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The Mazur-Ulam property for a Banach space which satisfies a separation condition

By

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Abstract

After some preparations in section 1, we recall the three well known concepts: the Choquet boundary, the Šilov boundary, and the strong boundary points in section 2. We need to define them by avoiding the confusion which appears because of the variety of names of these concepts; they sometimes differs from authors to authors. We describe the relationship between the three concepts emphasizing the case where a function space strongly separates the points in the underlying space. We study C-rich spaces, lush spaces, and extremely C-regular spaces concerning with the Mazur-Ulam property in section 3. We show that a uniform algebra and the uniform closure of the real part of a uniform algebra with the supremum norm are C-rich spaces, hence lush spaces. We prove that a uniformly closed subalgebra of the algebra of all complex-valued continuous functions on a locally compact Hausdorff space which vanish at infinity is extremely C-regular provided that it separates the points of the underlying space and has no common zeros. We exhibit a space of harmonic functions which has the Mazur-Ulam property (Corollary 3.8). The main concern in sections 4 through 6 is the Mazur-Ulam property. We exhibit a sufficient condition on a Banach space which has the Mazur-Ulam property and the complex Mazur-Ulam property (Propositions 4.11 and 4.12). In sections 5 and 6 we consider a Banach space with a separation condition $(*)$ (Definition 5.1). We prove that a real Banach space satisfying $(*)$ has the Mazur-Ulam property (Theorem 6.1), and a complex Banach space satisfying $(*)$ has the complex Mazur-Ulam property (Theorem 6.3). Applying these theorems and the results in the previous sections we prove that an extremely C-regular real (resp. complex) linear subspace has the (resp. complex) Mazur-Ulam property (Corollary 6.2 (resp. 6.4)) in section 6. As a consequence we prove that any closed subalgebra of the algebra of all complex-valued continuous functions defined on a locally compact Hausdorff space has the complex Mazur-Ulam property (Corollary 6.5).

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Contents

§ 1. Introduction

Tingley's problem asks whether every surjective isometry between the unit spheres of Banach spaces is linearly extended to a surjective isometry between the whole spaces.

Tingley [53] raised this problem in 1987. First solution of Tingley's problem seems to be due to Wang [54], who dealt with the space of all $\mathbb{K}$-valued ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) continuous functions which vanish at infinity on a locally compact Hausdorff space Y (cf. [55]). A considerable number of interesting results have shown that Tingley's problem has an affirmative answer in concrete spaces, and no counterexample is known. According to [58, p.730], Ding was the first to consider Tingley's problem between different type of spaces [20]. In fact, Ding [21, Corollary 2] proved that the real Banach space of all null sequences of real numbers satisfies what we now call the Mazur-Ulam property. Liu [32] also had an early contribution to the Tingley's problem on different types of spaces. Later, Cheng and Dong [15] formally introduced the concept of the Mazur-Ulam property.

Definition 1.1. A real Banach space B has the *Mazur-Ulam property* if for any real Banach space B', every surjective isometry from the unit sphere of B onto the unit sphere of B' admits an extension to a surjective real-linear isometry from B onto B'.

Tan [47, 48, 49] showed that the space $L^p(\mathbb{R})$ for σ-finite positive measure space has the Mazur-Ulam property. Boyko, Kadets, Martín and Werner introduced C-richness [11, Definition 2.3] and lushness [11, Definition 2.1] for subspaces of continuous functions and proved that a C-rich subspace is lush [11, Theorem 2.4]. Tan, Huang and Liu [50] introduced the notion of local GL (generalized lush) spaces and proved that every local GL space has the Mazur-Ulam property.

Tanaka [52] opened another direction in the study of Tingley's problem by exhibiting a positive solution for the Banach algebra of complex matrices. Mori and Ozawa [35] proved that the Mazur-Ulam property for unital C^*-algebras and real von Neumann algebras. Cueto-Avellaneda and Peralta [18] proved that the complex (resp. real) Banach space of all continuous maps taking values in a complex (resp. real) Hilbert space has the Mazur-Ulam property (cf. [19]). The results by Becerra-Guerrero, Cueto-Avellaneda, Fernández-Polo and Peralta [6] and Kalenda and Peralta [31] proved that any JBW*-triple has the Mazur-Ulam property. Peralta and Švarc [40] extends the results of Mori and Ozawa [35] for unital JB*-algebras.

The Mazur-Ulam property for a Banach space of dimension 2 has remained unsolved for many years. The final solution was exhibited by the remarkable outstanding advance of Banakh [4] who proved that any Banach space of dimension 2 has the Mazur-Ulam property. The problem on a Banach space of a finite dimension greater than 2 seems to be still open. The study of the Mazur-Ulam property is nowadays a challenging subject of study (cf. [13, 56]). Jiménez-Vargas, Morales-Compoy, Peralta and Ramírez [28, Theorems 3.8, 3.9] probably provided the first example of complex Banach spaces having the complex Mazur-Ulam property (cf. [38]). Hatori [26] formally introduced the concept of the complex Mazur-Ulam property.

Definition 1.2. A complex Banach space B is said to have the *complex Mazur-Ulam property*, emphasizing the term 'complex', if for any complex Banach space B', every surjective isometry from the unit sphere of B onto the unit sphere of B' admits an extension to a surjective real-linear isometry from B onto B'

Note that a complex Banach space has the complex Mazur-Ulam property provided that it has the Mazur-Ulam property as a real Banach space since a complex Banach space is a real Banach space simultaneously.

The complex Mazur-Ulam property for uniform algebras was proved in [26]. The existence of unit in a uniform algebra is a key point for the proof of the property. The complex Mazur-Ulam property for a uniformly closed algebra on a locally compact Hausdorff space is a problem in [26]. Cueto-Avellaneda, Hirota, Miura and Peralta [17] recently showed that each surjective isometry between the unit spheres of two uniformly closed algebras on locally compact Hausdorff spaces which separates the points without common zeros admits an extension to a surjective real linear isometry between these algebras. Very recently, Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [14] proved the complex Mazur-Ulam property for a commutative JB*-triple. Both results concerns the spaces of continuous functions without constants. Peralta [39] probably gives the first example of an infinite dimensional non-commutative C^*-algebra containing no unitaries and with the Mazur-Ulam property. Note that the Mazur-Ulam property, or even the complex Mazur-Ulam property for general non-unital C^*-algebras seems to be still missing.

In this paper we further study the problem on the complex Mazur-Ulam property. We introduce a separation condition which is named $(*)$ for a Banach space in section 5. We prove that a real (resp. complex) Banach space which satisfies the condition $(*)$ has the (resp. complex) Mazur-Ulam property. An extremely C-regular subspace satisfies the condition $(*)$. It was introduced by Fleming and Jamison [24, Definition 2.3.9], which is a generalization of an extremely regular subspace coined by Cengiz [16]. We prove that an extremely C-regular complex linear subspace has the complex Mazur-Ulam property (Corollary 6.4). As a consequence the complex Mazur-Ulam property for a closed subalgebra of the algebra of all complex-valued continuous functions defined on a locally compact Hausdorff space is established (Corollary 6.5).

We recall the notions of the Choquet boundary, the strong boundary, and the strong separation of the points in the underlying space in section 3. We are aware that many of results in section 3 are a part of folklore, but for the sake of self-contained exposition and for the convenience of the readers who might not be familiar with these concepts for function spaces without constants, we include as many complete proofs of results as possible.

For a real or complex Banach space B the unit sphere $\{a \in B : \|a\| = 1\}$ of B is denoted by $S(B)$ and the closed unit ball $\{a \in B : \|a\| \leq 1\}$ by $\mathrm{Ball}(B)$. The set of all maximal convex subsets of $S(B)$ is denoted by $\mathfrak{F}_B$. We denote by $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$) the set of all real (resp. complex) numbers. We denote the open unit disk in $\mathbb{C}$ by D, the closed unit disk by $\bar{D}$, and $\mathbb{T} = \{z \in \mathbb{K} : |z| = 1\}$. Throughout the paper Y denotes a locally compact Hausdorff space and X a compact Hausdorff space. The space of all $\mathbb{K}$-valued continuous functions on Y which vanish at infinity is denoted by $C_0(Y, \mathbb{K})$. If Y is compact, then we simply denote $C(Y, \mathbb{K})$ instead of $C_0(Y, \mathbb{K})$. The supremum norm on a subset W of Y is denoted by $\|\cdot\|_{\infty(W)}$ or $\|\cdot\|_\infty$. For a function $f \in C_0(Y, \mathbb{K})$ and $S \subset Y$, we denote the restriction of f on S by $f|S$. For $A \subset C_0(Y, \mathbb{K})$ and $S \subset Y$, we denote $A|S = \{f|S : f \in A\}$.

§ 2. Strong boundary points, Choquet boundary points and Šilov boundary points

§ 2.1. Function spaces which strongly separate the points in the underlying spaces

The definition of “strongly separate the points” in this paper is due to that of Araujo and Font [2]. Some authors use this term for a different notion (cf. Stout [46, p.36] and Miura [33, p.779])

Definition 2.1. Let E be a complex (resp. real) linear subspace of $C_0(Y, \mathbb{C})$ (resp. $C_0(Y, \mathbb{R})$). We say that E (resp. strongly) separates the points of Y, if for every pair $x, y \in Y$ with $x \neq y$ there exists $f \in E$ such that $f(x) \neq f(y)$ (resp. $|f(x)| \neq |f(y)|$). We say that E has no common zeros if for every $x \in Y$ there exists $f \in E$ such that $f(x) \neq 0$.

The following example exhibits an important space of functions which separates, but does not strongly separate the points of the underlying space.

Example 2.2. The space $C_0(Y, \mathbb{R})$ and $C_0(Y, \mathbb{C})$ strongly separate points of Y and have no common zeros by the Urysohn's lemma. Let B be a complex Banach space. For $a \in B$ we denotes $\hat{a} : \mathrm{Ball}(B^*) \setminus \{0\} \to \mathbb{C}$ by $\hat{a}(q) = q(a)$ for $q \in \mathrm{Ball}(B^*) \setminus \{0\}$. Then $\widehat{B} = \{\hat{a} : a \in B\}$ is a uniformly closed subspace of $C_0(\mathrm{Ball}(B^*) \setminus \{0\}, \mathbb{C})$ which separates the points of $\mathrm{Ball}(B^*) \setminus \{0\}$. On the other hand $|\hat{a}(p)| = |\hat{a}(-p)|$ for any $a \in B$ and $p \in \mathrm{Ball}(B^*) \setminus \{0\}$, that is, $\widehat{B}$ does not *strongly* separate the points of the underlying space $\mathrm{Ball}(B^*) \setminus \{0\}$. The situation is similar for a real Banach space.

In some cases, both separation conditions are equivalent. In fact, we have the following.

Proposition 2.3. *Supposes that E is a $\mathbb{K}$-linear subspace of $C(X,\mathbb{K})$, such that $1 \in E$, where X is a compact Hausdorff space. If E separates the points of X, then E strongly separates the points of X. Suppose that A is a subalgebra of $C_0(Y,\mathbb{K})$ which separates the points of Y. Then A strongly separates the points of Y.*

Proof. We prove the first assertion. Suppose that $x, y \in X$ with $x \neq y$. Then there exists $f \in E$ such that $f(x) \neq f(y)$. If $f(x) = 0$, then $|f(x)| \neq |f(y)|$. If $f(x) \neq 0$ then $|f|$ or $|f + f(x)|$ separates x and y, where $f + f(x) \in E$ since E contains constants.

Next suppose that A is a subalgebra of $C_0(Y,\mathbb{K})$. Suppose that $x, y \in Y$ with $x \neq y$. Then there exists $f \in A$ such that $f(x) \neq f(y)$. If $f(y) = 0$, then $f(x) \neq 0$ and $|f(x)| \neq |f(y)|$ follows. We may assume that $f(y) = 1$. Suppose that $|f(x)| = |f(y)|$. Then there exists a complex number λ with unit modulus such that $f(y) = \lambda f(x)$. As $\lambda \neq 1$, we infer that $|\lambda^2 + \lambda| < 2$. Hence we have

$$|f(x) + f(x)^2| = |\lambda + \lambda^2| < 2 = |f(y) + f(y)^2|.$$

Hence $f + f^2 \in A$ strongly separates x and y. □

Definition 2.4. We say that A is a *uniform algebra* on a compact Hausdorff space X if A is a closed subalgebra of $C(X,\mathbb{C})$ which contains constants and separates the points of X.

A uniform algebra on X not only separates the points of X but also strongly separates the points of X since A contains constants. For the theory of uniform algebras, see [12, 25, 46]. A uniform algebra is called a function algebra in [12].

§ 2.2. Strong boundary points

The author kindly points out that the definition of a strong boundary point sometimes differs from authors to authors. Stout in [46, Definition 7.6] says that $x_0 \in Y$ is a strong boundary point for a certain *subalgebra* of $C_0(Y,\mathbb{C})$ with some condition if for each open neighborhood U of x_0 there exists $f \in A$ such that

$$1 = f(x_0) = \|f\|_\infty \text{ and } |f(x)| < 1 \text{ for all } x \in Y \setminus U.$$

Araujo and Font [3, p. 80] follow this definition not only for a subalgebra but also for a linear subspace. Rao and Roy [43, Definition 8] recall the notion of a strong boundary point from [46] for a linear subspace. Note that a weak peak point (peak point in the weak sense in [12]) for a uniform algebra (function algebra in [12]) is also referred to as a strong boundary point in [12, p.96]. In a book of Gamelin [25] a weak peak set (resp. point) is referred to as a p-set (resp. point) or a generalized peak set (resp. point).

The definition of a strong boundary point for a $\mathbb{K}$-linear subspace E of $C_0(Y, \mathbb{K})$ due to Fleming and Jamison [24, Definition 2.3.9]: a point $x_0 \in Y$ is a strong boundary point for E if for each open neighborhood U of x_0, and each $\varepsilon > 0$, there exists $f \in E$ such that $1 = f(x_0) = \|f\|$, and $|f(x)| < \varepsilon$ for all $x \in Y \setminus U$, is stronger than that given in Definition 2.5. Note that the formula in the fourth line of [24, Definition 2.3.9] reads as $1 = f(s_0) = \|f\|$: not s but s_0.

In this paper, we adapt the definition of a strong boundary point for a subspaces of $C_0(Y, \mathbb{K})$ following Stout.

Definition 2.5. Let E be a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. We say that a point $x_0 \in Y$ is a *strong boundary point* for E if for each open neighborhood U of x_0 there exists $f \in E$ such that

$$1 = f(x_0) = \|f\|_\infty, \text{ and } |f(x)| < 1 \text{ for all } x \in Y \setminus U.$$

We say that a closed subset K of Y is a *peak set* for E if there exists a function $f \in E$ such that

$$f = 1 \text{ on } K \text{ and } |f| < 1 \text{ on } Y \setminus K.$$

We say that any such f *peaks* on K and f is a *peaking function* for K. A *weak peak set* (or *peak set in the weak sense*) for E is a non-empty intersection of peak sets for E. A point $y_0 \in Y$ is called a peak point for E if $\{y_0\}$ is a peak set for E. A point $y_0 \in Y$ is called a *weak peak point* (or *peak point in the weak sense*) for E if $\{y_0\}$ is a weak peak set for E.

The following exhibits simple examples which show that the condition of a *strong boundary point in the sense of Fleming and Jamison* is strictly stronger than that in Definition 2.5.

Example 2.6. Let $f_0 : (0, 1] \to \mathbb{R}$ be defined as

$$f_0(t) = \begin{cases} t, & 0 < t \leq \frac{1}{2}, \\ -\frac{3}{2}t + \frac{5}{4}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Let $g : (0, 1] \to \mathbb{R}$ be

$$g_0(t) = \begin{cases} 0, & 0 < t \leq \frac{1}{2} \\ -(t - \frac{1}{2})(t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Put $E = \{\lambda f_0 + \mu g_0 : \lambda, \mu \in \mathbb{K}\}$. Then E is a $\mathbb{K}$-linear subspace of $C_0((0, 1], \mathbb{K})$ which strongly separates the points of $(0, 1]$. The point $\frac{1}{2}$ is a strong boundary point (in the sense of Definition 2.5) for the space E while it does not satisfy the condition due to Fleming and Jamison [24, Definition 2.3.9].

Example 2.7. Let $E = \{f\colon f(t) = at+b, \exists a, b \in \mathbb{K}\} \subset C([0,1], \mathbb{K})$. Then 0 and 1 are strong boundary points in the sense of Definition 2.5. On the other hand, there is no strong boundary points in the sense of Fleming and Jamison.

In some situation, both are equivalent.

Proposition 2.8. *Let E be a $\mathbb{K}$-subalgebra of $C_0(Y, \mathbb{K})$. Suppose that $x \in Y$. Then x is a strong boundary point in the sense of Definition 2.5 if and only if it is a strong boundary point in the sense of Fleming and Jamison.*

Proof. Suppose that x is a strong boundary point in the sense of Definition 2.5. Suppose that U is an open neighborhood of x_0 and $\varepsilon > 0$. Then there exists a function $f \in E$ such that $1 = f(x_0) = \|f\|_\infty$, and $|f(x)| < 1$ for all $x \in Y \setminus U$. As f vanishes at infinity, there exists an $0 < M < 10$ such that $|f(x)| < M$ for all $x \in Y \setminus U$. Choose a large natural number n so that $M^n < \varepsilon$. Then $f^n \in E$ and $1 = f^n(x_0) = \|f^n\|_\infty$, and $|f^n(x)| < \varepsilon$ for all $x \in Y \setminus U$, that is, x_0 is a strong boundary point in the sense of Fleming and Jamison.

It is straightforward that a strong boundary point in the sense of Fleming and Jamison is one in the sense of Defintion 2.5 $\square$

Recall that the peripheral range $\mathrm{Ran}_\pi(f)$ of $f \in C_0(Y, \mathbb{C})$ is the set $\{z \in f(Y) : |z| = \|f\|_\infty\}$. A function $f \in E$ is a peaking function for a closed subsest of Y if and only if $\mathrm{Ran}_\pi(f) = \{1\}$.

Proposition 2.9. *Let E be a uniformly closed $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Let $x_0 \in Y$. Then the following are equivalent.*

(i) *The point x_0 is a strong boundary point for E,*

(ii) *for every open neighborhood U of x_0, there exists $f \in E$ such that*

$$f(x_0) = 1 = \|f\|_\infty \text{ and } |f(x)| < 1 \text{ for every } x \in Y \setminus U$$

and $\mathrm{Ran}_\pi(f) = \{1\}$,

(iii) *the point x_0 is a weak peak point for E*

Proof. We prove (i) implies (ii). Suppose (i) holds. Let U be an open neighborhood of x_0. Then there exists $f \in E$ with $f(x_0) = 1 = \|f\|_\infty$ and $|f| < 1$ on $Y \setminus U$. Put $U_n = \{y \in Y : |f(y) - 1| < 1/n\}$ for each positive integer n. Then $U \cap U_n$ is an open neighborhood of x_0. As $f^{-1}(1) = \bigcap_{n=1}^\infty U_n$, $f^{-1}(1)$ is a G_δ set, which is an intersection of a countable open sets. By (i) there exists a function $f_n \in E$ such that $f_n(x_0) = 1 = \|f_n\|_\infty$, $|f_n| < 1$ on $Y \setminus (U \cap U_n)$. Put $g = \sum_{n=1}^{\infty}(f_n/2^n)$. Then $g \in E$ since E is closed in $C_0(Y, \mathbb{K})$. For every $x \in Y \setminus U$, we have $x \in Y \setminus (U \cap U_1)$, hence $|f_1(x)| < 1$, so $|g(x)| < 1$. Suppose that $|g(y)| = 1$ for some $y \in Y$. Since $1 = |g(y)| \le \sum_{n=1}^{\infty} |f_n(y)|/2^n \le 1$, we infer that $|f_n(y)| = 1$ for every n. We conclude that $x \in \bigcap_{n=1}^{\infty} U_n$. Put $h = (f+g)/2$. Then $h(x_0) = 1 = \|h\|_\infty$ and $|h| < 1$ on $Y \setminus U$. Suppose that $|h(z)| = 1$ for $z \in Y$. Since

$$1 = |h(z)| = |(f(z) + g(z)|/2 \le (|f(z)| + |g(z)|)/2 \le 1 \tag{2.1}$$

and $\|f\|_\infty = \|g\|_\infty = 1$, we infer that $|g(z)| = 1$. Hence $z \in \bigcap_{n=1}^{\infty} U_n$. Since $f^{-1}(1) = \bigcap_{n=1}^{\infty} U_n$ we infer that $f(z) = 1$. By (2.1) we get $g(z) = 1$. Therefore $h(z) = 1$. Thus $\mathrm{Ran}_\pi(h) = \{1\}$.

To prove (ii) implies (iii), assume (ii). Let $y \in Y \setminus \{x_0\}$. There is an open neighborhood U_y of x_0 such that $y \notin U_y$. By (ii) there is a function $f_y \in E$ such that $f_y(x_0) = 1 = \|f_y\|_\infty$, $|f_y| < 1$ on the closed set $Y \setminus U_y$, and $\mathrm{Ran}_\pi(f_y) = \{1\}$. Then $f_y^{-1}(1)$ is a peak set which contains x_0 and $y \notin f_y^{-1}(1)$. Hence $\bigcap_{y \in Y \setminus \{x_0\}} f_y^{-1}(1) = \{x_0\}$ is a weak peak set, so x_0 is a weak peak point for E.

To prove (iii) implies (i), assume (iii): there exists a family of peak sets $\{K_\alpha\}$ such that $\bigcap_\alpha K_\alpha = \{x_0\}$. Suppose that U is an open neighborhood of $\{x_0\}$. Then $\bigcap_\alpha K_\alpha \subset U$. By considering the one point compactification, if necessary, we infer that there is a finite number of $\alpha_1, \ldots, \alpha_n$ such that $\bigcap_{k=1}^{n} K_{\alpha_k} \subset U$. Let $f_j \in E$ be a function which peaks on K_{α_j}. Then $f = \frac{1}{n}\sum_{j=1}^{n} f_j$ is in E and peaks on $\bigcap_{j=1}^{n} K_{\alpha_j}$. Hence $f(x_0) = 1 = \|f\|_\infty$. Since $Y \setminus U \subset Y \setminus \bigcap_{j=1}^{n} K_{\alpha_j}$, we have $|f| < 1$ on $Y \setminus U$. $\square$

Note that if E is a $\mathbb{K}$-linear subspace of $C(X, \mathbb{K})$ for a compact Hausdorff space X and $1 \in E$, which needs not to be uniformly closed, then (i) of Proposition 2.9 implies (ii) of Proposition 2.9. In fact, if $x_0 \in X$ is a strong boundary point and U is an open neighborhood and $f \in E$ satisfies $1 = f(x_0) = \|f\|_\infty$. Then $(f+1)/2$ satisfies the condition (ii). On the other hand if E is not uniformly closed nor $1 \notin E$, then (i) needs not imply (ii).

Example 2.10. Let $Y = \mathbb{T}$. For any positive integer n, let f_n be a continuous function on Y such that $f_n(z) = z$ for any $z \in Y$ with $|1 - z| \le 1/n$ and $|f_n(z)| < 1$ for any $z \in Y$ with $|1 - z| > 1/n$. Let E be a linear subspace generated by $\{1\} \cup \{f_n\}_{n=1}^{\infty}$. Note that E is a $\mathbb{K}$-linear subspace of $C(Y, \mathbb{K})$ which is not uniformly closed. Then 1 is a strong boundary point for E, while 1 does not satisfy the condition (ii) of Proposition 2.9

If the corresponding space E is an algebra, we have the following.

Corollary 2.11. *Let A be a closed subalgebra of $C_0(Y, \mathbb{K})$. Let $x_0 \in Y$. Then the following are equivalent.*

(i) *The point x_0 is a strong boundary point for A,*

(ii) *For every open neighborhood U of x_0, and $\varepsilon > 0$, there exists $f \in A$ such that*

$$f(x_0) = 1 = \|f\|_\infty \text{ and } |f(x)| < \varepsilon \text{ for every } x \in Y \setminus U,$$

(iii) *For every open neighborhood U of x_0, and $\varepsilon > 0$, there exists $f \in A$ such that*

$$f(x_0) = 1 = \|f\|_\infty \text{ and } |f(x)| < \varepsilon \text{ for every } x \in Y \setminus U$$

and $\mathrm{Ran}_\pi(f) = \{1\}$,

(iv) *The point x_0 is a weak peak point for A*

Proof. By Proposition 2.8 we have that (i) and (ii) are equivalent. The rest of the proof is easily followed by Proposition 2.9 □

To study subspace E in $C_0(Y, \mathbb{K})$ it is usefull to consider the addition of constant. We add constant functions in a subspace of $C_0(Y, \mathbb{K})$.

Definition 2.12. For a locally compact Hausdorff space Y, we denote by $Y_\infty = Y \cup \{\infty\}$ the one-point-compactification of Y. Let E be a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. For $f \in E$ we denote by $\dot{f}$ the unique extension of f on Y,

$$\dot{f}(y) = \begin{cases} f(y), & y \in Y \\ 0, & y = \infty. \end{cases}$$

Then $\dot{f}$ is continuous on Y_∞. We denote

$$\dot{E} + \mathbb{K} = \{F \in C(Y_\infty, \mathbb{K}) : F = \dot{f} + c, f \in E, c \in \mathbb{K}\}.$$

Then $\dot{E} + \mathbb{K}$ is a $\mathbb{K}$-linear subspace of $C(Y_\infty, \mathbb{K})$ which contains constants.

We may sometimes suppose that E is a closed subspace of $\dot{E} + \mathbb{K}$ without a confusion. It is easy to see that $\dot{E} + \mathbb{K}$ separates the points of Y_∞ and has no common zeros provided that E separates the points of Y and it has no common zeros. By a routine exercise we have that $\dot{E} + \mathbb{K}$ is closed in $C(Y_\infty, \mathbb{K})$ if E is closed in $C_0(Y, \mathbb{K})$. It is also a routine exercise to see that for $F \in \dot{E} + \mathbb{K}$, $F = \dot{f}$ for a $f \in E$ if and only if $F(\infty) = 0$.

Lemma 2.13. *Suppose that E is a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. If $x_0 \in Y$ is a strong boundary point for E, then x_0 is a strong boundary point for $\dot{E} + \mathbb{K}$.*

Proof. The proof is trivial and is omitted. □

The converse of the above lemma does not hold in general.

Example 2.14. Let E be the same space defined in Example 2.6. Then 1 is not a strong boundary point for E. On the other hand, 1 is a strong boundary point for $\dot{E}+\mathbb{K}$.

The converse of Lemma 2.13 holds for a closed subalgebra of $C_0(Y,\mathbb{K})$.

Proposition 2.15. *Suppose that A is a closed subalgebra of $C_0(Y,\mathbb{K})$. A point $x_0 \in Y$ is a strong boundary point for A if and only if it is a strong boundary point for $\dot{A}+\mathbb{K}$.*

Proof. Suppose that $x_0 \in Y$ is a strong boundary point for $\dot{A}+\mathbb{K}$. For any open neighborhood U of x_0 in Y, U may be considered as an open neighborhood of x_0 in Y_∞. Hence there exists a function $F \in \dot{A}+\mathbb{K}$ such that $F(x_0) = 1 = \|F\|_\infty$ and $|F| < 1$ on $Y_\infty \setminus U$. We note that $\infty \notin U$. Hence we may suppose that $|F(\infty)| < 1$. Put $\pi : \{z \in \mathbb{C} : |z| \le 1\} \to \{z \in \mathbb{C} : |z| \le 1\}$ by

$$\pi(z) = \frac{1-\overline{F(\infty)}}{1-F(\infty)} \cdot \frac{z-F(\infty)}{1-\overline{F(\infty)}z}, \quad z \in \{z \in \mathbb{C} : |z| \le 1\}$$

if $\mathbb{K} = \mathbb{C}$. We infer that $\pi(F(\infty)) = 0$ and $\pi(1) = 1$. As π is uniformly approximated by analytic polynomials (in fact, $\pi(rz)$ for any $0 < r < 1$ is uniformly approximated by the Taylor expansion on $\{z \in \mathbb{C} : |z| \le 1\}$, and $\pi(rz) \to \pi(z)$ uniformly on $\{z \in \mathbb{C} : |z| \le 1\}$) and as $\dot{A}+\mathbb{C}$ is uniformly closed algebra, we have $\pi \circ F \in \dot{A}+\mathbb{C}$. If $\mathbb{K} = \mathbb{R}$, then put $\pi : [-1,1] \to [-1,1]$ by

$$\pi(t) = \begin{cases} \frac{1}{1+F(\infty)}t - \frac{F(\infty)}{1+F(\infty)}, & -1 \le t \le F(\infty) \\ \frac{1}{1-F(\infty)}t - \frac{F(\infty)}{1-F(\infty)}, & F(\infty) \le t \le 1. \end{cases}$$

Then $\pi(F(\infty)) = 0$ and $\pi(1) = 1$. By the Weierstrass approximation theorem π is uniformly approximated by polynomials on $[-1,1]$. Hence $\pi \circ F \in \dot{A}+\mathbb{R}$. In any case we have that $\pi \circ F \in \dot{A}+\mathbb{K}$, $\pi \circ F(\infty) = 0$. Thus there exists $f \in A$ such that $F = \dot{f}$. Then we have that $\|f\|_\infty = \|F\|_\infty = 1$ and $f(x) = \pi \circ F(x) = \pi(1) = 1$. As $|\pi(z)| < 1$ for any $|z| < 1$, we have $|\pi \circ F| < 1$ on $Y_\infty \setminus U$. Therefore $|f| < 1$ on $Y \setminus U$. It follows that x_0 is a strong boundary point for A.

The converse statement is just Lemma 2.13 □

§ 2.3. The Choquet boundary and the Šilov boundary

The Choquet boundary was first mentioned by that name in a paper of Bishop and de Leeuw [7, p.306]. Definitions of Choquet boundary differ from case to case, although they are equivalent in the possible situation where the definitions can be applied. A definition of the Choquet boundary for $\mathbb{K}$-linear subspace of $C(X, \mathbb{K})$, for a compact Hausdorff space X, which contains constant functions is described by Phelps in [37, Section 6]. Let E be a $\mathbb{K}$-linear subspace of $C(X, \mathbb{K})$. Suppose that $1 \in E$. The state space of E is $K(E) = \{\phi \in E^* : \phi(1) = 1 = \|\phi\|\}$ (cf. [37, p.27]). The Choquet boundary in [37] is defined as the set of all $x \in Y$ such that the point evaluation τ_x is in $\mathrm{ext}(K(E))$, the set of all extreme points of the state space. Since it is easy to see that $\mathbb{T}\,\mathrm{ext}(K(E)) = \mathrm{ext}(\mathrm{Ball}(E^*))$, the two definitions of the Choquet boundary (Definition 2.16 and Definition in [37, p.29]) are equivalent provided that Y is compact and $1 \in E$. In section 8 of [37] Phelps describs an equivalent form of the Choquet boundary for a uniform algebra in terms of strong boundary points by referring a theorem of Bishop and de Leeuw. Browder [12, Section 2-2] exhibits a definition of the Choquet boundary for a uniform algebra in terms of mesures, which is also equivalent to that exhibited in Definition 2.16 in the case of a uniform algebra. Rao and Roy [43, p.176] defines the Choquet boundary for a uniformly closed complex linear subspace of $C(X, \mathbb{C})$ which separates the points of a compact Hausdorff space X, in a similar way as our Definition 2.16.

We are aware of the fact that some results in this subsection are a part of folklore, but for the sake of a self-contained exposition we have included as many proofs as possible of all the results stated.

2.3.1. Definition of the Choquet boundary.

We recall the notion of the *Choquet boundary* for a $\mathbb{K}$-subspace of $C_0(Y, \mathbb{K})$ from [24, Definition 2.3.7], which was stated by Novinger [36, p.274]. See also [44].

For a $\mathbb{K}$-linear subspace E of $C_0(Y, \mathbb{K})$ and $x \in Y$, τ_x denotes the point evaluation at x, that is, $\tau_x : E \to \mathbb{K}$ such that $\tau_x(f) = f(x)$ for $f \in E$. If E contains constants, then $\|\tau_x\| = 1$. In general, $\tau_x \in \mathrm{Ball}(E^*)$ and $\|\tau_x\|$ needs not be 1. For example, put

$$E = \{f \in P(\bar{D}) : f(0) = 0\}|(\bar{D} \setminus \{0\}),$$

where $P(\bar{D})$ is the disk algebra on the closed unit disk $\bar{D}$ in the complex plane. Let $x \in D$, the open disk. Then $\|\tau_x\| = |x|$ by the Schwarz lemma.

We define the Choquet boundary for a $\mathbb{K}$-linear subspace which needs not to be closed, not to separate the points of the underlying space, may have common zeros.

Definition 2.16. Suppose that E is a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. The Choquet boundary for E denoted by $\mathrm{Ch}(E)$ is the set of all $x \in Y$ such that the point evaluation τ_x is in $\mathrm{ext}(\mathrm{Ball}((E, \|\cdot\|\infty)^*))$, the set of all extreme points of $\mathrm{Ball}((E, \|\cdot\|_\infty)^*)$, where $(E, \|\cdot\|_\infty)$ denotes the normed linear space E with the uniform norm $\|\cdot\|_\infty$.

Note that even if E is a Banach space with some norm other than the uniform one, we consider the space $(E, \|\cdot\|_\infty)$ to define the Choquet boundary.

2.3.2. The representing measures and the Arens-Kelley theorem revisited

It is crucial for the foregoing discussion on the Choquet boundaries that measure theoretic arguments should be concerned. We begin by recalling the representing measures for bounded linear functionals. See [37].

Definition 2.17. Let E be a $\mathbb{K}$-linear subspace of $C_0(Y,\mathbb{K})$. Suppose that $\phi \in E^*$. We say that a complex regular Borel measure m on Y is a *representing measure* for ϕ if

$$\phi(f) = \int f dm, \quad f \in E$$

and $\|m\| = \|\phi\|$, where $\|m\| = |m|(Y)$ is the total valuation of m.

Existence of a representing measure for any $\phi \in E^*$ is as follows: let $\phi \in E^*$. By the Hahn-Banach extension theorem there is $\Phi \in C_0(Y,\mathbb{K})^*$ which extends ϕ and $\|\phi\| = \|\Phi\|$. Then by the Riesz-Kakutani theorem, there exists a complex regular Borel measure m such that $\Phi(f) = \int f dm$ for every $f \in C_0(Y,\mathbb{K})$ and $\|m\| = \|\Phi\|$. Hence m is a representing measure for ϕ. Recall that the *support* $\mathrm{supp}(m)$ of a complex regular Borel measure m is the set

$$\{x \in Y : |m|(G) > 0 \text{ for every open neighborhood } G \text{ of } x\},$$

where $|m|$ is the total valuation measure of m.

Versions of the Arens-Kelley theorem, which characterizes extreme points of the unit ball in the dual space of a subspace of $C_0(Y,\mathbb{K})$, have been obtained by a variety of authors. The Arens-Kelley theorem and the following corollary are well known, however for the sake of a self-contained exposition and for the convenience of the readers we include complete proofs.

For $x \in Y$ we denote the *point mass* at x by D_x: D_x is a complex regular Borel measure on Y such that $D_x(\{x\}) = 1 = \|D_x\|$.

The Arens-Kelley theorem . *Suppose that $\phi \in \mathrm{ext}\,\mathrm{Ball}(C_0(Y,\mathbb{K})^*)$. Then there exists a unique $x \in Y$ and $\lambda \in \mathbb{T}$ such that $\phi = \lambda\tau_x$. The representing measure for ϕ is only λD_x. Conversely, $\lambda\tau_x$ is an extreme point of $\mathrm{Ball}(C_0(Y,\mathbb{K})^*)$ for every $x \in Y$ and $\lambda \in \mathbb{T}$.*

Proof. Let $\phi \in \mathrm{ext}\,\mathrm{Ball}(C_0(Y,\mathbb{K})^*)$ and m a representing measure for ϕ. Let $y \in \mathrm{supp}(m)$ arbitrary. Suppose that $|m|(U) = 1$ for every open neighborhood U of y. By the regularity of m we infer that $|m|(\{y\}) = 1 = \|m\|$, hence $m = \lambda D_y$ for unimodular complex number λ.

Suppose that there exists an open neighborhood U_0 of y with $|m|(U_0) < 1$. As $y \in \mathrm{supp}(m)$, $0 < |m|(U_0)$ holds. By the definition of $|m|$ we have

$$|m|(U_0) = \sup\{\sum |m(G_j)| : G_j \text{ is a Borel set}, \cup_j G_j = U_0,\, G_i \cap G_j = \emptyset \text{ for } i \neq j\},$$

hence there is G_j such that $m(G_j) \neq 0$. By the (inner) regularity of m, there exists a compact subset $L \subset G_j \subset U_0$ with $0 < |m(L)|$. Also there exists an open set V with $L \subset V \subset U_0$ such that $|m|(V \setminus L) < |m(L)|/2$. By the Urysohn's lemma there exists $f_0 \in C_0(Y, \mathbb{R})$ such that $0 \leq f_0 \leq 1$, $f_0 = 1$ on L, and $f_0 = 0$ on $Y \setminus V$. Then we have

$$0 < |m(L)| \leq |m|(L) \leq |m|(V) \leq |m|(U_0) < 1.$$

Hence we also have $0 < |m|(Y \setminus V) < 1$. Put

$$\phi_1(g) = \frac{1}{|m|(V)} \int_V g dm, \quad g \in C_0(Y, \mathbb{K})$$

and

$$\phi_2(g) = \frac{1}{|m|(Y \setminus V)} \int_{Y \setminus V} g dm, \quad g \in C_0(Y, \mathbb{K}).$$

As m is a representing measure for ϕ we have $\phi = |m|(V)\phi_1 + |m|(Y \setminus V)\phi_2$, where $|m|(V) + |m|(Y \setminus V) = \|m\| = 1$. As $\phi \in \operatorname{ext} \operatorname{Ball}(C_0(Y, \mathbb{K})^*)$ we have $\phi = \phi_1 = \phi_2$. Then

$$\phi(f_0) = \phi_2(f_0) = \frac{1}{|m|(Y \setminus V)} \int_{Y \setminus V} f_0 dm = 0$$

since $f_0 = 0$ on $Y \setminus V$. On the other hand

$$\begin{aligned} |\phi(f_0)| = |\phi_1(f_0)| = \left| \frac{1}{|m|(V)} \int_V f_0 dm \right| &\geq \frac{1}{|m|(V)} \left(|\int_L f_0 dm| - |\int_{V \setminus L} f_0 dm| \right) \\ &\geq \frac{1}{|m|(V)} \left(|m(L)| - |m|(V \setminus L) \right) > \frac{|m(L)|}{2|m|(V)} > 0, \end{aligned}$$

hence $\phi(f_0) \neq 0$, which is a cotradiction.

Suppose that m is a representing measure for $\lambda\tau_x$. Let U be an open neighborhood of x. Then by the Urysohn's lemma there exists $f \in C_0(Y, \mathbb{K})$ such that $0 \leq f \leq 1$ on Y, $f(x) = 1$, and $f = 0$ on $Y \setminus U$. Since

$$1 = f(x) = \tau_x(f) = |\int f dm| \leq \int_U |f| d|m| \leq |m|(U) \leq 1,$$

we see that $\operatorname{supp}(m) \subset U$. As U can be arbitrary, we see that $\operatorname{supp}(m) = \{x\}$. Thus we infer that $m = \lambda D_x$.

Suppose conversely that $x_0 \in Y$ and $\lambda_0 \in \mathbb{T}$ and $\phi_0 = \lambda_0 \tau_x$. Suppose that $\phi_0 = (\phi_1 + \phi_2)/2$ for $\phi_j \in \operatorname{Ball}(C_0(Y, \mathbb{K})^*)$, $j = 1, 2$. Let μ_1 be a representing measure for ϕ_1. We prove that $\operatorname{supp}(\mu_1) = \{x_0\}$. Suppose not. As the support of a regular measure is not empty, there exists $y \in \operatorname{supp}(\mu_1) \setminus \{x_0\}$. Then by the Urysohn's lemma there exists $f \in C_0(Y, \mathbb{R})$ such that $f(y) = 0 \leq f \leq 1 = f(x_0)$. Put $U = \{z \in Y : f(z) < 1/2\}$. Then U is an open neighborhood of y, and $0 < |\mu_1|(U) < 1$ since $y \in \operatorname{supp}(\mu_1)$. Thus

$$|\phi_1(f)| \leq \int_U |f| d|\mu_1| + \int_{Y \setminus U} |f| d|\mu_1| \leq \frac{1}{2}|\mu_1|(U) + |\mu_1|(Y \setminus U) < 1.$$

It follows that

$$1 = |\phi_0(f)| \leq (|\phi_1(f)| + |\phi_2(f)|)/2 < 1,$$

which is a contradiction proving that $\mathrm{supp}(\mu_1) = \{x_0\}$. We infer that $\phi_1 = \lambda_1 \tau_{x_0}$ for $\lambda_1 \in \mathbb{T}$. In the same way we have that $\phi_2 = \lambda_2 \tau_{x_0}$ for $\lambda_2 \in \mathbb{T}$. Since

$$\lambda_0 = \phi_0(f) = (\phi_1(f) + \phi_2(f))/2 = (\lambda_1 + \lambda_2)/2$$

and $|\lambda_j| = 1$ for $j = 0, 1, 2$ we infer that $\lambda_0 = \lambda_1 = \lambda_2$. Thus $\phi_1 = \phi_2 = \phi_0$. We concluded that $\phi_0 \in \mathrm{ext}\,\mathrm{Ball}(C_0(Y, \mathbb{K})^*)$. $\square$

Corollary 2.18. *Suppose that E is a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Suppose that $\phi \in \mathrm{ext}\,\mathrm{Ball}(E^*)$. Then there exist $y \in Y$ and $\lambda \in \mathbb{T}$ such that $\phi = \lambda \tau_y$.*

Proof. Put

$$S = \{\varphi \in C_0(Y, \mathbb{K})^* : \varphi \text{ is a Hahn-Banach extension of } \phi\}.$$

Then S is a non-empty weak*-closed convex subset of $C_0(Y, \mathbb{K})$. Then the Krein-Milman theorem asserts that there exists a $\Phi \in \mathrm{ext}(S)$. Then Φ is an extreme point of $\mathrm{Ball}(C_0(Y, \mathbb{K})^*)$. In fact, suppose that $\Phi = (\Phi_1 + \Phi_2)/2$ for $\Phi_1, \Phi_2 \in \mathrm{Ball}(C_0(Y, \mathbb{K})^*)$. Then $\phi = \Phi|E = (\Phi_1|E + \Phi_2|E)/2$, and $\Phi_j|E \in \mathrm{Ball}(E^*)$ for $j = 1, 2$. As ϕ is an extreme point of $\mathrm{Ball}(E^*)$, we have $\Phi_1|E = \Phi_2|E = \phi$. Since

$$1 = \|\phi\| = \|\Phi_j|E\| \leq \|\Phi_j\| = 1,$$

we have $\|\Phi_j\| = 1$ for $j = 1, 2$. Hence $\Phi_j \in S$ for $j = 1, 2$. As Φ is an extreme point in S, we have $\Phi = \Phi_1 = \Phi_2$. Thus $\Phi \in \mathrm{ext}(C_0(Y, \mathbb{K})^*)$. By the Arens-Kelley theorem there exists $y \in Y$ and $\lambda \in \mathbb{T}$ such that $\Phi = \lambda D_y$. Thus we see that $\phi = \Phi|E = \lambda D_y$. $\square$

Note that $\lambda \tau_x$ needs not to be an extreme point of $\mathrm{Ball}(E^*)$ in general. In fact, $\tau_0 \notin \mathrm{ext}\,\mathrm{Ball}(P(\bar{D})^*)$ for the disk algebra on the closed unit disk $\bar{D}$ in the complex plane.

2.3.3. Relationship among three properties about a point x : (i) being a strong boundary point; (ii) being in the Choquet boundary; (iii) the representing measure for the point evaluation at x is unique.

Let E be a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$ and $x \in Y$. We study the relationship of the following (i), (ii) and (iii) :

(i) x is a strong boundary point for E,

(ii) $x \in \mathrm{Ch}(E)$,

(iii) the representing measure for τ_x is only D_x.

We recall the definition of the boundary.

Definition 2.19. Suppose that E is a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. A subset L of Y is said to be a boundary if for each $f \in E$ there exists a point $x \in L$ such that $|f(x)| = \|f\|_\infty$.

The following may be well known, for example, [37, Proposition 6.3] states about the case where E contains 1 and Y is compact. However for the sake of a self-contained exposition and for the convenience of the readers we include many proofs as possible.

Proposition 2.20. *Suppose that E is a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Then the Choquet boundary* $\mathrm{Ch}(E)$ *is a boundary.*

Proof. Let $f \in E$. We may assume that $\|f\|_\infty = 1$. Then there exists $y \in Y$ such that $|f(y)| = 1$. Put $L = \{\phi \in \mathrm{Ball}(E^*) : \phi(f) = f(y)\}$. As $\tau_y \in L$, L is non-empty weak*-closed convex subset of Ball(E^*). The Krein-Milman theorem asserts that there exists $\phi_0 \in \operatorname{ext} L$. Then ϕ_0 is an extreme point of Ball(E^*). In fact, suppose that $\phi_0 = (\phi_1 + \phi_2)/2$ for $\phi_1, \phi_2 \in \mathrm{Ball}(E^*)$. Then by

$$1 = |f(y)| = |\phi_1(f) + \phi_2(f)|/2 \leq (|\phi_1(f)| + |\phi_2(f)|)/2 \leq 1$$

we have $\phi_1(f) = \phi_2(f)$. By $\phi_0 = (\phi_1 + \phi_2)/2$ we infer that $f(y) = \phi_0(f) = \phi_1(f) = \phi_2(f)$. Thus $\phi_1, \phi_2 \in L$. As $\phi_0 \in \operatorname{ext} L$ we have $\phi_0 = \phi_1 = \phi_2$. Thus $\phi_0 \in \operatorname{ext} \mathrm{Ball}(E^*)$. By Corollary 2.18 there exists $x \in Y$ and a unimodular complex number λ such that $\phi_0 = \lambda\tau_x$ on E. Note that $x \in \mathrm{Ch}(E)$ since $\bar{\lambda}\phi_0 = \tau_x$ is an extreme point of Ball(E^*) for λ is a unimodular complex number. We have

$$|f(x)| = |\lambda\tau_x(f)| = |\phi_0(f)| = |f(y)| = \|f\|.$$

□

Proposition 2.21. *Let E be a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Suppose that $x \in Y$ is a strong boundary point for E. Then the representing measure for τ_x is only D_x*

Proof. Suppose that $x \in Y$ is a strong boundary point for E and m is its representing measure. Let U be an open neighborhood of x. Since x is a strong boundary point, there is a function $f \in E$ with $f(x) = 1 = \|f\|_\infty$ and $|f| < 1$ on $Y \setminus U$. Thus $1 = |\tau_x(f)| \leq \|\tau_x\| \leq 1$, so we have $1 = \|\tau_x\| = \|m\|$.

We prove $\mathrm{supp}\,(m) = \{x\}$. Suppose contrarily that there exists $y \in \mathrm{supp}(m) \setminus \{x\}$. Then there exists an open neighborhood V of y and an open neighborhood W of x such that $W \cap V = \emptyset$. There exists $g \in E$ such that $g(x) = 1 = \|g\|_\infty$, $|g| < 1$ on $Y \setminus W$. Since $Y \setminus W$ is closed set, there exists $\delta > 0$ such that $|g| \le 1 - \delta$ on $Y \setminus W$. As $W \cap V = \emptyset$ we have $|g| \le 1 - \delta$ on V. Since $y \in \operatorname{supp}(m)$ we have $|m|(V) > 0$. Since $\tau_x(g) = \int gdm$, we have

$$1 = |\tau_x(g)| \le \int_V |g|d|m| + \int_{Y \setminus V} |g|d|m| \le (1-\delta)|m|(V) + |m|(Y \setminus V) < 1,$$

which is a contradiction proving that $\operatorname{supp}(m) \setminus \{x\} = \emptyset$. As m is a regular measure, $\operatorname{supp}(m)$ is not empty, so $\operatorname{supp}(m) = \{x\}$. Thus $m = \lambda D_x$ for some unimodular complex number λ. As

$$1 = g(x) = \tau_x(g) = \int gdm = \int gd(\lambda D_x) = \lambda g(x),$$

we have that $\lambda = 1$ and $m = D_x$ □

Proposition 2.22. *Let E be a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Let $x \in Y$. Suppose that the representing measure for τ_x is only D_x. Then $x \in \operatorname{Ch}(E)$.*

Proof. Suppose that $x \in Y$ and the representing measure for τ_x is only D_x. Let $\tau_x = (\phi_1 + \phi_2)/2$ for $\phi_1, \phi_2 \in \operatorname{Ball}(E^*)$. Suppose that m_j is a representing measurer for ϕ_j for $j = 1, 2$. Then $(m_1 + m_2)/2$ is a representing measure for τ_x. Thus

$$|\tau_x(f)| = \left| \int fd(m_1 + m_2)/2 \right| \le \|f\|_\infty \|(m_1 + m_2)/2\|$$

for every $f \in E$. Thus

$$1 = \|\tau_x\| \le \|(m_1 + m_2)/2\| \le (\|m_1\| + \{m_2\|)/2 = 1.$$

Hence $\|(m_1 + m_2)/2\| = \|\tau_x\|$, so $(m_1 + m_2)/2$ is the representing measure for τ_x. Thus $D_x = (m_1 + m_2)/2$ and

$$1 = D_x(\{x\}) = (m_1(\{x\}) + m_2(\{x\}))/2.$$

As $|m_j(\{x\})| \le 1$ for $j = 1, 2$, we have that $m_j(\{x\}) = 1$ for $j = 1, 2$. As $\|m_j\| = 1$ for $j = 1, 2$ we infer that $m_1 = m_2 = D_x$ and $\tau_x = \phi_1 = \phi_2$. We conclude that $\tau_x \in \operatorname{ext}(\operatorname{Ball}(E^*))$, so $x \in \operatorname{Ch}(E)$. □

The following corollary is straightforward from Propositions 2.21, 2.22.

Corollary 2.23. *Suppose that E is a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Suppose that $x \in Y$ is a strong boundary point. Then $x \in \operatorname{Ch}(E)$.*

The converse of Proposition 2.22 does not hold in general. Let $P(\bar{D})$ be the disk algebra on the closed unit disk $\bar{D}$. Let

$$E = \{f \in P(\bar{D}) : f(i) = if(1)\}.$$

Then E is a uniformly closed $\mathbb{C}$-linear subspace of $C(\bar{D}, \mathbb{C})$ which separates the points of $\bar{D}$ and has no common zeros. In fact $f(z) = z \in E$ separates the points in $\bar{D}$ and $f(y) = y \neq 0$ for $y \in \bar{D}$ with $y \neq 0$. Put $g(z) = (z-1)(z-i)$. Then $g \in E$ and $g(0) = i \neq 0$. Thus E has no common zeros on $\bar{D}$. Furthermore we have the following.

Proposition 2.24. *Let $E = \{f \in P(\bar{D}) : f(i) = if(1)\}$. Then $1 \in \mathrm{Ch}(E)$, and D_1, the point mass at the point $1 \in \bar{D}$, and $-iD_i$ are representing measures for τ_1.*

Proof. Suppose that $\tau_1 = (p+q)/2$ for some $p, q \in \mathrm{Ball}(E^*)$. As $\tau_1(z) = 1$, we infer that $\|\tau_1\| = 1$. Since $1 = \tau_1(z) = (p(z) + q(z))/2$ and $|p(z)| \leq 1$, $|q(z)| \leq 1$ we infer that $p(z) = q(z) = 1$. Hence $\|p\| = \|q\| = 1$. Let m_p be a representing measure for p and m_q a representing measure for q. We show that $\mathrm{supp}(m_p) \subset \{1, i\}$. Suppose not; suppose that there exists $y \in \mathrm{supp}(m_p) \setminus \{1, i\}$.

Suppose that $|y| < 1$. As $\{1, i\}$ is a peak interpolation set for $P(\bar{D})$ (cf. [12, p.111] or [27, Lemma 4.1]), there is a function $f \in P(\bar{D})$ such that $f(1) = 1, = \|f\|$ and $f(i) = i$. By the maximum absolute value principle for analytic functions, we infer that $|f(y)| < 1$. Hence there is a positive integer n such that $|f^{4n+1}(y)| < 1/2$. As $if^{4n+1}(1) = f^{4n+1}(i)$ we have $f^{4n+1} \in E$. Then put $h = f^{4n+1}$.

Suppose that $|y| = 1$. As $\{1, i, y\}$ is a peak interpolation set [12, p.111], there exists $h \in P(\bar{D})$ such that $f(1) = 1 = \|f\|$, $h(i) = i$, and $|h(y)| < 1/2$.

Let U_y be an open neighborhood of y such that $|h| < 1/2$ on U_y. Since $y \in \mathrm{supp}(m_p)$ and the measure m_p is regular, we have $0 < |m_p|(U_y)$. Then we get

$$|p(h)| \leq |\int_{U_y} |h| d|m_p| + \int_{\bar{D} \setminus U_y} |h| d|m_p| \leq \frac{1}{2}|m_p|(U_y) + |m_p|(\bar{D} \setminus U_y) < 1.$$

Since $|q(h) \leq 1$, we get

$$1 = |\tau_1(h)| \leq (|p(h) + q(h)|)/2 < 1,$$

which is a contradiction proving that $\mathrm{supp}(m_p) \subset \{1, i\}$. Then we infer that there exists two complex numbers λ_p and μ_p with $|\lambda_p| + |\mu_p| = 1$ such that $m_p = \lambda_p D_1 + \mu_p D_i$ In the same way there exists two complex number λ_q and μ_q with $|\lambda_q| + |\mu_q| = 1$ such that $m_q = \lambda_q D_1 + \mu_q D_i$. Hence we obtain that $\frac{\lambda_p + \lambda_q}{2} D_1 + \frac{\mu_p + \mu_q}{2} D_i$ is a representing measure for $\frac{p+q}{2} = \tau_1$. Thus

$$f(1) = \frac{\lambda_p + \lambda_q}{2} f(1) + \frac{\mu_p + \mu_q}{2} f(i) = \left(\frac{\lambda_p + \lambda_q}{2} + i \frac{\mu_p + \mu_q}{2} \right) f(1)$$

for every $f \in E$. Hence

$$\frac{\lambda_p+\lambda_q}{2}+i\frac{\mu_p+\mu_q}{2}=\frac{\lambda_p+i\mu_p}{2}+\frac{\lambda_q+i\mu_q}{2}=1. \tag{2.2}$$

Since

$$\left|\frac{\lambda_p+i\mu_p}{2}\right| \le \frac{|\lambda_p|+|\mu_p|}{2}=\frac{1}{2}$$

and

$$\left|\frac{\lambda_q+i\mu_q}{2}\right| \le \frac{|\lambda_q|+|\mu_q|}{2}=\frac{1}{2},$$

we infer from (2.2) that $\lambda_p+i\mu_p=1$. Thus

$$p(f)=\lambda_p f(1)+\mu_p f(i)=(\lambda_p+i\mu_p)f(1)=f(1)=\tau_1(f)$$

for every $f \in E$. We have that $\tau_1=p$, so $\tau_1 \in \mathrm{ext}(\mathrm{Ball}(E^*))$. We conclude that $1 \in \mathrm{Ch}(E)$. It is evident that D_1 and $-iD_i$ are different representing measures for τ_1. □

The strong separation condition ensures uniqueness of the representing measure for the point evaluation at a Choquet boundary point. In fact, we have

Proposition 2.25. *Let E be a $\mathbb{K}$-linear subspace of $C_0(Y,\mathbb{K})$. Then the following is equivalent.*

(i) *E strongly separates the points of* $\mathrm{Ch}(E)$*,*

(ii) *for every $x \in \mathrm{Ch}(E)$, the representing measure for τ_x is only D_x .*

Proof. We prove (i) implies (ii). Suppose that $x \in \mathrm{Ch}(E)$ and m a representing measure for τ_x. Note that $\|\tau_x\|=1$ since $\tau_x \in \mathrm{ext}\,\mathrm{Ball}(E^*)$. We have $\|m\|=1$. Let $y \in \mathrm{supp}(m)$. We prove that for every open neighborhood U of y we have $|m|(U)=1$. (If it were proved, then $m=\lambda D_y$ for some complex number λ of modulus 1 since m is a regular measure. Then

$$f(x)=\tau_x(f)=\int f dm=\int f d\lambda D_y=\lambda f(y)=\lambda\tau_y(f) \tag{2.3}$$

for every $f \in E$. Then we have $\tau_y=\bar{\lambda}\tau_x$. Since τ_x is an extreme point of $\mathrm{Ball}(E^*)$, τ_y is also an extreme point of $\mathrm{Ball}(E^*)$. Thus $y \in \mathrm{Ch}(E)$. By the condition (i) that E strongly separates the points of $\mathrm{Ch}(E)$, we have from (2.3) that $\lambda=1$ and $x=y$. Thus $m=D_x$ follows.) Suppose not: suppose that there is an open neighborhood U_0 of y such that $|m|(U_0)\neq 1$. Then $|m|(U_0)<1$ as $\|m\|=1$. As m is regular and $y \in \mathrm{supp}(m)$, we have $0<|m|(U_0)$. Since $|m|(Y)=\|m\|=1$, $|m|(Y\setminus U_0)|>0$. Put

$$\varphi_1(f)=\frac{1}{|m|(U_0)}\int_{U_0} f dm, \quad f \in E,$$

$$\varphi_2(f) = \frac{1}{|m|(Y \setminus U_0)} \int_{Y \setminus U_0} f dm, \quad f \in E.$$

It follows that $\varphi_1, \varphi_2 \in \mathrm{Ball}(E^*)$ and $\tau_x = |m|(U_0)\varphi_1 + |m|(Y \setminus U_0)\varphi_2$, where $m|(U_0) > 0$, $|m|(Y \setminus U_0) > 0$ and $|m|(U_0) + |m|(Y \setminus U_0) = 1$. As $\tau_x \in \mathrm{ext}\,\mathrm{Ball}(E^*)$, we have that $\tau_x = \varphi_1$. In the same way we have

$$\tau_x(f) = \frac{1}{|m|(V)} \int_V f dm, \quad f \in E$$

for any open neighborhood V of y with $V \subset U_0$. Since f is continuous, for every $\varepsilon > 0$, there exists an open neighborhood V_ε of y such that $|f - f(y)| < \varepsilon$ on V_ε. Hence

$$|f(y) - f(x)| \leq \frac{1}{|m|(V_\varepsilon)} \int_{V_\varepsilon} |f(y) - f| d|m| \leq \varepsilon.$$

Hence $f(y) = f(x)$ for every $f \in E$, so $\tau_y = \tau_x$. As τ_x is an extreme point of $\mathrm{Ball}(E^*)$, so is τ_y. Hence $y \in \mathrm{Ch}(E)$. Since E strongly separates the points of $\mathrm{Ch}(E)$, we have that $x = y$. It follows that for any $y \in \mathrm{supp}(m)$, y coincides with x, that is, $\mathrm{supp}(m) = \{x\}$, which is a contradiction since we assume that $|m|(U_0) < 1$. We conclude that $|m|(U) = 1$ for any open neighborhood U of y.

We prove (ii) implies (i) by reductio ad absurdum. Suppose that E does not strongly separate the points of $\mathrm{Ch}(E)$: there exists a pair x and y of different points in $\mathrm{Ch}(E)$ such that the equation $|f(x)| = |f(y)|$ holds for every $f \in E$. As $\tau_x \in \mathrm{ext}\,\mathrm{Ball}(E^*)$, $\|\tau_x\| = 1$ holds, so there exists $f_0 \in E$ such that $f_0(x) = 1$. Then there exists a complex number λ_0 with $|\lambda_0| = 1$ such that $f_0(y) = \lambda_0 f_0(x) = \lambda_0$. For any $f \in E$ with $f(x) \neq 0$ there exists a complex number λ_f of unit modulus such that $f(y) = \lambda_f f(x)$. As $(f/f(x) + f_0)(x) = 2$, we have

$$\lambda_f + \lambda_0 = (f/f(x) + f_0)(y) = \lambda_{f/f(x)+f_0}(f/f(x) + f_0)(x) = 2\lambda_{f/f(x)+f_0}$$

for every $f \in E$ with $f(x) \neq 0$. As $|\lambda_f| = |\lambda_0| = |\lambda_{f/f(x)+f_0}| = 1$ we infer that $\lambda_f = \lambda_0$. Thus $f(y) = \lambda_0 f(x)$ for every $f \in E$ with $f(x) \neq 0$. This equation also holds for $f \in E$ with $f(x) = 0$. We conclude that $f(y) = \lambda_0 f(x)$ for every $f \in E$. Thus $\bar{\lambda_0} D_y$ is a representing measure for τ_x. As $x \neq y$ we have at least two representing measures D_x and $\bar{\lambda_0} D_y$ for τ_x. □

Corollary 2.26. *Let E be a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Suppose that E strongly separates the points of $\mathrm{Ch}(E)$. Then $x \in Y$ is in the Choquet boundary if and only if the representing measure for τ_x is only D_x.*

Proof. It is straightforward from Propositions 2.22, 2.25. □

If E is a subspace of $C(X, \mathbb{K})$ which contains constants or E is a subalgebra of $C_0(Y, \mathbb{K})$ which separates the points of Y, then the Choquet boundary points are characterized by the uniqueness of the representing measures for the corresponding point evaluations.

Corollary 2.27. *Supposes that X is a compact Hausdorff space and E is a $\mathbb{K}$-linear subspace of $C(X, \mathbb{K})$ which separates the points of X and contains constants. Then $x \in Y$ is in the Choquet boundary if and only if the representing measure for τ_x is only D_x.*

Proof. Proposition 2.3 asserts that E strongly separates the points of X. Then by Corollary 2.26 we have the conclusion. □

Corollary 2.28. *Suppose that E is a subalgebra of $C_0(Y, \mathbb{K})$ which separates the points of Y. Then $x \in Y$ is in the Choquet boundary if and only if the representing measure for τ_x is only D_x.*

Proof. Proposition 2.3 asserts that E strongly separates the points of X. Then by Corollary 2.26 we have the conclusion. □

We summarize the results to generalize a theorem of Bishop and de Leeuw on a characterization of the Choquet boundary for uniform algebras [37, p. 39], [12, Theorem 2.2.6] (cf. [42, Theorem 2.1], [43, Theorem 9]).

Theorem 2.29. *Suppose that A is a closed $\mathbb{K}$-subalgebra of $C_0(Y, \mathbb{K})$ which separates the points of Y and has no common zeros. Let $x \in Y$. The following are equivalent.*

(i) $x \in \mathrm{Ch}(A)$,

(i′) $x \in \mathrm{Ch}(\dot{A} + \mathbb{K})$,

(ii) *x is a strong boundary point for A,*

(ii′) *x is a strong boundary point for $\dot{A} + \mathbb{K}$,*

(iii) *the representing measure for the point evaluation τ_x on A is only D_x,*

(iv) *there exists a pair of $0 < \alpha < \beta \leq 1$ such that for every open neighborhood U of x there exists a function $f \in A$ such that $\|f\|_\infty \leq 1$, $|f(x)| \geq \beta$, and $|f| < \alpha$ on $Y \setminus U$,*

(v) *for every pair of $0 < \alpha < \beta \leq 1$ and for every open neighborhood U of x there exists a function $f \in A$ such that $\|f\|_\infty \leq 1$, $|f(x)| \geq \beta$, and $|f| < \alpha$ on $Y \setminus U$.*

Proof. If $\mathbb{K} = \mathbb{R}$, then by the Stone-Weierstrass theorem, $A = C_0(Y, \mathbb{R})$. It follows that the conditions in (i) through (v) hold for every $x \in Y$.

Suppose that $K = \mathbb{C}$. (i) $\leftrightarrow$ (iii) is just Corollary 2.28. (ii) $\to$ (i) is Corollary 2.23. (ii) $\leftrightarrow$ (ii$'$) is Proposition 2.15.

Since $\dot{A} + \mathbb{C}$ is a uniform algebra on Y_∞, (i$'$) $\to$ (ii$'$) follows from the Bishop-de Leeuw theorem (cf. [37, p.39], [12, Theorem 2.2.6]).

We prove (i) $\to$ (i$'$). Let $x \in \mathrm{Ch}(A)$. To distinguish the point evaluations on A and $\dot{A} + \mathbb{C}$, denote the point evaluation on $\dot{A} + \mathbb{C}$ by $\dot{\tau_x} : \dot{A} + \mathbb{C} \to \mathbb{C}$. The point evaluation on A is denoted τ_x as usual. Suppose that $\dot{\tau_x} = (p+q)/2$, where $p, q \in \mathrm{Ball}((\dot{A} + \mathbb{C})^*)$. As $1 = \dot{\tau_x}(1) = (p(1) + q(1))/2$ and $|p(1)| \le 1$, $|q(1)| \le 1$ we infer that $p(1) = q(1) = 1$. Hence for each $\dot{f} + \lambda \in \dot{A} + \mathbb{C}$ we have

$$\tau_x(f) + \lambda = \dot{\tau_x}(\dot{f} + \lambda) = (p(\dot{f} + \lambda) + q(\dot{f} + \lambda))/2 = (p(\dot{f}) + q(\dot{f}))/2 + \lambda.$$

Define $p' : A \to \mathbb{C}$ by $p'(g) = p(\dot{g})$ for $g \in A$, we have $p' \in \mathrm{Ball}(A^*)$. In the same way q' can be defined and $q' \in \mathrm{Ball}(A^*)$. By the above equality we have $\tau_x = (p' + q')/2$. As $\tau_x \in \mathrm{ext}(\mathrm{Ball}(A^*))$, $p' = q' = \tau_x$. It follows that $p = q = \dot{\tau_x}$, proving that $\dot{\tau_x} \in \mathrm{ext}(\mathrm{Ball}(\dot{A} + \mathbb{C})^*)$, so $x \in \mathrm{Ch}(\dot{A} + \mathbb{C})$.

We prove (i$'$) $\to$ (i). Suppose that $x \in \mathrm{Ch}(\dot{A}+\mathbb{C})$, in other wards, $\dot{\tau_x} \in \mathrm{ext}(\mathrm{Ball}(\dot{A}+\mathbb{C})^*)$. We have already proved that (i$'$) implies (ii$'$) and (ii$'$) implies (ii). Hence x is a strong boundary point for A. Thus we infer that $1 = \|\tau_x\| = |\dot{\tau_x}|\dot{A}\| = 1$. Let Δ_x denote a Hahn-Banach extension on $\dot{A} + \mathbb{C}$ of $\dot{\tau_x}|\dot{A}$. We show that $\Delta_x = \dot{\tau_x}$. By the Riesz-Kakutani theorem there exists m_x of a representing measure of Δ_x on Y_∞. Note that $\|m_x\| = \|\Delta_x\| = \|\dot{\tau_x}|\dot{A}\| = 1$ and $\Delta_x(\dot{f} + \lambda) = \int (\dot{f} + \lambda) dm_x$ for $\dot{f} + \lambda \in \dot{A} + \mathbb{C}$. Note also that $1 = \dot{\tau_x}(1) = \int 1 dm_x$ ensures that m_x is a probability measure. As x is a strong boundary point for A, by Proposition 2.9 there exists a family $\{K_\alpha\}$ of peak sets for A such that $\bigcap_\alpha K_\alpha = \{x\}$. Denote $f_\alpha \in A$ the corresponding peaking function for K_α. By the bounded convergence theorem for the probability measure m_x we get

$$1 = \Delta_x(f_\alpha^n) = \int f_\alpha^n dm_x \to m_x(K_\alpha)$$

as $n \to \infty$ since $f_\alpha = 1$ on K_α and $|f_\alpha| < 1$ on $Y \backslash K_\alpha$. We see that $m_x(K_\alpha) = 1$ for every peak set K_α for A which contains x. Let U be an open neighborhood of x in Y. Although Y needs not be compact, by considering that U and $K_\alpha' s$ are subsets of compact space Y_∞, there exists a finite number of $K_{\alpha_1}, \dots, K_{\alpha_n}$ such that $U \supset \bigcap_{j=1}^n K_{\alpha_j}$. Then we have $1 = m_x(\bigcap_{j=1}^n K_{\alpha_j}) \le m_x(U) \le 1$. As U is arbitrary open neighborhood of x, we get $m_x(\{x\}) = 1$ since m_x is a regular measure. Thus $m_x = D_x$. Hence $\Delta_x(\dot{f} + \lambda) = \int \dot{f} + \lambda dD_x = \dot{\tau_x}(\dot{f} + \lambda)$ for every $\dot{f} + \lambda \in \dot{A} + \mathbb{C}$. We get that $\Delta_x = \dot{\tau_x}$.

We prove that $\tau_x \in \mathrm{ext}(\mathrm{Ball}(A^*))$. Suppose that $\tau_x = (p+q)/2$ for some $p, q \in \mathrm{Ball}(A^*)$. Let $\check{p} : \dot{A} \to \mathbb{C}$ be defined as $\check{p}(\dot{f}) = p(f)$, $\dot{f} \in \dot{A}$ and $\check{q} : \dot{A} \to \mathbb{C}$ be defined as

$\check{q}(\dot{f}) = q(f)$, $\dot{f} \in \dot{A}$. Note that it is well defined since $f \mapsto \dot{f}$ is a bijection from A onto $\dot{A}$. Note also that

$$\left(\frac{\check{p}+\check{q}}{2}\right)(\dot{f}) = \left(\frac{p+q}{2}\right)(f) = \tau_x(f) = \dot{\tau_x}(\dot{f}), \quad f \in A,$$

so $\dot{\tau_x}|\dot{A} = (\check{p}+\check{q})/2$. Let $\dot{p}$ and $\dot{q}$ be Hahn-Banach extensions of $\check{p}$ and $\check{q}$ on $\dot{A}+\mathbb{C}$ respectively. As $(\dot{p}+\dot{q})/2$ is an extension of $(\check{p}+\check{q})/2 = \dot{\tau_x}|\dot{A}$, we have

$$1 = \|(p+q)/2\| = \|(\check{p}+\check{q})/2\| \leq \|(\dot{p}+\dot{q})/2\| \leq (\|\dot{p}\|+\|\dot{q}\|)/2 = 1.$$

Thus $\|(\dot{p}+\dot{q})/2\| = 1$, so $(\dot{p}+\dot{q})/2$ is a Hahn-Banach extension of $\dot{\tau_x}|\dot{A}$. By the result in the previous paragraph that a Hahn-Banach extension of $\dot{\tau_x}|\dot{A}$ is always $\dot{\tau_x}$ we have

$$\dot{\tau_x} = (\dot{p}+\dot{q})/2.$$

As $\dot{\tau_x} \in \mathrm{ext}(\mathrm{Ball}(\dot{A}+\mathbb{C}^*))$ we have $\dot{p} = \dot{q} = \dot{\tau_x}$. For every $f \in A$ we have $\dot{\tau_x}(\dot{f}) = \tau_x(f)$, $\dot{p}(\dot{f}) = \check{p}(\dot{f}) = p(f)$, and $\dot{q}(\dot{f}) = \check{q}(\dot{f}) = q(f)$ for every $f \in A$, we have $\tau_x(f) = p(f) = q(f)$ for every $f \in A$. We conclude that $p = q = \tau_x$, $\tau_x \in \mathrm{ext}(\mathrm{Ball}(A^*))$. It follows that $x \in \mathrm{Ch}(A)$.

Suppose that iv) holds. We prove (ii$'$). For every open neighborhood (as a subset of Y_∞) U of x there exists a function $\dot{f} \in \dot{A} \subset \dot{A}+\mathbb{C}$ such that $\|\dot{f}\|_\infty \leq 1$, $|f(x)| \geq \beta$, and $|f| < \alpha$ on $Y_\infty \setminus U$. Then by [12, Theorem 2.3.4] we have x is a strong boundary point for $\dot{A}+\mathbb{C}$; (ii$'$) holds.

Suppose that (ii$'$) holds. We prove (iv). Let U be an open neighborhood of x. As we have already pointed out that (ii$'$) is equivalent to (ii), there exists $f \in A$ such that $f(x) = 1 = \|f\|_\infty$ and $|f| < 1$ on $Y \setminus U$. As $Y \setminus U$ is closed, there exists $\delta < 1$ such that $|f| < \delta$ on $Y \setminus U$. Put $\alpha = \delta$ and $\beta = (1+\alpha)/2$. Then we have that $\|f\| = 1 = f(x) > \beta$ and $|f| < \alpha$ on $Y \setminus U$; iv) holds.

(v) $\to$ (iv) is trivial.

We prove (ii) $\to$ (v). Suppose that x is a strong boundary point for A. Let α, β be any pair such that $0 < \alpha < \beta \leq 1$. Let U be an arbitrary open neighborhood of x. Then there exists $f \in A$ such that $f(x) = 1 = \|f\|$ and $|f| < 1$ on $Y \setminus U$. As $Y \setminus U$ is closed, there exists $\delta < 1$ such that $|f| < \delta$ on $Y \setminus U$. For a sufficiently large positive integer n the inequality $\delta^n < \alpha$ holds. Thus $f^n \in A$ satisfies $\beta \leq |f^n(x)| = 1 = \|f^n\|_\infty$ and $|f^n| < \alpha$ on $Y \setminus U$. □

Even if E is a strongly separating $\mathbb{K}$-linear subspace of $C_0(Y,\mathbb{K})$, a point $y \in \mathrm{Ch}(\dot{E}+\mathbb{K})$ needs not be a point in $\mathrm{Ch}(E)$.

Example 2.30. Let E be the space defined in Example 2.6. Then $1 \notin \mathrm{Ch}(E)$ while $1 \in \mathrm{Ch}(\dot{E}+\mathbb{K})$. The reason is as follows. The space E is strongly separating.

Hence $t \in (0,1]$ is in $\mathrm{Ch}(E)$ (resp. $\mathrm{Ch}(\dot{E}+\mathbb{K})$) if and only if the representing measure for τ_t is unique. It is trivial that D_1 and $-D_{\frac{1}{4}}$ are both representing measure for τ_1. Thus $1 \notin \mathrm{Ch}(E)$. On the other hand 1 is a strong boundary point for $\dot{E}+\mathbb{K}$. Thus $1 \in \mathrm{Ch}(\dot{E}+\mathbb{K})$.

2.3.4. The Šilov boundary

According to [3, p.80] the existence of Šilov boundary for a subalgebra of $C(X,\mathbb{K})$ which separates the points of X and contains constant is given by Šilov [45]. A simple proof of the existence of the Šilov boundary for a $\mathbb{K}$-linear subspace of $C(X,\mathbb{K})$ which separates the points of X and contains constants for a compact Hausdorff space is exhibited by Bear [5]. In [37, Proposition 6.4] Phelps showed that the closure of the Choquet boundary is the Šilov boundary. For further references on Šilov boundary see [3, 8, 9, 30, 43] for example.

Definition 2.31. Let E be a $\mathbb{K}$-linear subspace of $C_0(Y,\mathbb{K})$. We say that a closed subset K of Y is a *Šilov boundary* for E if it is the smallest closed boundary in the sense that it is a boundary for E and $K \subset L$ for any *closed boundary* (a boundary for E which is a closed subset of Y) L for E.

Araujo and Font [3, Theorem 1] proved that the closure of the Choquet boundary for E is the Šilov boundary for E if E is a $\mathbb{K}$-linear subspace of $C_0(Y,\mathbb{K})$ which strongly separates the points of Y. The following slightly generalizes it.

Proposition 2.32. *Let E be a $\mathbb{K}$-linear subspace of $C_0(Y,\mathbb{K})$. If E strongly separates the points of* $\mathrm{Ch}(E)$, *then the closure* $\overline{\mathrm{Ch}(E)}$ *of the Choquet boundary in Y is the Šilov boundary for E.*

Proof. Let K be a closed boundary for E. Let $x \in \mathrm{Ch}(E)$. We prove that $x \in K$. As K is a boundary, the restriction $E|K$ of E on K is uniformly closed $\mathbb{K}$-subspace of $C_0(K,\mathbb{K})$. The restriction map $T : E \to E|K$ by $T(f) = f|K$, $f \in E$ is a bijection and an isometry. We define $\dot{\tau_x} : E|K \to \mathbb{K}$ by $\dot{\tau_x}(F) = \tau_x(T^{-1}(F))$, $F \in E|K$. Then

$$T^* \circ \dot{\tau_x}(f) = \dot{\tau_x}(T(f)) = \tau_x(T^{-1}(T(f))) = \tau_x(f), \quad f \in E,$$

so we infer that $T^* \circ \dot{\tau_x} = \tau_x$ on E.

We prove that $\dot{\tau_x} \in \mathrm{ext}(\mathrm{Ball}(E|K)^*)$. Suppose that $\dot{\tau_x} = (p+q)/2$ for $p, q \in (E|K)^*$. Then

$$\tau_x = T^* \circ \dot{\tau_x} = \frac{T^* \circ p + T^* \circ q}{2}.$$

As $\tau_x \in \mathrm{ext}\, E^*$ we have $T^* \circ p = T^* \circ q = \tau_x$. Hence $p = q = \dot{\tau_x}$ since T^* is a bijection. It follows that $\dot{\tau_x} \in \mathrm{ext}(\mathrm{Ball}(E|K)^*)$.

By Corollary 2.18 there exist $y \in K$ and $\lambda \in \mathbb{T}$ such that $\dot{\tau_x} = \lambda\tau_y|(E|K)$, where $\tau_y|(E|K) : E|K \to \mathbb{K}$ by $(\tau_y|(E|K))(F) = F(y)$ for each $F \in E|K$. It is well defined since $y \in K$. We have

$$(T^* \circ (\tau_y|(E|K)))(f) = (\tau_y|(E|K))(T(f)) = (\tau_y|(E|K))(f|K) = f(y) = \tau_y(f), \quad f \in E,$$

so

$$\tau_x(f) = (T^* \circ \dot{\tau_x})(f) = (T^* \circ (\lambda\tau_y)|(E|K)))(f) = \lambda\tau_y(f), \quad f \in E.$$

Thus $\tau_x = \lambda\tau_y$ on E. It follows that D_x and λD_y are representing measures for τ_x. By Proposition 2.25 we see that a representing measure for τ_x is only D_x since we assume E strongly separates the points of $\mathrm{Ch}(E)$. We conclude that $x = y$ and $\lambda = 1$. Thus $x \in K$, $\mathrm{Ch}(E) \subset K$, so the closure $\overline{\mathrm{Ch}(E)}$ of $\mathrm{Ch}(E)$ is a subset of K. As $\mathrm{Ch}(E)$ is a boundary for E (Proposition 2.20) so is $\overline{\mathrm{Ch}(E)}$. We conclude that $\overline{\mathrm{Ch}(E)}$ is the Šilov boundary. □

It is not always the case that the Šilov boundary exists. A simple example is as follows.

Example 2.33. Let $E = \{f \in C(\mathbb{T}, \mathbb{C}) : f(\lambda) = \lambda f(1),\ \lambda \in \mathbb{T}\}$. Then E is $\mathbb{C}$-linear subspace of $C(\mathbb{T}, \mathbb{C})$ which separates the points in $\mathbb{T}$. It is easy to see that $\mathrm{Ch}(E) = \mathbb{T}$. On the other hand $\{\lambda\}$ is a closed boundary for E, for each $\lambda \in \mathbb{T}$. Thus there is no smallest closed boundary for E.

Even if the Šilov boundary exists, it needs not coincide with the closure of the Choquet boundary.

Example 2.34. Let $X = [0,1] \cup \{2\}$. Let $E = \{f \in C(X, \mathbb{K}) : f(2) = -f(1)\}$. It is evident that $[0,1]$ is the Šilov boundary and $\mathrm{Ch}(E) = X$. Note that $\mathrm{Ch}(E)$ separates, but does not strongly separate, 1 and 2. Note also that $-D_1$ and D_2 are representing measures for τ_2

Corollary 2.35. *Let E be a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$ which separates the points of Y, If Y is compact and E contains constants, or E is a subalgebra of $C_0(Y, \mathbb{K})$, then the closure $\overline{\mathrm{Ch}(E)}$ of $\mathrm{Ch}(E)$ in Y is the Šilov boundary.*

Proof. If Y is compact and E contains constants, then E strongly separates the points of Y by Proposition 2.3. If E is a subalgebra of $C_0(Y, \mathbb{K})$, then by Proposition 2.3 asserts that E strongly separates the points of Y. Hence by Proposition 2.32 we have the conclusion. □

Note that the case of $1 \in E$ is described in [37, Proposition 6.4]. Note also that if Y is compact and E is a strongly separating space, then [43, Proposition 6] described the above corollary.

The following is a well known example that shows the Choquet boundary needs not to be closed even if the Šilov boundary exists.

Example 2.36. Let $A = \{f \in P(\bar{D}) : f(0) = f(1)\}|\mathbb{T}$, where $p(\bar{D})$ is the disk algebra on the closed unit disk $\bar{D}$. Then A is a uniform algebra on the unit circle $\mathbb{T}$. Then $\mathrm{Ch}(A) = \mathbb{T} \setminus \{1\}$ since every point $\lambda \in \mathbb{T} \setminus \{1\}$ is a peak point with the peaking function $(z + \lambda)/2$ while 1 is not a peak point by the maximum absolute value principle for analytic functions. Note that the Šilov boundary is $\mathbb{T}$.

§ 3. C-rich spaces, lush spaces and extremely C-regular spaces

§ 3.1. C-richness, lushness, the numerical index and the Mazur-Ulam property.

A C-rich subspace was introduced by Boyko, Kadets, Martín and Werner [11].

Definition 3.1 ([11])**.** A closed $\mathbb{K}$-linear subspace E of $C(X, \mathbb{K})$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, is called *C-rich* if for every nonempty open subset U of X and $\varepsilon > 0$, there exists a positive function h_ε of norm 1 with support inside U such that the distance from h_ε to E is less than ε.

Suppose that X is a compact Hausdorff space without isolated points. Suppose also that $p_1, \dots, p_n \in C(X, \mathbb{K})^*$. Then $E = \bigcap_{j=1}^n p_j^{-1}(0)$ is a C-rich subspace of $C(X, \mathbb{K})$ [11, Proposition 2.5]. Furthermore, if X is perfect, then every subspace of $C(X, \mathbb{K})$ of codimension finite is C-rich since in this case C-richness is equivalent to richness [29, Proposition 1.2]. Another example is a uniform algebra. Recall that a *uniform algebra* A on a compact Hausdorff space X if A is a closed subalgebra of $C(X, \mathbb{C})$ which contains constants and separates the points of X.

Proposition 3.2. *Let A be a uniform algebra on a compact Hausdorff space X and S the Šilov boundary. Then $A|S = \{f \in C(S, \mathbb{C}) \colon F = f \text{ on } S \text{ for some } F \in A\}$ is a C-rich $\mathbb{C}$-subspace of $C(S, \mathbb{C})$.*

Proof. Let U be an open subset of S and $\varepsilon_0 > 0$ arbitrary. We may suppose that U is a proper subset of S. Let $0 < \varepsilon < \min\{1/2, \varepsilon_0/(\sqrt{5} + 1)\}$. It is known that the Choquet boundary $\mathrm{Ch}(A)$ for A is dense in S and each point $x \in \mathrm{Ch}(A)$ is a strong boundary point [12] (cf. Theorem 2.29 and Corollary 2.35). Hence there exist $p \in U \cap \mathrm{Ch}(A)$ and $f \in A|S$ such that $f(p) = 1 = \|f\|_\infty$ and $|f| < 1$ on $S \setminus U$. Since A is closed under the multiplication, we may suppose that $|f| < 1/2$ on $S \setminus U$. Put

$$\Delta = \{z \in \bar{D} : \mathrm{Re}\, z \geq 0,\ |\mathrm{Im}\, z| \leq \varepsilon\}.$$

Then by the well known Carathéodory theorem (cf. [41]) there is a homeomorphism $\pi_\varepsilon : \bar{D} \to \Delta$ such that π_ε is analytic from D onto the interior of Δ. We may assume that $\pi_\varepsilon(1) = 1$ and $\pi_\varepsilon(\{z \in \bar{D} : |z| < 1/2\}) \subset \{z \in \Delta : \operatorname{Re} z < \varepsilon\}$. As π_ε is uniformly approximated by analytic polynomials on $\bar{D}$, we have $g = \pi_\varepsilon \circ f \in A|S$. Note that $0 \le \operatorname{Re} g \le 1$ on S. By Urysohn's lemma there exists a continuous function $h : S \to [0,1]$ such that

$$h(y) = \begin{cases} 0, & \text{if } \operatorname{Re} g(y) \le \varepsilon, \\ 1, & \text{if } \operatorname{Re} g(y) \ge 2\varepsilon. \end{cases}$$

If $y \in S \setminus U$, then $|f(y)| < 1/2$, so $\operatorname{Re} g(y) = \operatorname{Re}(\pi_\varepsilon \circ f)(y) < \varepsilon$. Thus $hg = 0$ on $S \setminus U$. As $g(p) = \pi_\varepsilon \circ f(p) = 1$ we have that $hg(p) = 1$. As $\|g\|_\infty = g(p) = 1 = h(p) = \|h\|_\infty$ we have $\|hg\|_\infty = 1$. We show that $\|hg - g\| \le \sqrt{5}\varepsilon$. Let $y \in S$. If $\operatorname{Re} g(y) \ge 2\varepsilon$, then $h(y) = 1$. Hence $(hg - g)(y) = 0$. Suppose that $\operatorname{Re} g(y) \le 2\varepsilon$. As $0 \le h(y) \le 1$, $|h(y) - 1| \le 1$. As $g(y) \in \Delta$ and $\operatorname{Re} g(y) \le 2\varepsilon$ we infer that $|g(y)| \le \sqrt{5}\varepsilon$. Hence

$$|(hg - g)(y)| = |g(y)||h(y) - 1| \le \sqrt{5}\varepsilon.$$

As $\operatorname{Re} g \ge 0$ on X, we have $0 \le h \operatorname{Re} g \le 1$. Put $h_0 = h \operatorname{Re} g$. Then h_0 is a positive function of norm 1 since $h_0(p) = 1$. As $h = 0$ on $S \setminus U$, we have $h_0 = 0$ on $S \setminus U$. Thus the support of h_0 is inside of U. We have

$$\|h_0 - g\|_\infty \le \|hg - g\|_\infty + \|h\|_\infty \|\operatorname{Re} g - g\|_\infty \le \sqrt{5}\varepsilon + \varepsilon < \varepsilon_0.$$

Thus $d(h_0, A|S) < \varepsilon_0$ $\square$

In the rest of the section $\overline{B}$ for a subset $B \in C_0(Y, \mathbb{K})$ denotes the uniform closure of B. In the same way as the proof of Proposition 3.2 we see the following.

Proposition 3.3. *Suppose that A is a uniform algebra on a compact Hausdorff space X and S its Šilov boundary. Then $\overline{\operatorname{Re} A|S}$ is a C-rich $\mathbb{R}$-subspace of $C(X, \mathbb{R})$.*

Proof. Let U be an open subset U of S and $\varepsilon_0 > 0$ arbitrary. In fact, a given U and $\varepsilon_0 > 0$, h_0 and $g \in A$ are the same functions as in the proof of Proposition 3.2 we have $\|h_0 - \operatorname{Re} g\|_\infty \le \varepsilon_0$. $\square$

A lush space was introduced by Boyko, Kadets, Martín and Werner [11].

Definition 3.4. Let B be a $\mathbb{K}$-Banach space. Let $\delta > 0$ and $p \in S(B^*)$. The *slice* denoted by $SL(\operatorname{Ball}(B), p, \delta)$ is

$$\{a \in \operatorname{Ball}(B) : \operatorname{Re} p(a) > 1 - \delta\}.$$

B is said to be $\mathbb{K}$-*lush* if for every $a, b \in S(B)$ and $\varepsilon > 0$, there exists $q \in S(B^*)$ such that $b \in SL(\mathrm{Ball}(B), q, \varepsilon)$ and

$$d(a, \mathrm{co}(\mathbb{T}SL(\mathrm{Ball}(B), q, \varepsilon))) < \varepsilon.$$

where $\mathrm{co}(\cdot)$ stands the convex hull.

Boyko, Kadets, Martín and Werner [11, Theorem 2.4] proved that a C-rich $\mathbb{K}$-subspace of $C(X, \mathbb{K})$ for a compact Hausdorff space X is $\mathbb{K}$-lush. Tan, Huang and Liu introduced a local-Gl-space in [50], which is a real Banach space. They proved that $\mathbb{R}$-lush space is a local-GL-space [50, Example 3.6] and every local-GL-space has the Mazur-Ulam property. Let us briefly recall that a real Banach space B is GL-*space* if for every $a \in S(B)$ and every $0 < \varepsilon < 1$ there exists a $p \in S(B^*)$ such that

$$d(b, SL(\mathrm{Ball}(B), p, \varepsilon)) + d(-b, SL(\mathrm{Ball}(B), p, \varepsilon)) < 2 + \varepsilon$$

for all $b \in S(B)$. A real Banach space B is said to be a *local* GL-*space* if for every separable subspace E of B , there exists a GL-subspace E' such that $E \subset E' \subset B$.

Corollary 3.5. *Every uniform algebra is $\mathbb{C}$-lush. The uniform closure of the real part of a uniform algebra is $\mathbb{R}$-lush and consequently it has the Mazur-Ulam property.*

Before proving Corollary 3.5 we show two lemmas to prove it.

Lemma 3.6. *Let A be a uniform algebra on a compact Hausdorff space X and S a boundary for A. Then*

$$\|\mathrm{Re}\, f\|_{\infty(X)} = \|\mathrm{Re}\, f\|_{\infty(S)}$$

for every $f \in A$.

Proof. Suppose that $\|\mathrm{Re}\, f_0\|_{\infty(X)} \neq |\mathrm{Re}\, f_0\|_{\infty(S)}$ for some $f_0 \in A$, whence we have $\|\mathrm{Re}\, f_0\|_{\infty(X)} > |\mathrm{Re}\, f_0\|_{\infty(S)}$. There exists $y_0 \in X$ such that $|\mathrm{Re}\, f_0(y_0)| = \|\mathrm{Re}\, f_0\|_{\infty(X)}$. We may assume $\mathrm{Re}\, f_0(y_0) > 0$. (If $\mathrm{Re}\, f_0(y_0) < 0$, then replace f_0 by $-f_0$.) Hence

$$\mathrm{Re}\, f_0(y_0) > \|\mathrm{Re}\, f_0\|_{\infty(S)},$$

and

$$\|\exp \mathrm{Re}\, f_0\|_{\infty(X)} \geq \exp \mathrm{Re}\, f_0(y_0) > \exp \|\mathrm{Re}\, f_0\|_{\infty(S)} \geq \|\exp \mathrm{Re}\, f_0\|_{\infty(S)}.$$

As $|\exp f_0(y)| = \exp \mathrm{Re}\, f_0(y)$ for every $y \in X$, we have $\|\exp f_0\|_{\infty(X)} = \|\exp \mathrm{Re}\, f_0\|_{\infty(X)}$ and $\|\exp f_0\|_{\infty(S)} = \|\exp \mathrm{Re}\, f_0\|_{\infty(S)}$. Hence we get

$$\|\exp f_0\|_{\infty(X)} > \|\exp f_0\|_{\infty(S)},$$

which is against that S is a boundary and $\exp f_0 \in A$. Thus we have that $\|\mathrm{Re}\, f\|_{\infty(X)} = \|\mathrm{Re}\, f\|_{\infty(S)}$ for every $f \in A$. □

Lemma 3.7. *Let A be a uniform algebra on a compact Hausdorff space X and S a boundary for A. Then we have $\overline{\mathrm{Re}\, A|S} = \overline{\mathrm{Re}\, A}|S$.*

Proof. Since the inclusion $\overline{\mathrm{Re}\, A|S} \supset \overline{\mathrm{Re}\, A}|S$ is obvious, we need to prove the reverse inclusion. Let $u \in \overline{\mathrm{Re}\, A|S}$ arbitrary. Then there exists a sequence $\{u_n\}$ in $\mathrm{Re}\, A|S$ such that $\|u_n - u\|_{\infty(S)} \to 0$ as $n \to \infty$. Then by the axion of choice there is a sequence $\{U_n\}$ in $\mathrm{Re}\, A$ such that $U_n|S = u_n$ for every positive integer n. Then we have by Lemma 3.6 that $\|u_n - u_m\|_{\infty(S)} = \|U_n - U_m\|_{\infty(X)}$ for every n and m. Hence $\{U_n\}$ is a Cauchy sequence since so is the sequence $\{u_n\}$. There is $U \in \overline{\mathrm{Re}\, A}$ such that $\|U_n - U\|_{\infty(X)} \to 0$ as $n \to \infty$. Then

$$\|u_n - U|S\|_{\infty(S)} \leq \|U_n - U\|_{\infty(X)} \to 0$$

as $n \to \infty$, so that $u = U|S \in \overline{\mathrm{Re}\, A}|S$. We conclude that $\overline{\mathrm{Re}\, A|S} \subset \overline{\mathrm{Re}\, A}|S$. □

Proof of Corollary 3.5 . Let A be a uniform algebra on X and S the Šilov boundary for A. Then by Proposition 3.2, $A|S$ is C-rich $\mathbb{C}$-subspace of $C(S, \mathbb{C})$. Then by [11, Theorem 2.4] we see that $A|S$ is $\mathbb{C}$-lush. Note that lushness is invariant under the isometries by definition of lushness. Therefore A is $\mathbb{C}$-lush since S is a closed boundary for A and the restriction map is obviously an isometry of A onto $A|S$ by definition of a boundary.

By Proposition 3.3 $\overline{\mathrm{Re}\, A|S}$ is C-rich. Then [11, Theorem 2.4] ensures that $\overline{\mathrm{Re}\, A|S}$ is $\mathbb{R}$-lush. As the Šilov boundary is a boundary, we have $\overline{\mathrm{Re}\, A|S} = \overline{\mathrm{Re}\, A}|S$ by Lemma 3.7. Hence $\overline{\mathrm{Re}\, A}|S$ is $\mathbb{R}$-lush. Consider the restriction map $I \colon \overline{\mathrm{Re}\, A} \to \overline{\mathrm{Re}\, A}|S$. As $\|\mathrm{Re}\, f\|_{\infty(X)} = \|\mathrm{Re}\, f\|_{\infty(S)}$ for every $f \in A$, the map I is a surjective isometry. Since lushness is invariant under the isometries, we see that $\overline{\mathrm{Re}\, A}$ is $\mathbb{R}$-lush. Tan, Huang and Liu proved that $\mathbb{R}$-lush space is a local-GL-space [50, Example 3.6] and every local-GL-space has the Mazur-Ulam property [50, Theorem 3.8]. Hence $\overline{\mathrm{Re}\, A}$ has the Mazur-Ulam property.

□

Recall that a uniform algebra A on a compact Hausdorff space X is a Dirichlet algebra provided that $\overline{\mathrm{Re}\, A} = C(X, \mathbb{R})$. Several uniform algebras including the disk algebra on the unit circle is a Dirichlet algebra. Uniform algebras needs not be Dirichlet in many cases. The ball algebra and the polydisk algebra on the ball and the polydisk of dimension 2 or greater are not Dirichlet algebras even on the Šilov boundaries respectively. For further information see [12, 25, 46]. Let

$$E = \{u \in C(\bar{D}, \mathbb{R}) \colon u \text{ is harmonic on } D\}.$$

By solving the Dirichlet problem, any real-valued continuous function on $\mathbb{T}$ is extended to a continuous function on $\bar{D}$ which is harmonic on D. Hence $E|\mathbb{T} = C(\mathbb{T}, \mathbb{R})$ and E is isometric to $E|\mathbb{T}$ since every function in E takes the maximum value and the minimum value on $\mathbb{T}$ as it is harmonic on D. By a theorem of Fang and Wang [23, Theorem 3.2] the Banach space $C(\mathbb{T},\mathbb{R})$ has the Mazur-Ulam property, hence E has the Mazur-Ulam property. This is also proved by Corollary 3.5 as follows. Let $A(\bar{D})$ be the disk algebra on the closed unit disk $\bar{D}$. It is trivial that $\operatorname{Re} A(\bar{D}) \subset E$. Since the uniform limit of a sequence of harmonic functions are harmonic, we infer that E is uniformly closed. Thus we have $\overline{\operatorname{Re} A(\bar{D})} \subset E$. Conversely suppose that $U \in E$. As the restriction $A(\bar{D})|\mathbb{T}$ (the disk algebra on the unit circle) is a Dirichlet algebra on $\mathbb{T}$. There is a sequence $\{U_n\}$ of functions in $\operatorname{Re} A(\bar{D})$ such that $\|U_n|\mathbb{T} - U|\mathbb{T}\|_{\infty(\mathbb{T})} \to 0$ as $n \to \infty$. As U and every U_n are harmonic on the open unit disk D, we have by the maximum value principle of harmonic functions that

$$\|U_n - U\|_{\infty(\bar{D})} = \|U_n|\mathbb{T} - U|\mathbb{T}\|_{\infty(\mathbb{T})} \to 0$$

as $n \to \infty$. We have proved that $U \in \overline{\operatorname{Re} A(\bar{D})}$. It follows that $\overline{\operatorname{Re} A(\bar{D})} = E$. By Corollary 3.5 we have that E has the Mazur-Ulam property. In general we have the following.

Corollary 3.8. *Suppose that $A(\Omega)$ is a uniform algebra on a compact subset Ω of the complex plane $\mathbb{C}$ which consists of complex-valued continuous functions on Ω which is analytic on the interior of Ω. Then every function in $\overline{\operatorname{Re} A(\Omega)}$ is harmonic on the interior of Ω. In particular, $\overline{\operatorname{Re} A(\Omega)}$ has the Mazur-Ulam property.*

Proof. The uniform limit of a sequence of harmonic functions is harmonic. Hence every function in $\overline{\operatorname{Re} A(\Omega)}$ is harmonic on the interior of Ω. By Corollary 3.5 we have the conclusion. □

It seems not to be known if a $\mathbb{C}$-lush space has the complex Mazur-Ulam property or not. We proved that a uniform algebra has the complex Mazur-Ulam property in [26].

According to [22] the numerical index of a Banach space was introduced by Lumer in 1968. For the algebra of all bounded linear operators $L(B)$ on a Banach space B, the numerical index is

$$n(B) = \inf\{\nu(\mathcal{A}) : \mathcal{A} \in L(B), \|\mathcal{A}\| = 1\},$$

where $\nu(\mathcal{A})$ is the numerical radius given by

$$\nu(\mathcal{A}) = \sup\{|p(\mathcal{A}(a))| : a \in S(B), p \in S(B^*), p(a) = 1\}.$$

Boyko, Kadets, Martín and Werner [11, Proposition 2.2] showed that the numetrical index of a lush space is 1. Hence we see that

Corollary 3.9. *Let A be a uniform algebra. Then $n(A) = n(\overline{\mathrm{Re}\, A}) = 1$. Let $E = \{u \in C(\bar{D}, \mathbb{R})\colon u$ is harmonic on $D\}$. Then $n(E) = 1$.*

Proof. By Corollary 3.5 A is $\mathbb{C}$-lush and $\overline{\mathrm{Re}\, A}$ is $\mathbb{R}$-lush. By Proposition 2.2 in [11] we have the conclusion.

As we have shown (just before Corollary 3.8) that $E = \overline{\mathrm{Re}\, A(\bar{D})}$, where $A(\bar{D})$ is the disk algebra on the closed unit disk. Hence we have $n(E) = 1$. □

§ 3.2. Extremely regular spaces and extremely C-regular spaces.

The concept of an extremely regular space was given by Cengiz [16]. Extremely C-regular spaces were introduced by Fleming and Jamison [24, Definition 2.3.9].

Definition 3.10. A $\mathbb{K}$-linear subspace E of $C_0(Y, \mathbb{K})$ is said to be *extremely C-regular (resp. regular)* if for each x in the Choquet boundary[1] $\mathrm{Ch}(E)$ (resp. $x \in Y$) satisfies the condition that for each $\varepsilon > 0$ and each open neighborhood U of x, there exists $f \in E$ such that $f(x) = 1 = \|f\|_\infty$, and $|f| < \varepsilon$ on $Y \setminus U$.

We may say that a $\mathbb{K}$-linear subspace E of $C_0(Y, \mathbb{K})$ is extremely C-regular if every point in the Choquet boundary is a *strong boundary point in the sense of Fleming and Jamison.*

Suppose that m is a complex regular Borel continuous measure on Y. Then $E = \{f \in C_0(Y, \mathbb{C}) : \int f dm = 0\}$ is an extremely regular closed subspace of $C_0(Y, \mathbb{C})$ (see [16, Theorem]).

Theorem 3.11. *Suppose that E is a uniformly closed extremely C-regular $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. The following are equivalent.*

(i) *$x \in \mathrm{Ch}(E)$,*

(ii) *x is a strong boundary point for E,*

(iii) *the representing measure for the point evaluation τ_x on E is only D_x,*

Proof. Since E is extremely C-regular, E strongly separates the points in $\mathrm{Ch}(E)$. By Corollary 2.26 we have that (i) $\leftrightarrow$ (iii). The implication (i) $\rightarrow$ (ii) also follows from the definition of the extremely C-regularity. Suppose that (ii) holds. Then by Corollary 2.23 (i) holds. □

Abrahamsen, Nygaard and Põldvere [1] introduced a *somewhat regular subspaces* of $C_0(Y, \mathbb{K})$, which is a generalization of extremely regular subspaces.

[1] The definition is given in Definition 2.16

Definition 3.12 (Definition 2.1 in [1]). We call a $\mathbb{K}$-linear subspace E of $C_0(Y,\mathbb{K})$ *somewhat regular*, if for every non-empty open subset $V$of Y and $0<\varepsilon$, there exists $f\in E$ such that there exists $x_0\in V$ with $f(x_0)=1=\|f\|_\infty$ and $|f|\le\varepsilon$ on $Y\setminus V$.

Proposition 3.13. *A closed subalgebra A of $C_0(Y,\mathbb{C})$ which separates the points of Y and has no common zeros is an extremely C-regular subspace of $C_0(Y,\mathbb{C})$. In particular, a uniform algebra on a compact Hausdorff space X is an extremely C-regular subspace of $C(X,\mathbb{C})$. If the Choquet boundary $\mathrm{Ch}(A)$ is closed in Y, then $A|\mathrm{Ch}(A)$ is an extremely regular subspace of $C_0(\mathrm{Ch}(A),\mathbb{C})$. Let S be the Šilov boundary for A. Then $A|S$ is a somewhat regular subspace of $C_0(S,\mathbb{C})$.*

Proof. Let $x\in\mathrm{Ch}(A)$. Suppose that U is an open neighborhood of x. Letting $\beta=1$ and $\varepsilon=\alpha$ for (v) of Theorem 2.29 we assert that there exists $g\in A$ such that $|g(x)|=1=\|g\|_\infty$ and $|g|<\epsilon$ on $Y\setminus U$. Then $f=\overline{g(x)}g$ is the required function which proves that A is extremely C-regular. If $\mathrm{Ch}(A)$ is closed, then by the definition of extreme regularity, we have that $A|\mathrm{Ch}(A)$ is an extremely regular subspace.

Let S be a Šilov boundary. We prove that $A|S$ is somewhat regular. Let V be a non-empty subset of S and $1>\varepsilon>0$ arbitrary. By Corollary 2.35 there exists $x_0\in\mathrm{Ch}(A)\cap V$. Then there exists $f\in A|S$ such that $f(x_0)=1=\|f_0\|_\infty$ and $|f|\le\varepsilon$. Thus $A|S$ is somewhat regular. $\square$

Proposition 3.14. *Let A be a uniform algebra on a compact Hausdorff space X. Then the space of the real parts $\mathrm{Re}\,A$ of A is an extremely C-regular subspace of $C_{\mathbb{R}}(X)$.*

Proof. We prove that $\mathrm{Ch}(\mathrm{Re}\,A)=\mathrm{Ch}(A)$. Let $p\in\mathrm{Ch}(A)$. As p is a strong boundary point (see Theorem 2.29 and preceding comments), for every open neighborhood of p and $\varepsilon>0$ there exists a function $f\in A$ such that

$$f(p)=1=\|f\|_\infty \text{ and } |f|<\varepsilon \text{ on } X\setminus U$$

We asserts that

(3.1)

$$\mathrm{Re}\,f(p)=1\le\|\mathrm{Re}\,f\|_\infty\le\|f\|_\infty=1 \text{ and } |\mathrm{Re}\,f(x)|\le|f(x)|<\varepsilon \text{ for every } x\in X\setminus U$$

Thus p is a strong boundary point and by Corollary 2.23 we infer that $p\in\mathrm{Ch}(\mathrm{Re}\,A)$.

Suppose conversely that $p\in X\setminus\mathrm{Ch}(A)$. Then there is a representing measure $\mu\ne D_p$ on X for the point evaluation τ_p. Note that μ is a probability measure (cf. [12, p.81]). Then we have

$$\int\mathrm{Re}\,g d\mu=\mathrm{Re}\int g d\mu=\mathrm{Re}\,g(p)$$

for all $g \in A$. It means that μ is a representing measure for τ_p which is not D_p. By Corollary 2.27 we have that $p \in X \setminus \mathrm{Ch}(\mathrm{Re}\, A)$.

By (3.1) we see that $\mathrm{Re}\, A$ is an extremely C-regular subspace of $C_{\mathbb{R}}(X)$. □

Proposition 3.15. *Suppose that E is an extremely C-regular subspace of $C_0(Y, \mathbb{K})$ for a locally compact Hausdorff space Y. Then the uniformly closure $\bar{E}$ of E is also an extremely C-regular subspace of $C_0(Y, \mathbb{K})$.*

Proof. Every $\phi \in (E, \|\cdot\|_\infty)^*$ is uniquely extended to $\bar{\phi} \in (\bar{E}, \|\cdot\|_\infty)^*$ with $\|\phi\| = \|\bar{\phi}\|$. Then the map $\phi \to \bar{\phi}$ from $(E, \|\cdot\|_\infty)^*$ onto $(\bar{E}, \|\cdot\|_\infty)^*$ is a surjective $\mathbb{K}$-linear isometry. Since $\bar{\tau}_y$ for $\tau_y \in (E, \|\cdot\|_\infty)^*$ is $\tau_y \in (\bar{E}, \|\cdot\|_\infty)^*$, we see that τ_y is in $\mathrm{ext}(\mathrm{Ball}((E, \|\cdot\|_\infty)^*))$ if and only if τ_y is in $\mathrm{ext}(\mathrm{Ball}((\bar{E}, \|\cdot\|_\infty)^*))$ for $y \in Y$. Hence $\mathrm{Ch}(E) = \mathrm{Ch}(\bar{E})$. The rest of the proof is clear. □

Applying Propositions 3.14 and 3.15 we have

Corollary 3.16. *For a uniform algebra A on a compact Hausdorff space X, the uniform closure $\overline{\mathrm{Re}\, A}$ of the real parts $\mathrm{Re}\, A$ of A is an extremely C-regular subspace of $C_{\mathbb{R}}(X)$.*

§ 3.3. Some properties of closed subalgebras of $C_0(Y, \mathbb{C})$.

In this subsection Y is an *infinite* locally compact Hausdorff space. Abrahamsen, Nygaard and Põldvere [1] showed that extremely regular spaces play a role in recent theory of Banach spaces by exhibiting that they involve the Daugavet property, the symmetric strong diameter 2 property and so on under some additional assumptions. We say B has the *symmetric strong diameter 2 property* (SSD2P) if for every $\varepsilon > 0$ and every finite collection of slices $S_1, \ldots, S_m$, there exists $x_i \in S_i$ for $i = 1, \ldots, m$ and $y \in \mathrm{Ball}(B)$ with $\|y\| > 1 - \varepsilon$ and $x_i \pm y \in S_i$ for all $i = 1, \ldots, m$ [1, Definition 1.3]. Recall that a Banach space is *almost square* (ASQ) if for any finite number of $x_1, \ldots, x_n \in S(B)$, there exists a sequence $\{y_k\} \subset \mathrm{Ball}(B)$ such that $\|x_i \pm y_k\| \to 1$ and $\|y_k\| \to 1$ as $k \to \infty$ for all $1 \leq j \leq n$ [1, definition 1.3]. Recall that a Banach space B has the *Daugavet property* if every rank-one operator $\mathcal{A}$ on B satisfies that $\|1 + \mathcal{A}\| = 1 + \|\mathcal{A}\|$ [1, Definition 1.4]. Recall that a linear surjection $\mathcal{T} : N_1 \to N_2$ for normed linear space N_1 and N_2 is called an ε-isometry if

$$(1 - \varepsilon)\|a\| \leq \|\mathcal{T}(a)\| \leq (1 + \varepsilon)\|a\|$$

for every $a \in N_1$ [1]. We have

Corollary 3.17. *Let A be a closed subalgebra of $C_0(Y, \mathbb{C})$ which separates the points of Y and has no common zeros. Then we have*

(1) *A has the SSD2P,*

(2) *A is ASQ if the Šilov boundary is non-compact,*

(3) *A has the Daugavet property if the Šilov boundary of A is perfect,*

(4) *A contains an ε-isometric copy of c_0 if $0 < \varepsilon < 1$.*

Proof. Put $A_0 = A|S$, where S is the Šilov boundary. Then the restriction map is a surjective isometry from A onto A_0. The SSD2P, ASQ, the Daugavet property and to contain an ε-isometric copy of c_0 are inherited by an isometry, so it is enough to prove results for A_0.

By Proposition 3.13 that A_0 is a somewhat regular. Then by [1, Theorems 2.2, 2.5, 2.6, 3.1] we have the conclusions. □

Note that Wojtaszczyk [57, Theorem 2] proved that the Daugavet equation (DE) : $\|1 + \mathcal{A}\| = 1 + \|\mathcal{A}\|$ holds for a weakly compact operator $\mathcal{A}$ on a uniform algebra on X such that the strong boundary points are dense in X and X has no isolated points. As the strong boundary point coincides with the Choquet boundary points (Theorem 2.29) and they are dense in the Šilov boundary (Proposition 2.35), the hypothesis on the uniform algebra in the theorem of Wojtaszczyk can be seen that A is a uniform algebra on a perfect X, where X is the Šilov boundary for A. As A and $A|S$ are isometric, where A is a uniform algebra and S is the Šilov boundary for A, Wojtaszczyk in fact proved that the Daugavet equation holds for a weakly compact operators on a uniform algebra of which Šilov boundary is perfect.

At the end of the section we note that C-richness implies the somewhat regularity.

Proposition 3.18. *Let E be a $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Suppose that for every nonempty open subset U of Y and $\varepsilon > 0$, there exists a function $h_\varepsilon \in C_0(Y, \mathbb{R})$ such that $0 \le h_\varepsilon \le 1 = \|h\|_\infty$ with support inside U and that the $d(h_\varepsilon, E) < \varepsilon$. Then E is somewhat regular. We also have*

(1) *E has the SSD2P,*

(2) *E is ASQ if Y is non-compact,*

(3) *E has the Daugavet property if Y is perfect,*

(4) *E contains an ε-isometric copy of c_0 if $0 < \varepsilon < 1$.*

In particular, if Y is compact (hence E is C-rich), then E is somewhat regular and (1), (3) and (4) hold.

Proof. Let U be a nonempty open subset of X and $\varepsilon > 0$ arbitrary. We prove that there exists $f \in E$ and $x_0 \in U$ such that $f(x_0) = 1 = \|f\|$ and $|f| \leq \varepsilon$. To prove it we may assume that $\varepsilon < 1$. Put $\varepsilon_0 = \varepsilon/(1+\varepsilon)$. Then there exists $h \in C_0(Y, \mathbb{R})$ such that $0 \leq h \leq 1$ on Y, $h = 0$ on $Y \setminus U$, $\|h\|_\infty = 1$, and the distance between h and E is less than ε_0. Hence there exists $g \in E$ such that $\|h - g\|_\infty < \varepsilon_0$. Then $\|g\|_\infty > \|h\|_\infty - \varepsilon_0 = 1 - \varepsilon_0$. As $h = 0$ on $Y \setminus U$, we infer that $|g| < \varepsilon_0$ on $Y \setminus U$. Choose $x_0 \in Y$ so that $\|g\|_\infty = |g(x_0)|$. Put $f = g/g(x_0)$. Then $f \in E$, $f(x_0) = 1 = \|f\|_\infty$ and $|f| < \varepsilon_0/(1-\varepsilon_0) = \varepsilon$ on $Y \setminus U$. As U and ε are arbitrary, we have that E is somewhat regular. Then by [1, Theorems 2.2, 2.6, 3.1] we have the conclusion. □

§ 4. Sets of representatives

§ 4.1. Is the homogeneous extension linear?

Let B be a real or complex Banach space. Any singleton $\{a\}$ of $a \in S(B)$ is convex subset of $S(B)$. Applying Zorn's lemma, there exists a maximal convex subset of $S(B)$ which contains $\{a\}$. Hence $S(B)$ is a union of all maximal convex subsets of $S(B)$. We denote the set of all maximal convex subsets of $S(B)$ by $\mathfrak{F}_B$. Suppose that the map $T : S(B_1) \to S(B_2)$ is a surjective isometry with respect to the metric induced by the norm, where B_1 and B_2 are both real Banach spaces or both complex Banach spaces. The homogeneous extension $\widetilde{T} : B_1 \to B_2$ of T is defined as

$$\widetilde{T}(a) = \begin{cases} \|a\| T\left(\frac{a}{\|a\|}\right), & 0 \neq a \in B_1 \\ 0, & a = 0. \end{cases}$$

By the definition $\widetilde{T}$ is a bijection which satisfies $\|\widetilde{T}(a)\| = \|a\|$ for every $a \in B_1$ and it is positively homogeneous. The Tingley's problem asks if $\widetilde{T}$ is real-linear or not.

In [26] we introduced *a set of representatives* which plays a role in study on complex Mazur-Ulam property. For the convenience of the readers we recall it here. Suppose that $F \in \mathfrak{F}_B$ for a real or complex Banach space B. It is well known that there exists an extreme point p in the closed unit ball $\mathrm{Ball}(B^*)$ of the dual space B^* of B such that $F = p^{-1}(1) \cap S(B)$ (cf. [52, Lemma 3.3], [27, Lemma 3.1]). Recall that $\mathrm{ext}(\mathrm{Ball}(B^*))$ denotes the set of all extreme points of $\mathrm{Ball}(B^*)$. Put

$$Q = \{q \in \mathrm{ext}(\mathrm{Ball}(B^*)) : q^{-1}(1) \cap S(B) \in \mathfrak{F}_B\}.$$

We define an equivalence relation $\sim$ in Q. Recall that we write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ if B is a complex Banach space, where $\mathbb{C}$ denotes the space of all complex numbers, and $\mathbb{T} = \{\pm 1\}$ if B is a real Banach space.

In Definition 4.1 through Definition 4.4 B is a real or complex Banach space.

Definition 4.1 (Definition 2.1 in [26]). Let $p_1, p_2 \in Q$. We denote $p_1 \sim p_2$ if there exists $\gamma \in \mathbb{T}$ such that $p_1^{-1}(1) \cap S(B) = (\gamma p_2)^{-1}(1) \cap S(B)$.

Note that $\gamma p \in Q$ if $\gamma \in \mathbb{T}$ and $p \in Q$. It is a routine argument to show that the binary relation $\sim$ is an equivalence relation on Q.

Definition 4.2 (Definition 2.3 in [26]). A set of all representatives with respect to the equivalence relation $\sim$ is simply called a set of representatives for $\mathfrak{F}_B$.

Note that a set of representatives exists due to the axiom of choice. Note also that a set of representatives P for $\mathfrak{F}_B$ is a norming family for B in the sense that $\|a\| = \sup_{p \in P} |p(a)|$ for $a \in B$. Hence it is a uniqueness set for B.

Lemma 4.3 (Lemma 2.5 in [26]). *Let P be a set of representatives for $\mathfrak{F}_B$. For $F \in \mathfrak{F}_B$ there exists a unique $(p, \lambda) \in P \times \mathbb{T}$ such that $F = \{a \in S(B) : p(a) = \lambda\}$. Conversely, for $(p, \lambda) \in P \times \mathbb{T}$ we have $\{a \in S(B) : p(a) = \lambda\}$ is in $\mathfrak{F}_B$.*

For each set of representatives P, Lemma 4.3 gives a bijective correspondence between $\mathfrak{F}_B$ and $P \times \mathbb{T}$.

Definition 4.4 (Definition 2.6 in [26]). For $(q, \lambda) \in Q \times \mathbb{T}$, we denote $F_{q,\lambda} = \{a \in S(B) : q(a) = \lambda\}$. A map

$$I_B : \mathfrak{F}_B \to P \times \mathbb{T}$$

is defined by $I_B(F) = (p, \lambda)$ for $F = F_{p,\lambda} \in \mathfrak{F}_B$.

By Lemma 4.3 the map I_B is well defined and bijective. An important theorem of Cheng, Dong and Tanaka states that a surjective isometry between the unit spheres of Banach spaces preserves maximal convex subsets of the unit spheres. This was first exhibited by Cheng and Dong in [15, Lemma 5.1] and a complete proof was given by Tanaka [51, Lemma 3.5].

In the following $T : S(B_1) \to S(B_2)$ is a surjective isometry between both real Banach spaces or both complex Banach spaces B_1 and B_2. We denote by P_j a set of representatives for $\mathfrak{F}_{B_j}$ for $j = 1, 2$. Applying the theorem of Cheng, Dong and Tanaka, a bijection $\mathfrak{T} : \mathfrak{F}_{B_1} \to \mathfrak{F}_{B_2}$ is well defined.

Definition 4.5 (Definition 2.7 in [26]). The map $\mathfrak{T} : \mathfrak{F}_{B_1} \to \mathfrak{F}_{B_2}$ is defined by $\mathfrak{T}(F) = T(F)$ for $F \in \mathfrak{F}_{B_1}$. The map $\mathfrak{T}$ is well defined and bijective. Put

$$\Psi = I_{B_2} \circ \mathfrak{T} \circ I_{B_1}^{-1} : P_1 \times \mathbb{T} \to P_2 \times \mathbb{T}.$$

Define two maps

$$\phi : P_1 \times \mathbb{T} \to P_2$$

and

$$\tau : P_1 \times \mathbb{T} \to \mathbb{T}$$

by

$$\Psi(p,\lambda) = (\phi(p,\lambda), \tau(p,\lambda)),\ (p,\lambda) \in P_1 \times \mathbb{T}. \tag{4.1}$$

If $\phi(p,\lambda) = \phi(p,\lambda')$ for every $p \in P_1$ and $\lambda, \lambda' \in \mathbb{T}$ we simply write $\phi(p)$ instead of $\phi(p,\lambda)$.

An equivalent form of (4.1) is as follows:

$$T(F_{p,\lambda}) = F_{\phi(p,\lambda),\tau(p,\lambda)}, \quad (p,\lambda) \in P_1 \times \mathbb{T}. \tag{4.2}$$

Note that

$$\phi(p,-\lambda) = \phi(p,\lambda),\ \tau(p,-\lambda) = -\tau(p,\lambda) \tag{4.3}$$

for every $(p,\lambda) \in P_1 \times \mathbb{T}$ (cf. [26]). The reason is as follows. First it is well known that $T(-F) = -T(F)$ for every $F \in \mathfrak{F}_{B_1}$ (cf. [34, Proposition 2.3]). Hence

$$\begin{aligned} F_{\phi(p,-\lambda),\tau(p,-\lambda)} = T(F_{p,-\lambda}) = T(-F_{p,\lambda}) \\ = -T(F_{p,\lambda}) = -F_{\phi(p,\lambda),\tau(p,\lambda)} = F_{\phi(p,\lambda),-\tau(p,\lambda)} \end{aligned}$$

for every $p \in P_1$ since $F_{p,-\lambda} = -F_{p,\lambda}$ by the definition of $F_{p,\lambda}$. Since the map I_{B_2} is a bijection we have (4.3).

Rewriting (4.2) we get an essential equation in our argument.

$$\phi(p,\lambda)(T(a)) = \tau(p,\lambda), \quad a \in F_{p,\lambda}. \tag{4.4}$$

Looking at this equation we will prove that the homogeneous extension $\widetilde{T}$ of T is real-linear. Before describing a precise argument in the later subsections, we exhibit a rough picture of the argument. Under the Hausdorff distance condition which will be given in Definition 4.6, we have

$$\phi(p,\lambda) = \phi(p,\lambda'), \quad p \in P_1$$

for every λ and λ' in $\mathbb{T}$, and

$$\tau(p,\lambda) = \tau(p,1) \times \begin{cases} \lambda, & \text{for some } p \in P_1 \\ \overline{\lambda}, & \text{for other } p\text{'s} \end{cases}$$

for $\lambda \in \mathbb{T}$. We get from (4.4), under the Hausdorff distance condition on B_1, that

$$\phi(p)(T(a)) = \tau(p,1) \times \begin{cases} p(a), & \text{for some } p \in P_1 \\ \overline{p(a)}, & \text{for other } p\text{'s} \end{cases} \tag{4.5}$$

for $a \in F_{p,p(a)}$. We emphasize that (4.5) holds for $a \in S(B_1)$ with $|p(a)| = 1$. It is crucial to prove (4.5) for all $a \in S(B_1)$. If the equation (4.5) holds for any $a \in S(B_1)$ without the restriction that $a \in F_{p,p(a)}$, then applying the definition of $\widetilde{T}$ we get

$$\phi(p)(\widetilde{T}(a)) = \tau(p,1) \times \begin{cases} p(a), & \text{for some } p \in P_1 \\ \overline{p(a)}, & \text{for other } p\text{'s} \end{cases} \tag{4.6}$$

for *every* $a \in B_1$, from which we infer that

$$\phi(p)(\widetilde{T}(a+rb)) = \phi(p)(\widetilde{T}(a)) + \phi(p)(r\widetilde{T}(b))$$

for every pair $a, b \in B_1$ and every real number r. By a further consideration, we will conclude that $\widetilde{T}$ is real-linear. It means that we will arrive at the final positive solution for Tingley's problem if (4.5) holds for all $a \in S(B_1)$. We will apply the version of additive Bishop's lemma (Proposition 5.5) to prove the equation (4.5) for any $a \in S(B_1)$.

§ 4.2. The Hausdorff distance condition.

Recall that the Hausdorff distance $d_H(K, L)$ between non-empty closed subsets K and L of a metric space with metric $d(\cdot,\cdot)$ is defined by

$$d_H(K, L) = \max\{\sup_{a\in K} d(a, L), \sup_{b\in L} d(b, K)\}.$$

Definition 4.6 (Definition 3.2 in [26]). Let B be a complex Banach space and P a set of representatives for $\mathfrak{F}_B$. We say that B satisfies the Hausdorff distance condition if the equality

$$d_H(F_{p,\lambda}, F_{p',\lambda'}) = 2$$

holds for every pair (p, λ) and (p', λ') in $P \times \mathbb{T}$ such that $p \neq p'$.

By Lemma 3.1 in [26], $d_H(F_{p,\lambda}, F_{p'\lambda'}) = 2$ provided that $p \neq p'$ and $F_{p,\lambda} \cap F_{p',-\lambda'} \neq \emptyset$. We can formulate the notion of the condition of the Hausdorff distance in terms of Q.

Lemma 4.7. *A complex Banach space B satisfies the Hausdorff distance condition if and only if $d_H(F_{q,\lambda}, F_{q',\lambda'}) = 2$ for every pair q and q' of Q with $q \not\sim q'$.*

A proof is a routine argument and is omitted.

Lemma 4.8 (Lemma 3.4 in [26]). *Let B_j be a complex Banach space for $j = 1, 2$ and $T : S(B_1) \to S(B_2)$ a surjective isometry. Suppose that B_1 satisfies the Hausdorff* *distance condition. Let P_1 be a set of representatives for $\mathfrak{F}_{B_1}$. Then we have $\phi(p,\lambda) = \phi(p,\lambda')$ for every $p \in P_1$ and $\lambda, \lambda' \in \mathbb{T}$. Put*

$$P_1^+ = \{p \in P_1 : \tau(p,i) = i\tau(p,1)\}$$

and

$$P_1^- = \{p \in P_1 : \tau(p,i) = \bar{i}\tau(p,1)\}.$$

Then P_1^+ and P_1^- are possibly empty disjoint subsets of P_1 such that $P_1^+ \cup P_1^- = P_1$. Furthermore we have

$$\tau(p,\lambda) = \lambda\tau(p,1), \quad p \in P_1^+, \lambda \in \mathbb{T}$$

and

$$\tau(p,\lambda) = \bar{\lambda}\tau(p,1), \quad p \in P_1^-, \lambda \in \mathbb{T}.$$

Proof. See the proof of [26, Lemma 3.4] □

§ 4.3. The set $M_{p,\alpha}$ and the Mazur-Ulam property

We exhibit the definition of $M_{p,\alpha}$ for a real or complex Banach space. The case of a complex Banach space is in [26, Definition 4.1]. We denote $\bar{\mathbb{D}} = \{z \in \mathbb{K} : |z| \leq 1\}$, where $\mathbb{K} = \mathbb{R}$ if the corresponding Banach space is a real one and $\mathbb{K} = \mathbb{C}$ if the corresponding Banach space is a complex one.

Definition 4.9. Let B be a real or complex Banach space and P a set of representatives for $\mathfrak{F}_B$. For $p \in P$ and $\alpha \in \bar{\mathbb{D}}$ we denote

$$M_{p,\alpha} = \{a \in S(B) : d(a, F_{p,\alpha/|\alpha|}) \leq 1 - |\alpha|, d(a, F_{p,-\alpha/|\alpha|}) \leq 1 + |\alpha|\},$$

where we read $\alpha/|\alpha| = 1$ if $\alpha = 0$.

Lemma 4.10 (cf. Lemma 4.2 in [26])**.** *Suppose that B_j is a real or complex Banach space for $j = 1, 2$, and $T : S(B_1) \to S(B_2)$ is a surjective isometry.*

If B_j is a real Banach space for $j = 1, 2$, then we have

$$T(M_{p,\alpha}) = \tau(p,1)M_{\phi(p),\alpha}$$

for every $(p,\alpha) \in P_1 \times \mathbb{T}$.

If B_j is a complex Banach space $j = 1, 2$ and B_1 satisfies the Hausdorff distance condition, then we have

$$T(M_{p,\alpha}) = \begin{cases} \tau(p,1)M_{\phi(p),\alpha}, & p \in P_1^+ \\ \tau(p,1)M_{\phi(p),\overline{\alpha}}, & p \in P_1^- \end{cases}$$

for every $(p,\alpha) \in P_1 \times \mathbb{T}$. Here P_1^+ and P_1^- are defined as in Lemma 4.8.

Proof. According to the definition of the map Ψ we have

$$T(F_{p,\frac{\alpha}{|\alpha|}}) = F_{\phi(p,\frac{\alpha}{|\alpha|}),\tau(p,\frac{\alpha}{|\alpha|})}$$

and

$$T(F_{p,-\frac{\alpha}{|\alpha|}}) = F_{\phi(p,-\frac{\alpha}{|\alpha|}),\tau(p,-\frac{\alpha}{|\alpha|})}$$

Suppose that B_j is a real Banach space. Then by the definition $\mathbb{T} = \{\pm 1\}$. By (4.3) we have $\phi(p,1) = \phi(p,-1)$ for every $p \in P_1$. Hence $\phi(p,\lambda)$ does not depend on the second term for a real Banach space. We also have $\tau(p,-1) = -\tau(p,1)$ for every $p \in P_1$ by (4.3). It follows that

$$T(F_{p,\frac{\alpha}{|\alpha|}}) = F_{\phi(p),\frac{\alpha}{|\alpha|}\tau(p,1)} = \tau(p,1)F_{\phi(p),\frac{\alpha}{|\alpha|}}$$

and

$$T(F_{p,-\frac{\alpha}{|\alpha|}}) = F_{\phi(p),-\frac{\alpha}{|\alpha|}\tau(p,1)} = \tau(p,1)F_{\phi(p),-\frac{\alpha}{|\alpha|}}$$

As T is a surjective isometry we have

$$\begin{aligned} d(a,F_{p,\frac{\alpha}{|\alpha|}}) &= d(T(a),F_{\phi(p,\frac{\alpha}{|\alpha|}),\tau(p,\frac{\alpha}{|\alpha|})}) = d(T(a),F_{\phi(p),\frac{\alpha}{|\alpha|}\tau(p,1)}) \\ &= d(T(a),\tau(p,1)F_{\phi(p),\frac{\alpha}{|\alpha|}}) = d(\tau(p,1)T(a),F_{\phi(p),\frac{\alpha}{|\alpha|}}) \end{aligned}$$

and

$$\begin{aligned} d(a,F_{p,-\frac{\alpha}{|\alpha|}}) &= d(T(a),F_{\phi(p,-\frac{\alpha}{|\alpha|}),\tau(p,-\frac{\alpha}{|\alpha|})}) = d(T(a),F_{\phi(p),-\frac{\alpha}{|\alpha|}\tau(p,1)}) \\ &= d(T(a),\tau(p,1)F_{\phi(p),-\frac{\alpha}{|\alpha|}}) = d(\tau(p,1)T(a),F_{\phi(p),-\frac{\alpha}{|\alpha|}}). \end{aligned}$$

As T is a bijection we conclude that

$$\tau(p,1)T(M_{p,\alpha}) = M_{\phi(p),\alpha}$$

for every $p \in P_1$ and $\alpha \in \bar{\mathbb{D}}$, so

$$T(M_{p,\alpha}) = \tau(p,1)M_{\phi(p),\alpha}$$

for every $p \in P_1$ and $\alpha \in \bar{\mathbb{D}}$

A proof for the case where B_j is a complex Banach space is in [26, Proof of Lemma 4.2]. □

4.3.1. A sufficient condition for the Mazur-Ulam property : the case of a real Banach space.

Proposition 4.11. *Let B be a real Banach space and P a set of representatives for $\mathfrak{F}_B$. Suppose that*

$$M_{p,\alpha} = \{a \in S(B) : p(a) = \alpha\} \tag{4.7}$$

for every $p \in P$ and $-1 \leq \alpha \leq 1$. Then B has the Mazur-Ulam property.

Proof. Let B_2 be a real Banach space and $T : S(B_1) \to S(B_2)$ a surjective isometry. We first prove the following equation (4.8) for every $p \in P_1$ and $a \in S(B_1)$ without assuming that $|p(a)| = 1$;

$$\phi(p)(T(a)) = \tau(p, 1)p(a) \tag{4.8}$$

for every $p \in P_1$ and $a \in S(B_1)$ with $|p(a)| \leq 1$. Let $p \in P_1$ and $a \in S(B_1)$. Put $\alpha = p(a)$. Then by (4.7) $a \in M_{p,\alpha}$. We have by Lemma 4.10 that

$$\phi(p)(T(a)) = \alpha\tau(p, 1) = \tau(p, 1)p(a).$$

It follows that for the homogeneous extension $\widetilde{T}$ of T we have

$$\phi(p)(\widetilde{T}(c)) = \phi(p)\left(\|c\|T\left(\frac{c}{\|c\|}\right)\right) = \|c\|\tau(p, 1)p\left(\frac{c}{\|c\|}\right) = \tau(p, 1)p(c)$$

for every $0 \neq c \in B_1$. As the equality $\phi(p)(\widetilde{T}(0)) = \tau(p, 1)p(0)$ holds, we obtain for $a, b \in B_1$ and a real number r that

$$\phi(p)(\widetilde{T}(a + rb) = \tau(p, 1)p(a + rb) = \tau(p, 1)p(a) + r\tau(p, 1)p(b)$$

and

$$\phi(p)(\widetilde{T}(a) + r\widetilde{T}(b)) = \phi(p)(\widetilde{T}(a)) + r\phi(p)(\widetilde{T}(b)) = \tau(p, 1)p(a) + r\tau(p, 1)p(b).$$

It follows that

$$\phi(p)(\widetilde{T}(a + rb)) = \phi(p)(\widetilde{T}(a) + r\widetilde{T}(b))$$

for every $p \in P_1$, $a, b \in B_1$, and every real number r. As $\phi(P_1) = P_2$ is a norming family we see that $\widetilde{T}$ is real-linear on B_1. As the homogeneous extension is a norm-preserving bijection as is described in the subsection 4.1 we complete the proof. □

4.3.2. A sufficient condition for the complex Mazur-Ulam property : the case of a complex Banach space.

The case of a complex Banach space is exhibited in Proposition 4.4 in [26].

Proposition 4.12 (Proposition 4.4 in [26]). *Let B be a complex Banach space and P a set of representatives for $\mathfrak{F}_B$. Assume the following two conditions:*

(i) *B satisfies the Hausdorff distance condition,*

(ii) *$M_{p,\alpha} = \{a \in S(B) : p(a) = \alpha\}$ for every $p \in P$ and $\alpha \in \mathbb{D}$.*

Then B has the complex Mazur-Ulam property.

§ 5. Banach spaces which satisfy the condition $(*)$

Definition 5.1. Let B be a real or complex Banach space. We say that B satisfies the condition $(*)$ whenever there exists a set of representative P for $\mathfrak{F}_B$ with the condition : for every $p \in P$, $\varepsilon > 0$, and a closed subset F of P with respect to the relative topology induced by the weak*-topology on B^* such that $p \notin F$, there exists $a \in S(B)$ such that $p(a) = 1$ and $|q(a)| \leq \varepsilon$ for all $q \in F$.

Example 5.2. Suppose that E is a uniformly closed C-regular $\mathbb{K}$-linear subspace of $C_0(Y, \mathbb{K})$. Put $P = \{\tau_x : x \in \mathrm{Ch}(E)\}$. By Theorem 3.11 every point in $\mathrm{Ch}(E)$ is a strong boundary point. Hence $F_{p,\lambda}$ for any $(p, \lambda) \in P \times \mathbb{T}$ is a maximal convex set. Since $p \not\sim q$ for $p, q \in P$ with $p \neq q$, we have that P is a set of representatives. The condition $(*)$ holds with P. We proved that a closed subalgebra of $C_0(Y, \mathbb{C})$ which separates the points of Y and has no common zeros is extremely C regular. Thus such an algebra satisfies the condition $(*)$.

Throughout the section we assume that B is a real or complex Banach space and P is a set of representatives for $\mathfrak{F}_B$ for which the conditon $(*)$ is satisfied.

Lemma 5.3. *Let $p_1, p_2 \in P$ with $p_1 \neq p_2$. Let $\mu_1, \mu_2 \in \mathbb{T}$. For every $\varepsilon > 0$ and an open neighborhood U of $\{p_1, p_2\}$ with respect to the relative topology on P induced by the weak*-topology on B^*, there exists $h \in B$ such that $\|h\| \leq 1 + \varepsilon$, $p_j(h) = \mu_j$ for $j = 1, 2$, and $|q(h)| \leq \varepsilon$ for every $q \in P \setminus U$.*

Proof. We may assume that $\varepsilon \leq 1/3$. Let V_j be an open neighborhood of p_j for $j = 1, 2$ in P such that $V_1 \cup V_2 \subset U$ and $V_1 \cap V_2 = \emptyset$. Choose any positive real number δ with $0 < \frac{4\delta}{1-\delta^2} < \varepsilon$. By the condition $(*)$ there exists $f_j \in B$ such that $p_j(f_j) = 1 = \|f_j\|$ and $|q(f_j)| \leq \delta$ for every $q \in P \setminus V_j$ for $j = 1, 2$. As $V_1 \cap V_2 = \emptyset$ and $p_2 \in V_2$ we have $p_2 \in P \setminus V_1$, so $|p_2(f_1)| \leq \delta$. We also have that $|p_1(f_2)| \leq \delta$. Hence we infer that $0 < 1 - |p_1(f_2)p_2(f_1)|$. Put

$$h_1 = \frac{f_1 - p_2(f_1)f_2}{1 - p_1(f_2)p_2(f_1)}.$$

Then we infer that $p_1(h_1) = 1$ and $p_2(h_1) = 0$. By a simple calculation we have

$$\|h_1\| \leq \frac{\|f_1\| + |p_2(f_1)|\|f_2\|}{1 - |p_1(f_2)||p_2(f_1)|} \leq 1/(1 - \delta).$$

For $q \in P \setminus V_1$ we have

$$|q(h_1)| \leq \frac{|q(f_1)| + |p_2(f_1)||q(f_2)|}{1 - |p_1(f_2)||p_2(f_1)|} \leq 2\delta/(1 - \delta^2).$$

In a similar way, we have $p_2(h_2) = 1$, $p_1(h_2) = 0$, $\|h_2\| \le 1/(1-\delta)$ and $|q(h_2)| \le 2\delta/(1-\delta^2)$ for every $q \in P \setminus U_2$, where

$$h_2 = \frac{f_2 - p_1(f_2)f_1}{1 - p_1(f_2)p_2(f_1)}.$$

Put $h = \mu_1 h_1 + \mu_2 h_2$. Then $p_j(h) = \mu_j$ for $j = 1, 2$. We prove that $\|h\| \le 1 + \varepsilon$. Let $q \in V_1$. Then $q \in P \setminus V_2$. Hence

$$|q(h)| \le |q(h_1)| + |q(h_2)| \le \|h_1\| + 2\delta/(1-\delta^2) \le 1/(1-\delta) + 2\delta/(1-\delta^2) < 1 + \varepsilon.$$

Let $q \in V_2$. Then we have $q \in P \setminus V_1$, and $|q(h)| < 1 + \varepsilon$ follows. For $q \in P \setminus (V_1 \cup V_2)$. We infer that

$$|q(h)| \le |q(h_1)| + |q(h_2)| \le 4\delta/(1-\delta^2) < \varepsilon.$$

In particular, we have $|q(h)| < \varepsilon$ for every $q \in P \setminus U$ since $V_1 \cup V_2 \subset U$. Since P is a norming family we infer that $\|h\| \le 1 + \varepsilon$. □

Proposition 5.4. *Let $p_1, p_2 \in P$ with $p_1 \neq p_2$, and $\mu_1, \mu_2 \in \mathbb{T}$. Then there exists $f \in S(B)$ such that $p_j(f) = \mu_j$ for $j = 1, 2$.*

Proof. The idea of proof comes from the proof of the Bishop's $\frac{1}{4} - \frac{3}{4}$ criterion (cf. [12, Theorem 2.3.2]). We define inductively a sequence $\{U_n\}$ of open (with respect to the relative topology on P induced by the weak*-topology) neighborhoods of $\{p_1, p_2\}$, and a sequence $\{h_n\}$ in B as follows : let ε be as $0 < \varepsilon \le 1/3$. Let U_1 be any open neighborhood of $\{p_1, p_2\}$. Then by Lemma 5.3 there exists $h_1 \in B$ such that $\|h_1\| \le 1 + \varepsilon$, $p_l(h_1) = \mu_l$ for $l = 1, 2$, and $|q(h_1)| \le \varepsilon$ for every $q \in P \setminus U_1$. Having defined $U_1, \cdots, U_{n-1}$ and $h_1, \cdots, h_{n-1}$, set

$$U_n = \{\{q \in U_{n-1} : |q(h_j)| < 1 + 2^{-n}\varepsilon, 1 \le j \le n-1\}.$$

By Lemma 5.3 there exists $h_n \in B$ such that $\|h_n\| \le 1 + \varepsilon$, $p_l(h_n) = \mu_l$ for $l = 1, 2$, and $|q(h_n)| \le \varepsilon$ for every $q \in P \setminus U_n$. Now let

$$\mathfrak{h} = \sum_{n=1}^{\infty} \frac{h_n}{2^n}.$$

In fact the series converges and $\mathfrak{h} \in B$ since $\|h_n\| \le 1 + \varepsilon$ for every n. We have that

$$p_l(\mathfrak{h}) = p_l\left(\sum_{n=1}^{\infty} \frac{h_n}{2^n}\right) = \sum_{n=1}^{\infty} \frac{p_l(h_n)}{2^n} = \mu_l$$

for $l = 1, 2$.

To prove $\|\mathfrak{h}\| \leq 1$, it is enough to observe $|q(\mathfrak{h})| \leq 1$ for every $q \in P$ since P is a norming family. We consider three cases: i) $q \in P \setminus \bigcup_{n=1}^{\infty} U_n$; ii) there exists n such that $q \in U_n \setminus U_{n+1}$; iii) $q \in \bigcap_{n=1}^{\infty} U_n$. It is possible since $U_n \supset U_{n+1}$ for every n.

Suppose that i) occurs. We have $|q(h_n)| \leq \varepsilon$ for every n, hence $|q(\mathfrak{h})| \leq \varepsilon \leq 1/3$.

Suppose that ii) occurs. If $q \in U_1 \setminus U_2$, we have $q \notin U_m$ for $m \geq 2$ since $\{U_n\}$ is decreasing. Thus $|q(h_1)| \leq \|h_1\| \leq 1+\varepsilon$ and $|q(h_m)| \leq \varepsilon$ for every $m \geq 2$. Therefore we have that

$$|q(\mathfrak{h})| \leq \frac{1+\varepsilon}{2} + \sum_{m=2}^{\infty} \frac{\varepsilon}{2^m} = 1/2 + \varepsilon < 1$$

since $\varepsilon \leq 1/3$. If $q \in U_n \setminus U_{n+1}$ for some $n \geq 2$, then $|q(h_j)| < 1+2^{-n}\varepsilon$ for $1 \leq j \leq n-1$. Since $q \notin U_{n+1}$ we have that $q \notin U_k$ for $k \geq n+1$. Hence $|q(h_k)| \leq \varepsilon$ for every $k \geq n+1$. Therefore we get

$$\begin{aligned}|q(\mathfrak{h})| \leq \sum_{j=1}^{n-1} \frac{|q(h_j)|}{2^j} + \frac{|q(h_n)|}{2^n} + \sum_{k=n+1}^{\infty} \frac{|q(h_k)|}{2^k} \\ \leq (1+2^{-n}\varepsilon)(1-2^{-(n-1)}) + (1+\varepsilon)2^{-n} + 2^{-n}\varepsilon \leq 1\end{aligned}$$

since $\varepsilon \leq 1/3$.

Suppose that iii) occurs. We have $|q(h_j)| < 1+2^{-n}\varepsilon$ for all $n > j$. Hence $|q(h_j)| \leq 1$ for all j, so $|q(\mathfrak{h})| \leq 1$. We conclude that $|q(\mathfrak{h})| \leq 1$ for every $q \in P$.

As P is a norming family we infer that $\|\mathfrak{h}\| \leq 1$. Thus $\|\mathfrak{h}\| = 1$ since $1 = |p_1(\mathfrak{h})| \leq \|\mathfrak{h}\|$. □

The following proposition is a version of an additive Bishop's lemma. The proof of one we proved in [27, Lemma 5.3] requires the existence of the constants in the target algebra. Proposition 5.5 is a generalization of Lemma 5.3 in [27] which is valid for every Banach space with the condition (∗).

We read $\frac{\alpha}{|\alpha|} = 1$ provided that $\alpha = 0$.

Proposition 5.5. *Let* $p \in P$ *and* $f \in \mathrm{Ball}(B)$*. For every* $0 < r < 1$ *there exist* H *and* H' *in* B *such that*

$$\|H + rf\| = 1, p(H + rf) = \frac{p(f)}{|p(f)|}, \|H\| \leq 1 - r|p(f)|, \tag{5.1}$$

and

$$\|H' + rf\| = 1, p(H' + rf) = -\frac{p(f)}{|p(f)|}, \|H'\| \leq 1 + |p(f)|. \tag{5.2}$$

Proof. Put $\alpha = p(f)$. The proof of of (5.1) is similar to that of Lemma 5.3 in [27], but a small revision is required since B needs not to be closed under the multiplication.

The proof of (5.2) is different from that of Lemma 5.3 in [27]. It requires substantial changes.

We first prove (5.1). Let ε be any real number such that $0 < \varepsilon < 1 - r|\alpha|$. Put

$$F_0 = \{q \in P : |r\alpha - rq(f)| \geq \varepsilon/4\}$$

and

$$F_n = \{q \in P : \varepsilon/2^{n+2} \leq |r\alpha - rq(f)| \leq \varepsilon/2^{n+1}\}$$

for a positive integer n. By the condition $(*)$, for every positive integer n there exists $u_n \in B$ such that

$$p(u_n) = 1 = \|u_n\|$$

and

$$|q(u_n)| \leq \min\left\{\frac{1-r}{1-r|\alpha|}, \frac{1}{2^{n+1}}\right\}$$

for every $q \in F_0 \cup F_n$. Put

$$u = \sum_{n=1}^{\infty} \frac{u_n}{2^n}.$$

Then $u \in \mathrm{Ball}(B)$ since $\|u_n\| = 1$ for every n. Since $p(u) = \sum_{n=1}^{\infty} \frac{p(u_n)}{2^n} = 1$, we see that $\|u\| = 1$.

Letting $H = \left(\frac{\alpha}{|\alpha|} - r\alpha\right) u$, we prove that $\|H + rf\| \leq 1$ in the following three cases: (i) $q \in F_0$; (ii) $q \in F_n$ for some $n \geq 1$; (iii) $q \in P \setminus \bigcup_{k=0}^{\infty} F_k$.

(i) Let $q \in F_0$. We have

$$|q(H + rf)| \leq \left|\frac{\alpha}{|\alpha|} - r\alpha\right| |q(u)| + r|q(f)| \leq (1 - r|\alpha|)\frac{1-r}{1-r|\alpha|} + r = 1.$$

(ii) Let $q \in F_n$ for some $n \geq 1$. In this case we have

$$|q(u)| \leq \sum_{m \neq n} \frac{|q(u_m)|}{2^m} + \frac{|q(u_n)|}{2^n} \leq 1 - \frac{1}{2^n} + \frac{1}{2^n 2^{n+1}}.$$

As $0 < \varepsilon < 1 - r|\alpha|$ we have

$$\begin{aligned} |q(H + rf)| &\leq (1 - r|\alpha|)|q(u)| + r|\alpha| + |rq(f) - r\alpha| \\ &\leq (1 - r|\alpha|)(1 - 1/2^n + 1/(2^n 2^{n+1})) + r|\alpha| + \varepsilon/2^{n+1} \\ &< (1 - r|\alpha|)(1 - 1/2^n + 1/(2^n 2^{n+1}) + 1/2^{n+1}) + r|\alpha| < 1 \end{aligned}$$

(iii) Let $q \in P \setminus \bigcup_{n=0}^{\infty} F_n$. In this case we have $q(f) = \alpha$, hence

$$|q(H + rf)| \leq (1 - r|\alpha|)|q(u)| + r|\alpha| \leq 1 - r|\alpha| + r|\alpha| = 1.$$

By (i), (ii) and (iii) we have $\|H+rf\| \le 1$ since P is a norming family. As we will see that $p(H+rf)=\alpha/|\alpha|$, it will follow that $\|H+rf\|=1$.

By simple calculations we have

$$\|H\| = \left\| \left(\frac{\alpha}{|\alpha|} - r\alpha \right) u \right\| = \left| \frac{\alpha}{|\alpha|} - r\alpha \right| \|u\| = 1 - r|\alpha|$$

and

$$p(H+rf) = p\left(\left(\frac{\alpha}{|\alpha|} - r\alpha \right) u + rf \right) = \left(\frac{\alpha}{|\alpha|} - r\alpha \right) p(u) + rp(f) = \frac{\alpha}{|\alpha|}.$$

We have completed the proof of (5.1).

Next we prove (5.2). Suppose that $|\alpha|=1$. Put $H'=-(1+r)f$. Then we have $\|H'\| \le 1+r \le 1+|\alpha|$. We also have that $p(H'+rf) = -p(f) = -\frac{\alpha}{|\alpha|}$ and $\|H'+rf\| = \|-f\| \le 1$. Since $1 = |p(H'+rf)| \le \|H'+rf\|$, we infer that $\|H'+rf\|=1$. Thus (5.2) holds if $|\alpha|=1$. Suppose that $\alpha=0$. As $\|-f\| \le 1$ and $p(-f)=0=\alpha$, we have by (5.1) that there exists $H \in B$ such that $\|H-rf\|=1$, $p(H-rf)=\frac{\alpha}{|\alpha|}$ and $\|H\| \le 1-r|\alpha| = 1$. Letting $H'=-H$ we have that $\|H'+rf\|=1$, $p(H'+rf)=-1= -\frac{\alpha}{|\alpha|}$ and $\|H'\|=\|H\| \le 1 = 1+|\alpha|$. Thus (5.2) holds if $\alpha=0$.

We assume $0<|\alpha|<1$. To prove (5.2) we apply induction. Define a sequence $\{a_n\}$ by $a_1=1/3$, $a_{n+1}=(a_n+1)/2$ for every positive integer n. Put $I_1=(0,a_1]$ and $I_n=(a_{n-1},a_n]$ for each positive integer $n \ge 2$. For each positive integer n, put

$$C_n = \{g \in \mathrm{Ball}(B) : |p(g)| \in I_n\}.$$

By induction on n we prove that for any $g \in C_n$ and $0<s<1$, there exists $H_{g,s} \in B$ such that

$$\|H_{g,s}+sg\|=1, p(H_{g,s}+sg) = -\frac{p(g)}{|p(g)|}, \|H_{g,s}\| \le 1+|p(g)|. \tag{5.3}$$

If it will be proved, then (5.3) will hold for every $g \in \mathrm{Ball}\, B$ such that $0<|p(g)|<1$ since $\lim_{n\to\infty} a_n = 1$. Combining with the results for $\alpha=0$ or $|\alpha|=1$, it will follows that (5.2) holds for every $f \in \mathrm{Ball}(B)$ and $0<r<1$.

Suppose that $g_1 \in C_1$ and $0<s_1<1$. Put $\alpha_1=p(g_1)$. Then $0<|\alpha_1| \le a_1$. Let ε_1 be as $0<\varepsilon_1<1-\sqrt{s_1}$ and

$$K_1 = \{q \in P : |\alpha - q(g_1)| \ge \varepsilon_1/2\}.$$

By the condition $(*)$ there exists $v_1 \in S(B)$ such that $p(v_1)=1=\|v_1\|$ and $|q(v_1)| \le \varepsilon_1/2$ for every $q \in K_1$. Look at $\|-2\sqrt{s_1}\alpha_1 v_1 + \sqrt{s_1}g_1\|$. If $q \in K_1$, then $|q(v_1)| \le \varepsilon_1/2$. Therefore

$$|q(-2\sqrt{s_1}\alpha_1 v_1 + \sqrt{s_1}g_1)| \le 2\sqrt{s_1}|\alpha_1||q(v_1)| + \sqrt{s_1}|q(g_1)| \le 2\sqrt{s_1}|\alpha_1|\varepsilon_1/2 + \sqrt{s_1} < 1$$

since $|\alpha_1| \le 1/3$. If $q \in P \setminus K_1$, then $|\alpha_1 - q(g_1)| < \varepsilon_1/2$. Thus

$$|q(-2\sqrt{s_1}\alpha_1 v_1 + \sqrt{s_1}g_1)| \le \sqrt{s_1}|\alpha_1 - 2\alpha_1 q(v_1)| + \sqrt{s_1}|q(g_1) - \alpha_1| \le 3\sqrt{s_1}|\alpha_1| + \sqrt{s_1}\varepsilon_1/2 < 1$$

since since $|\alpha_1| \le 1/3$ and $\varepsilon_1 < 1 - \sqrt{s_1}$. We conclude that

$$\| - 2\sqrt{s_1}\alpha_1 v_1 + \sqrt{s_1}g_1\| \le 1$$

since P is a norming family. Put $g_1' = -2\sqrt{s_1}\alpha_1 v_1 + \sqrt{s_1}g_1$ Then $p(g_1') = -\sqrt{s_1}\alpha_1$. Applying (5.1) with $\sqrt{s_1}$ and g_1' instead of r and f respectively, we find $H_+ \in B$ such that

$$\|H_+ + \sqrt{s_1}g_1'\| = 1, \quad p(H_+ + \sqrt{s_1}g_1') = \frac{p(g_1')}{|p(g_1')|} = -\frac{\alpha_1}{|\alpha_1|}$$

and

$$\|H_+\| \le 1 - \sqrt{s_1}|p(g_1')| = 1 - s_1|\alpha_1|.$$

Letting $H_{g_1,s_1} = H_+ - 2s_1\alpha_1 v_1$ we infer that

$$H_{g_1,s_1} + sg_1 = H_+ - 2s_1\alpha_1 v_1 + s_1 g_1 = H_+ + \sqrt{s_1}(-2\sqrt{s_1}\alpha_1 v_1 + \sqrt{s_1}g_1) = H_+ + \sqrt{s_1}g_1'.$$

Hence

$$\|H_{g_1,s_1} + s_1 g_1\| = 1, \quad p(H_{g_1,s_1} + s_1 g_1) = -\frac{\alpha_1}{|\alpha_1|}.$$

We also have

$$\|H_{g_1,s_1}\| \le \|H_+\| + 2s_1|\alpha_1|\|v_1\| \le 1 - s_1|\alpha_1| + 2s_1|\alpha_1| = 1 + s_1|\alpha_1| \le 1 + |\alpha_1|.$$

We have proved that (5.3) holds for $n = 1$

Suppose that (5.3) holds for every $1 \le n \le m$. Let $g_{m+1} \in C_{m+1}$ and $0 < s_{m+1} < 1$ arbitrary. Put $\alpha_{m+1} = p(g_{m+1})$. Put $0 < \varepsilon_{m+1} < 1 - \sqrt{s_{m+1}}$ and

$$K_{m+1} = \{q \in P : |\alpha_{m+1} - q(g_{m+1})| \ge \varepsilon_{m+1}/2\}.$$

By the condition $(*)$ there exists $v_{m+1} \in B$ such that

$$p(v_{m+1}) = 1 = \|v_{m+1}\|$$

and

$$|q(v_{m+1})| \le \frac{\varepsilon_{m+1}}{2(a_{m+1} - a_m)}$$

for every $q \in K_{m+1}$. Put

$$f_{m+1} = \sqrt{s_{m+1}}\left(\left(a_m \cdot \frac{\alpha_{m+1}}{|\alpha_{m+1}|} - \alpha_{m+1}\right) v_{m+1} + g_{m+1}\right).$$

Note that $\left|a_m \cdot \frac{\alpha_{m+1}}{|\alpha_{m+1}|} - \alpha_{m+1}\right| = |\alpha_{m+1}| - a_m$ since $a_m < |\alpha_{m+1}| \leq a_{m+1}$. Suppose that $q \in K_{m+1}$. Then

$$\begin{aligned}|q(f_{m+1})| &\leq \sqrt{s_{m+1}}(|\alpha_{m+1}| - a_m)|q(v_{m+1})| + \sqrt{s_{m+1}}|q(g_{m+1})| \\ &\leq \frac{\sqrt{s_{m+1}}(|\alpha| - a_m)\varepsilon_{m+1}}{2(a_{m+1} - a_m)} + \sqrt{s_{m+1}} \leq \sqrt{s_{m+1}}(\varepsilon_{m+1}/2 + 1) < 1\end{aligned}$$

since $|\alpha_{m+1}| \leq a_{m+1}$. Suppose that $q \in P \setminus K_{m+1}$. Then

$$\begin{aligned}|q(f_{m+1})| &\leq \sqrt{s_{m+1}}\left(\left|a_m \cdot \frac{\alpha_{m+1}}{|\alpha_{m+1}|} - \alpha_{m+1}\right| |q(v_{m+1})| + |\alpha_{m+1}| + |q(g_{m+1}) - \alpha_{m+1}|\right) \\ &\leq \sqrt{s_{m+1}}((|\alpha_{m+1}| - a_m) + |\alpha_{m+1}| + \varepsilon_{m+1}/2) = \sqrt{s_{m+1}}(2|\alpha_{m+1}| - a_m + \varepsilon_{m+1}/2) \\ &\leq \sqrt{s_{m+1}}(2a_{m+1} - a_m + \varepsilon_{m+1}/2) = \sqrt{s_{m+1}}(1 + \varepsilon_{m+1}/2) < 1\end{aligned}$$

Therefore we have that $\|f_{m+1}\| \leq 1$ since P is a norming family. By a calculation we have $p(f_{m+1}) = \sqrt{s_{m+1}}a_m \cdot \frac{\alpha_{m+1}}{|\alpha_{m+1}|}$, hence $|p(f_{m+1})| = \sqrt{s_{m+1}}a_m < a_m$. It means that $|p(f_{m+1})| \in I_k$ for some $1 \leq k \leq m$. By the hypothesis of induction there exists $H_k \in B$ such that

$$\|H_k + \sqrt{s_{m+1}}f_{m+1}\| = 1, \quad p(H_k + \sqrt{s_{m+1}}f_{m+1}) = -\frac{p(f_{m+1})}{|p(f_{m+1})|} = -\frac{\alpha_{m+1}}{|\alpha_{m+1}|}$$

and

$$\|H_k\| \leq 1 + |p(f_{m+1})| = 1 + \sqrt{s_{m+1}}a_m$$

Put

$$H_{g_{m+1},s_{m+1}} = H_k + s_{m+1}\left(a_m \cdot \frac{\alpha}{|\alpha|} - \alpha\right)v_{m+1}.$$

Then $H_{g_{m+1},s_{m+1}} + s_{m+1}g_{m+1} = H_k + \sqrt{s_{m+1}}f_{m+1}$ and

$$\|H_{g_{m+1},s_{m+1}} + s_{m+1}g_{m+1}\| = 1, \quad p(H_{g_{m+1},s_{m+1}} + s_{m+1}g_{m+1}) = -\frac{\alpha_{m+1}}{|\alpha_{m+1}|}.$$

We also have

$$\begin{aligned}\|H_{g_{m+1},s_{m+1}}\| &\leq \|H_k\| + \left\|s_{m+1}\left(a_m \cdot \frac{\alpha_{m+1}}{|\alpha_{m+1}|} - \alpha_{m+1}\right)v_{m+1}\right\| \\ &\leq 1 + \sqrt{s_{m+1}}a_m + s_{m+1}(|\alpha_{m+1}| - a_m) < 1 + \sqrt{s_{m+1}}a_m + \sqrt{s_{m+1}}(|\alpha_{m+1}| - a_m) \\ &\leq 1 + |\alpha_{m+1}|.\end{aligned}$$

We conclude by induction that (5.3) holds if $0 < |p(g)| < 1$.

□

§ 6. Banach spaces which satisfy the condition $(*)$ and the Mazur-Ulam property

Theorem 6.1. *Let B be a real Banach space which satisfies the condition $(*)$. Then B has the Mazur-Ulam property.*

Proof. Let P be a set of representative for $\mathfrak{F}_B$ with which the condition $(*)$ is satisfied. We prove that (4.7) of Proposition 4.11 holds. Let $p \in P$ and $-1 \leq \alpha \leq 1$ arbitrary. In the same way as [26, Lemma 4.3] we infer that

$$M_{p,\alpha} \subset \{a \in S(B) : p(a) = \alpha\}.$$

We prove the inverse inclusion. Suppose that $f \in S(B)$ with $p(f) = \alpha$. Let $0 < r < 1$ arbitrary. By Proposition 5.5 there exists $H \in B$ such that $H + rf \in F_{p,\frac{\alpha}{|\alpha|}}$ and $\|H\| \leq 1 - r|\alpha|$. Thus

$$\|H + rf - f\| \leq 1 - r|\alpha| + 1 - r.$$

We infer that $d(f, F_{p,\frac{\alpha}{|\alpha|}}) \leq 1 - |\alpha|$. We also have by Proposition 5.5 that there exists $H' \in B$ such that $H' + rf \in F_{p,-\frac{\alpha}{|\alpha|}}$ and $\|H'\| \leq 1 + |\alpha|$. Hence

$$\|H' + rf - f\| \leq 1 + |\alpha| + 1 - r.$$

We infer that $d(f, F_{p,-\frac{\alpha}{|\alpha|}}) \leq 1 + |\alpha|$. Thus $f \in M_{p,\alpha}$. Hence $\{f \in S(B) : p(f) = \alpha\} \subset M_{p,\alpha}$. It follows from Proposition 4.11 that B has the Mazur-Ulam property. $\square$

Corollary 6.2. *Let E be a uniformly closed extremely C-regular $\mathbb{R}$-linear subspace of $C_0(Y, \mathbb{R})$ for a locally compact Hausdorff space Y. Then E has the Mazur-Ulam property. In particular, $C_0(Y, \mathbb{R})$ itself and a uniformly closed extremely regular subspace of $C_0(Y, \mathbb{R})$ has the Mazur-Ulam property.*

Proof. We prove that E satisfies the condition $(*)$. Put $P = \{\tau_x : x \in \mathrm{Ch}(E)\}$. By Theorem 3.11 every point in $\mathrm{Ch}(E)$ is a strong boundary point. Hence $F_{p,\lambda}$ for any $(p, \lambda) \in P \times \{\pm 1\}$ is a maximal convex set. Since $p \not\sim q$ for $p, q \in P$ with $p \neq q$, we have that P is a set of representatives. The condition $(*)$ holds with P. Then by Theorem 6.1 we see that E has the Mazur-Ulam property. $\square$

Theorem 6.3. *Let B be a complex Banach space which satisfies the condition $(*)$. Then B has the complex Mazur-Ulam property.*

Proof. Let P be a set of representative for $\mathfrak{F}_B$ with which the condition $(*)$ is satisfied. We prove (i) and (ii) of Proposition 4.12. Let $F_{p,\lambda}, F_{p',\lambda'} \in \mathfrak{F}_B$ such that

$p \neq p'$. By Proposition 5.4 there exists $f \in S(B)$ such that $p(f) = -\lambda$ and $p'(f) = \lambda'$. Then $f \in F_{p',\lambda'}$. For any $g \in F_{p,\lambda}$ we infer that

$$2 = |p(f) - p(g)| \leq \|f - g\| \leq 2,$$

hence $d(f, F_{p,\lambda}) = 2$. Hence $d_H(F_{p,\lambda}, F_{p',\lambda'}) = 2$. As a pair $F_{p,\lambda}$ and $F_{p',\lambda'}$ with $p \neq p'$ is arbitrary, we see that B satisfies the Hausdorff distance condition holds; (i) of Proposition 4.12 holds.

We prove (ii) of Proposition 4.12. Let $p \in P$ and $\alpha \in \bar{D}$ arbitrary. Let $0 < r < 1$ be arbitrary. By Proposition 5.5 there exists $H \in B$ with $H + rf \in F_{p,\frac{\alpha}{|\alpha|}}$ and $\|H\| \leq 1 - r|\alpha|$. There also exists $H' \in B$ with $H' + rf \in F_{p,-\frac{\alpha}{|\alpha|}}$ and $\|H'\| \leq 1 + |\alpha|$. Hence

$$\|H + rf - f\| \leq 1 - r|\alpha| + 1 - r,$$

and

$$\|H' + rf - f\| \leq 1 + |\alpha| + 1 - r.$$

As r is arbitrary we infer that $d(f, F_{p,\frac{\alpha}{|\alpha|}}) \leq 1 - |\alpha|$ and $d(f, F_{p,-\frac{\alpha}{|\alpha|}}) \leq 1 + |\alpha|$. It follows that $f \in M_{p,\alpha}$. Thus $\{f \in S(B) : p(f) = \alpha\} \subset M_{p,\alpha}$. The inverse inclusion

$$M_{p,\alpha} \subset \{f \in S(B) : p(f) = \alpha\}$$

is by [26, Lemma 4.3]. We conclude that (ii) of Proposition 4.12 holds.

It follows from Proposition 4.12 that B has the complex Mazur-Ulam property. □

Corollary 6.4. *A uniformly closed extremely C-regular $\mathbb{C}$-linear subspace of $C_0(Y, \mathbb{C})$ for a locally compact Hausdorff space has the complex Mazur-Ulam property. In particular, $C_0(Y, \mathbb{C})$ itself and a uniformly closed extremely regular $\mathbb{C}$-linear subspace of $C_0(Y, \mathbb{C})$ has the complex Mazur-Ulam property.*

Proof. The proof that E satisfies the condition $(*)$ is essentially the same as that for Corollary 6.2 (cf. Example 5.2). Then by Theorem 6.3 we see that E has the complex Mazur-Ulam property. □

Hatori [26, Theorem 4.5] proved that a uniform algebra has the complex Mazur-Ulam property. In the proof of Theorem 4.5 in [26], it is crucial that a uniform algebra contains the constants. Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [14, Corollary 3.2] proved that $C_0(Y, \mathbb{C})$ satisfies the complex Mazur-Ulam property. Cueto-Avellaneda, Hirota, Miura and Peralta [17, Theorem 2.1] have proved that each surjective isometry between the unit spheres of two uniformly closed algebras on locally compact Hausdorff spaces which separates the points without common zeros admits an extension to a surjective real linear isometry between these algebras. The following generalizes a both theorems of Cueto-Avellaneda, Hirota, Miura and Peralta [17, Theorem 2.1] and Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [14, Corollary 3.2].

Corollary 6.5. *A (non-zero) closed subalgebra of $C_0(Y,\mathbb{C})$ has the complex Mazur-Ulam property. In particular, a uniform algebra has the complex Mazur-Ulam property.*

Proof. Let A be a non-zero closed subalgebra of $C_0(Y,\mathbb{C})$. Let

$$C = \{y \in Y : f(y) = 0 \text{ for all } f \in A\}.$$

As we assume $A \neq \{0\}$, there is a non-zero function in A, so C is a proper subset of Y. For any pair of points x and y in $Y \setminus C$, we denote $x \sim y$ if $f(x) = f(y)$ for all $f \in A$. Then $\sim$ is an equivalence relation on $Y \setminus C$. Let Y_0 be the quotient space induced by the relation $\sim$. Then Y_0 is a locally compact, possibly compact, Hausdorff space induced by the quotient topology. We may suppose that A is a closed subalgebra of $C_0(Y_0,\mathbb{C})$ which separates the points in Y_0 and has no common zeros. Then by Proposition 3.13 A is a uniformly closed extremely C-regular $\mathbb{C}$-linear subspace of $C_0(Y_0,\mathbb{C})$. It follows by Corollary 6.4 that A has the complex Mazur-Ulam property.

Suppose that A is a uniform algebra on a compact Hausdorff space. Then A is a closed subalgebra of $C(X,\mathbb{C})$. Then by the first part we have that A has the complex Mazur-Ulam property. □

§ 7. Final remarks

In section 5 we introduced the condition $(*)$ for Banach spaces and we proved that an extremely C-regular closed subspace of $C_0(Y,\mathbb{K})$ satisfies the condition $(*)$ in section 6. On the other hand we have not enough examples of Banach spaces which satisfy the condition. Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [14] proved that every *commutative* JB* triple satisfies the complex Mazur-Ulam property. The author does not know if a commutative JB* triple satisfies the condition $(*)$ or not. If it would satisfy the condition $(*)$, we would get an alternative proof of a theorem of Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [14, Theorem 3.1]. It is interesting to exhibit enough examples of Banach spaces which satisfy the condition $(*)$. We have proved the *complex* Mazur-Ulam property especially for a closed subalgebra of $C_0(Y,\mathbb{C})$ which separates the points of Y and has no common zeros, we expect it also has the Mazur-Ulam property. It is trivial that the complex Mazur-Ulam property follows the Mazur-Ulam property. The author does not know the converse statement : does the Mazur-Ulam property follow the complex Mazur-Ulam property?

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