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Applications of the Quotient Lifting Property

By

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Abstract

We review the "Quotient Lifting Property" of Banach spaces and survey results involving this property. We discuss the existence of lifts of operators and conditions for uniqueness of liftings.

§ 1. Introduction

This note contains a survey of results on the Quotient Lifting Property (QLP) for Banach spaces. Given a Banach space X and a closed subspace J, the QLP for the pair (X, J) was considered in [1] and [4]. We start with the definition.

Definition 1.1. (cf. [1]) The pair (X, J) has the QLP if for every Banach space Y and every bounded operator $S : Y \to X/J$ there exists a bounded operator T from Y to X lifting S while preserving the norm, i.e. $\|T\| = \|S\|$ and $\pi \circ T = S$.

For the trivial subspaces, $J = X$ or $J = \{0\}$, we have that X/J is trivial or X, and the QLP holds for (X, J). If (X, J) has the QLP then given the identity operator,

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I on X/J, there exists a norm 1 lifting, $\tilde{I} : X/J \to X$, such that $\pi \circ \tilde{I} = I$. For $x \in X$, we have

$$\|\pi \circ \tilde{I}(x + J)\| = \|x + J\| \leq \|\tilde{I}(x + J)\| \leq \|x + J\|,$$

then $\tilde{I}$ is an isometry into X. Furthermore, the operator $P = \tilde{I} \circ \pi$ is a contractive projection with kernel equal to J. Thus, if (X, J) has the QLP then J is a complemented subspace of X. This implies that (ℓ_∞, c_0) does not have the QLP, since c_0 is not complemented in ℓ_∞. The next proposition gives a necessary and sufficient condition for a subspace of a Banach space to define a pair with this property.

Proposition 1.2. *(cf. [1]) A subspace of X, J, is the kernel of a contractive projection on X if and only if the pair (X, J) has the QLP.*

Proof. Let $P : X \to X$ be a contractive projection with kernel equal to J. We define $\tilde{I} : X/J \to X$ by $\tilde{I}(x+J) = P(x)$. Since $x - P(x) \in Ker(P)$, we have $\pi(P(x)) = P(x) + J = x + J = I(x+J)$. Then $\tilde{I}$ is a lift of the identity on X/J. Conversely, given $\tilde{I}$, a lift of the identity on X/J, $P = \tilde{I} \circ \pi$ is a projection with kernel equal to J. □

Therefore, given a bi-contractive projection on X, P,

$$(X, Ker(P)) \text{ and } (X, Ran(P)) \text{ have the QLP,}$$

with $Ker(P)$ and $Ran(P)$ denoting the kernel and the range of P, respectively. M-projections and L-projections are examples of bi-contrac-tive projections, see [12]. Further, if X is M-embedded, i.e. X is an M-ideal in X^{**} then $X^{***} = X^* \oplus_1 X^\perp$ and (X^{***}, X^*) has the QLP. We recall that $X^\perp$ stands for all elements in X^* vanishing on X. Hence, (ℓ_∞^*, ℓ_1) has the QLP. We recall that c_0 is an M-ideal in ℓ_∞ (cf. [12] p. 3), but (ℓ_∞, c_0) does not have the QLP. This also shows that (X^{**}, X) can fail the QLP. It is an interesting problem to determine conditions on a Banach space X under which the pair (X^{**}, X) has the QLP. Some partial results can be found in [4] and [9].
For completeness of exposition, we recall the definitions of metric projection, metric complement and proximinality of a subspace. We denote by $\mathcal{P}(J)$ the collection of all subsets of J, see [1] and [5].

Definition 1.3. (see [5]) Given a closed subspace J of a Banach space X, the metric projection onto J is a set valued function $P_J : X \to \mathcal{P}(J)$ given by

$$P_J(x) = \{j \in J : \|j - x\| = dist(x, J)\},$$

with $dist(x, J)$ denoting the distance from x to J. The subspace J is proximinal in X if and only if for every $x \in X$, $P_J(x)$ is nonempty.

The next theorem, from [1], reviews some conditions for the QLP to hold.

Theorem 1.4. *(cf. [1]) Let X be a Banach space and J a closed subspace of X. Then*

1. *If (X, J) has the QLP then J is proximinal in X.*
2. *If J is proximinal in X, then (X,J) has the QLP if and only if the metric projection onto J has a linear selection, i.e. there exists a linear map p such that for $x \in X$, $p(x) \in P_J(x)$.*

The pair (ℓ_∞, c_0) also shows that proximinality of the subspace is not sufficient for the property.

§ 2. Uniqueness of Lifts for the QLP

In this section we investigate the uniqueness of liftings for pairs of spaces with the QLP. More precisely, we investigate conditions under which a closed subspace of a Banach space X, J, is the kernel of a unique contractive projection. It is easy to see that this is equivalent to say that $Id : X/Y \to X/Y$ has a unique norm preserving lifting. A particular case deals with the pair (X^{**}, X). The problem is formulated in terms of the uniqueness of contractive projections on X^{**} with kernel equal to X. We recall that a Banach space X is said to be very smooth if each unit vector x has a unique norming functional in X^{***}, cf [11]. A closed space $J \subset X$, with X very smooth, is the range of at most one contractive projection in X^{**}, see Theorem 2 in [11]. Theorem 4 in [11] formulates that if the norm of X is Fréchet differentiable, then X is very smooth. From Theorem III.4.6 in [12], every M-embedded space can be renormed to be very smooth. We recall that, given J a subspace of X, $J^\perp = \{\tau \in X^* : \tau(j) = 0, \text{ for all } j \in J\}$.

Proposition 2.1. *Let X be a very smooth Banach space and let $J \subset X$ be a reflexive subspace. If $(X^*, J^\perp)$ has the QLP, then $J^\perp$ is the kernel of a unique projection of norm one on X^*.*

Proof. If P, Q are two norm one projections in X^* such that $ker(P) = J^\perp = ker(Q)$. Results in [6] (p. 102) imply the existence of natural identifications of $J^{\perp\perp}$ with $\left(X^*/J^\perp\right)^*$ and with J^{**}. The reflexivity assumption on J implies J is the range of the contractive projections: P^* and Q^*. Since X is very smooth, then $P^* = Q^*$, and $P = Q$. □

§ 2.1. Strictly Contractive Projections

The study of uniqueness of projections with a given range leads to the class of strictly contractive projections, considered in [3] and [10].

Definition 2.2. (cf. [10]) Let p be a contractive projection on a Banach space X. Then p is strictly contractive if $\|px\| < \|x\|$, for every x such that $p(x) \neq x$.

Orthogonal projections on a Hilbert space are strictly contractive. L-projections on a Banach space (i.e. $\|x\| = \|P(x)\| + \|x - P)x)\|$, for every x) are strictly contractive. It is straightforward to check that contractive projections on a strictly convex space are strictly contractive. Given P a contractive projection on X (strictly convex) and $x \in X$, of norm 1 and such that $Px \neq x$, we have

$$\|P\frac{x + Px}{2}\| = \|Px\| \leq \|\frac{x + Px}{2}\| < 1.$$

We now consider the definition of a quotient lifting property, requiring uniqueness of liftings.

Definition 2.3. A pair of Banach spaces with the QLP, (X, J) is said to have the quotient unique-lifting property (QULP) if the identity $I : X/J \to X/J$ has a unique norm preserving lift, $\tilde{I} : X/J \to X$ such that $\pi \circ \tilde{I} = I$.

We prove that the uniqueness of lifting of isomorphisms onto the quotient space is equivalent to the uniqueness of lifting for the identity operator on X/J.

Proposition 2.4. *Let X and J be Banach spaces with J a closed subspace of X. Then (X, J) has the QULP for invertible operators on X/J if and only if there exists a unique norm 1 operator $\tilde{I} : X/J \to X$ such that $\pi \circ \tilde{I} = I$.*

Proof. Given a Banach space Y and a bounded operator $S : Y \to X/J$, $\tilde{S}_0 = \tilde{I} \circ S$ is a norm preserving lift of S, with $\tilde{I}$ the lift of the identity on X/J. Any other lift of S,

$\tilde{S}_1$, is equal to $\tilde{S}$ plus an operator with values in J. If a bijective operator S from Y onto X/J admits two different lifts, $\tilde{S}_0$ and $\tilde{S}_1$, then the identity I on X/J also admits two different lifts. More precisely, we define $\tilde{I}_0(S(y)) = \tilde{S}_0(y)$ and $\tilde{I}_1(S(y)) = \tilde{S}_1(y)$. Then, $\pi \circ \tilde{I}_i(S(y)) = \pi \circ \tilde{S}_i(y) = S(y)$, with $i \in \{0, 1\}$. The reversed implication is clear. □

Example 2.5.

(i) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K}$ be a closed subspace. Then the quotient space $\mathcal{H}/\mathcal{K}$ is isometric to the orthogonal complement $\mathcal{K}^\perp$. The identity $\mathcal{K}^\perp$ has a unique lift from $\mathcal{K}^\perp$ to $\mathcal{H}$, the inclusion operator. Therefore $(\mathcal{H}, \mathcal{K})$ has the QULP.

(ii) Let $X = \mathbb{R} \oplus_1 \mathbb{R}$ and $J = Span\{(1, -1)\}$. Then the identity operator on X/J has two different lifts $\tilde{I}_1((x, y) + J) = (x + y, 0)$ and $\tilde{I}_2((x, y) + J) = (0, x + y)$. We observe that $\pi \circ \tilde{I}_1((x, y) + J) = (x + y, 0) + J = (x, y) + J$. Similar observation applies to $\tilde{I}_2$. It remains to show that both $\tilde{I}_1$ and $\tilde{I}_2$ have norm 1. We have

$$|x + y| = \{|x - t| + |y + t|, min\{x, -y\} \leq t \leq max\{x, -y\}\}$$

and then

$$|x + y| = min_{t\in\mathbb{R}}\{|x - t| + |y + t|\} = \|((x, y) + J\|.$$

Therefore (X, J) does not have the QULP.

From the existence of strictly contractive projections we can infer information about the QULP.

Proposition 2.6. *Let (X, J) be a pair of spaces with the QLP. Let $\tilde{I}$ be a lift of the identity operator on X/J and P the projection $\tilde{I} \circ \pi$. The following statements are equivalent :*

1. *The range of P is equal to $\{x \in X : \|x\| = \|x + J\|)\}$.*

2. *The projection P is strictly contractive.*

Proof. We prove that the statement 1. implies 2. For $x \in X$ such that $Px \neq x$, $\|Px\| = \|\tilde{I}(x + J)\| \leq \|x + J\| < \|x\|$, and P is strictly contractive.
We show that 2. implies 1. Let x_0 be a point in the range of P. Then

$$\|P(x_0)\| = \|x_0\| \leq \|x_0 + J\|$$

therefore $x_0 \in \{x \in X : \|x\| = \|x + J\|)\}$, and the range of P is contained in $\{x \in X : \|x\| = \|x + J\|)\}$. Let x_0 be such that $\|x_0\| = \|x_0 + J\|$. Then $\|Px_0\| = \|\tilde{I}(x_0 + J)\| = \|x_0\|$, because $\tilde{I}$ is an isometry. Since P is strictly contractive we have $P(x_0) = x_0$. □

A known result asserts that two projections p and q with the same range, such that $I - p$ and $I - q$ are contractive and at least one of them is strictly contractive are equal, cf. Theorem 2 in [10].

Corollary 2.7. *Let (X, J) be a pair of spaces with the QLP. Let $\tilde{I}$ be a lift of the identity operator on X/J. If $P = \tilde{I} \circ \pi$ is strictly contractive then the identity on X/J admits a unique norm preserving lift.*

Proof. If $\tilde{I}_1$ is a lift of the identity, different from $\tilde{I}$, the corresponding projection $Q = \tilde{I}_1 \circ \pi$, and P, both have kernel J. Since P is strictly contractive, Theorem 2 in [10] with $q = I - P$ and $p = I - Q$ implies that $Q = P$. Hence $\tilde{I} = \tilde{I}_1$. □

If P is an L-projection, then $(X, Ker(P))$ and $(X, Ker(I - P))$ have the QULP.

Corollary 2.8. *Let X be strictly convex and J a closed subspace such that (X, J) has the QLP then (X, J) has the QULP.*

Proof. Since (X, J) has the QLP then the identity operator on X/J has a lift $\tilde{I}$. Then the projection $\pi \circ \tilde{I}$ is strictly contractive. Corollary 2.7 implies the uniqueness of $\tilde{I}$. This completes the proof. □

Remark. Let τ be a norm one attaining functional in X^*, then $(X, Ker(\tau))$ has the QLP. See [4] on p.8. If τ attains its norm at a single vector then $(X, ker(\tau))$ has the QULP. Furthermore, given J a codimension 1 subspace of X, (X, J) has the QULP if and only if there exists a functional with kernel J that attains its norm at a single vector. If every functional on X has this property then X is strictly convex. This suggests that the QULP for finite codimension subspaces of X may lead to interesting results.

§ 3. QLP through Renorming

In this section we show that every Banach space with a complemented subspace can be renormed to define a pair with the QLP.

Theorem 3.1. *Let X be a Banach space and J a complemented subspace of X. Then X can be renormed so that (X, J) has the QLP.*

Proof. We denote by P a projection with range equal to J. We define a new norm on X as follows:

$$\|x\|_{new} = \|x - P(x)\| + \|P(x)\|.$$

We have $\|x\| \leq \|x\|_{new} \leq (1 + 2\|P\|)\|x\|$. These two norms are equivalent. We set $X_n = (X, \|\cdot\|_{new})$ and J_n the corresponding subspace. We show that (X_n, J_n) has the QLP. Let I be the identity on X_n/J_n and $\tilde{I}(x + J_n) = x - P(x)$. It is clear that $\pi \circ \tilde{I} = I$. Let $x + J_n \in X_n/J_n$, then

$$\begin{aligned}\|\tilde{I}(x + J_n)\| &= inf_{j \in J_n} \|x + j\|_n \\ &= inf_{j \in J_n} (\|x + j - P(x) - j\| + \|P(x) + j\|) \\ &= \|x - P(x)\|.\end{aligned}$$

Then $\tilde{I}$ has norm equal to 1. □

Remark. The projection $\tilde{I} \circ \pi$ is strictly contractive. Given x such that $\tilde{I} \circ \pi(x) \neq x$ (i.e. $Px \neq x$) we have

$$\begin{aligned}\|\tilde{I} \circ \pi(x)\|_{new} &= \|x - Px\|_{new} \\ &= \|x - Px\| < \|x - Px\| + \|Px\| = \|x\|_{new}.\end{aligned}$$

Corollary 2.7 implies that (X_{new}, J_{new}) has the QULP.

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