$ satisfying $v_\xi = \tau(\xi) v_{\xi_0}$ for all $\xi \in K_1$. Combining this fact with the conclusion in (4.1) we arrive at

$$\Phi(\xi \otimes \eta) = \widehat{\xi} \otimes \tau(\xi) v_{\xi_0}(\eta) = \overline{\tau(\xi)}\widehat{\xi} \otimes v_{\xi_0}(\eta),$$

for all $\xi \in K_1$, $\eta \in H_1$. Defining $u : K_1 \to K_1$ by $u(\xi) = \overline{\tau(\xi)}\widehat{\xi}$, we get a well-defined bijection which inherits the linearity from that of Φ. This concludes the proof of the first statement.

Let us briefly comment the second case.

Step 4. Let us assume now, by Steps 1 and 2, that for each $\xi \in K_1$, there exists a $\widehat{\eta} \in H_2$, depending on ξ, such that $\Phi(L_\xi) = R_{\widehat{\eta}}$. Consequently, for each $\eta \in H_1$ there exists a unique $u_\xi(\eta) \in K_2$ such that

$$\Phi(\xi \otimes \eta) = u_\xi(\eta) \otimes \widehat{\eta}. \tag{4.2}$$

Now the mapping $u_\xi : H_1 \to K_2$ is a conjugate-linear bijection –essentially because the mapping $(\xi, \eta) \mapsto \xi \otimes \eta$ is sesquilinear. By repeating or adapting the arguments in the first part of the proof, we find two conjugate-linear bijections $u : H_1 \to K_2$ and $v : K_1 \to H_2$ such that $\Phi(\xi \otimes \eta) = u(\eta) \otimes v(\xi)$, for all $x \in K_1, \eta \in H_1$. $\square$

It should be commented that there is certain margin to consider weaker hypotheses in the above theorem, but the current statement is enough for our purposes.

We have already gathered the required machinery to study bijections preserving triple transition probabilities between subsets of minimal tripotents of two type 1 Cartan factors.

Theorem 4.4. *Let $\Phi : \mathcal{U}_{min}(M) \to \mathcal{U}_{min}(N)$ be a bijective transformation preserving triple transition pseudo-probabilities (i.e., $TTP(\Phi(v), \Phi(e)) = \varphi_{\Phi(e)}(\Phi(v)) = \varphi_e(v) = TTP(v, e)$, for all e, v in $\mathcal{U}_{min}(M)$), where $M = B(H_1, K_1)$ and $N = B(H_2, K_2)$ are type 1 Cartan factors with $dim(H_j)$, $dim(K_j) \geq 2$. Then there exists a (unique) triple isomorphism $T : M \to N$ whose restriction to $\mathcal{U}_{min}(M)$ is Φ.*

Proof. By applying Theorem 2.3 we find a bijective linear mapping

$$T_0 : soc(B(H_1, K_1)) \to soc(B(H_2, K_2))$$

whose restriction to $\mathcal{U}(B(H_1, K_1))$ is Φ. Theorem 4.3 asserts that one of the next statements holds:

(a) There are bijective linear mappings $u : K_1 \to K_2$, and $v : H_1 \to H_2$ such that $\Phi(\xi \otimes \eta) = u(\xi) \otimes v(\eta)$ $(\xi \in K_1, \eta \in H_1)$.

(b) There are bijective conjugate-linear mappings $u : H_1 \to K_2$, $v : K_1 \to H_2$ such that $\Phi(\xi \otimes \eta) = u(\eta) \otimes v(\xi)$ $(\xi \in K_1,\ \eta \in H_1)$.

The hypothesis affirming that Φ maps minimal tripotents to minimal tripotents can be now applied to deduce that in any of the previous cases the mappings u and v are isometries. Therefore, u and v are surjective linear isometries in case (a) and surjective conjugate-linear isometries in case (b). So, by defining $T(x) = uxv^*$ and $T(x) = ux^*v^*$ $(x \in B(H_1, K_1))$ in cases (a) and (b), respectively, we get the desired triple isomorphism. □

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Note added in proofs. Months after this paper was accepted, the question every bijection between the sets of minimal tripotents in two atomic JBW*-triples automatically preserves orthogonality was possitively solved in [A.M. Peralta, Preservers of triple transition pseudo-probabilities in connection with orthogonality preservers and surjective isometries, *Results Math.* **78** (2023), no. 2, Paper No. 51, 23 pp.], so the main conclusion of this paper also holds for bijections preserving triple transition pseudo-probabilities between sets of minimal tripotents in two atomic JBW*-triples.

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