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Almost disjointness preserving functionals on Banach lattices of differentiable functions

By

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Abstract

Let $C^1[0,1]$ be the space of all continuously differentiable function on $[0,1]$. When define the order $f \geq g$ by

$$f(0) \geq g(0) \quad \text{and} \quad f' \geq g' \text{ pointwise on } [0,1],$$

and the norm is defined by $\|f\|_\sigma = |f(0)| + \|f'\|_\infty$, the space $C^1[0,1]$ is a Banach lattice. We will give the representation of bounded ε-disjointness preserving linear functionals of $C^1[0,1]$.

§ 1. Preliminaries and Definitions

Recall that an operator T between Banach lattices E and F is called *disjointness preserving* if $Tx \perp Ty$ whenever $x \perp y$. Jarosz [7] gave a complete analysis of linear disjointness preserving operators between $C(X)$-spaces, when X is a compact Hausdorff space. A similar result was shown on algebras of differentiable functions [8] or on Köthe spaces [13]. Brown and Wong [4] gave a full description of (bounded or unbounded, real or complex) disjointness preserving linear funcitonls of continuous function space $C_0(X)$ defined on a locally compact space. Moveover, the inverse of disjointness preserving operators are studied in [1, 10]. Order bounded disjointness preserving operators have many applications in the dynamical systems and differential equation (see [3] and

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references therein). The disjointness preserving operator on differential function spaces were studied by many authors.

Theorem 1.1. *(see [10, Theorem 6.6] and [2, Theorem 6.2]) Suppose that X, Y are open subsets of $\mathbb{R}^m, \mathbb{R}^n$, respectively. Let $T : C^p(X, \mathbb{R}^l) \to C^p(Y, \mathbb{R}^l)$ be a disjointness preserving bijection. If Y has only finitely many connected components, then T^{-1} is disjointness preserving. Moreover, there exist a diffeomorphism $\tau : Y \to X$ of class C^p and a map $J : Y \to L(\mathbb{R}^l)$ such that*

$$(Tf)(y) = (Jy)f(\tau(y)), \quad y \in Y, f \in C^p(X, \mathbb{R}^l).$$

In this paper, we will investigate the almost disjointness preserving operators. Let $\epsilon \geq 0$. Suppose that E and F are Banach lattices, an operator $T : E \to F$ is said to be *ε-disjointness preserving* if for any disjoint $x, y \in E$,

$$\| |Tx| \wedge |Ty| \| \leq \varepsilon \max\{\|x\|, \|y\|\}.$$

The 0-disjointness preserving operators are precisely the disjointness preserving operators. Note that this definition is very different from the Dolinar ε-disjointness preserving operators (see [6]). Oikhberg and Tradacete [14] studied the stability of ε-disjointness preservers on some Banach lattices. For example, if F is a Banach lattice having the Fatou property with constant ϱ, then for any positive ε-disjointness preserving operator $T : c_0 \to F$, there exists a disjointness preserving operator $S : c_0 \to F$ such that $0 \leq S \leq T$ and $\|T - S\| \leq 256\varrho\varepsilon$.

Let $C^1[0,1]$ be the space of all continuously differentiable function $f : [0,1] \to \mathbb{R}$. The order $f \geq g$ is defined by

$$f(0) \geq g(0) \quad \text{and} \quad f' \geq g' \text{ pointwise on } [0,1].$$

The norm on $C^1[0,1]$ is defined by $\|f\|_\sigma = |f(0)| + \|f'\|_\infty$. Then $(C^1[0,1], \geq, \|\cdot\|_\sigma)$ is a Banach lattice (see [11, p.11]). For any $f \in C^1[0,1]$, denote by $\operatorname{coz}(f)$ the set $\{x \in X : f(x) \neq 0\}$ in the following.

Let $\mathbb{R} \oplus_1 C[0,1]$ be the Banach lattice with the norm $\|(r,h)\| = |r| + \|h\|_\infty$ and the canonical order, that is, $(r_1, h_1) \leq (r_2, h_2)$ if and only if $r_1 \leq r_2$ and $h_1 \leq h_2$. Then the mapping

$$\pi : f \in C^1[0,1] \mapsto (f(0), f') \in \mathbb{R} \oplus_1 C[0,1]$$

is a lattice isomorphism and surjective isometry. For any $f, g \in C^1[0,1]$, it follows that $(f \vee g)(0) = \max\{f(0), g(0)\}, (f \wedge g)(0) = \min\{f(0), g(0)\}$,

$$(f \vee g)'(t) = \max\{f'(t), g'(t)\} \quad \text{for all } t \in [0,1],$$

and

$$(f \wedge g)'(t) = \min\{f'(t), g'(t)\} \quad \text{for all } t \in [0,1].$$

Therefore, one can give the representation of $|f|$ as follows

$$|f|(t) = (f \vee (-f))(t) = |f(0)| + \int_0^t |f'(s)|ds, \quad \text{for all } t \in [0,1].$$

This implies that $f \perp g$ if and only if $|f| \wedge |g| = 0$ (see [11, p.2]) if and only if

$$f(0)g(0) = 0 \quad \text{and} \quad f'g' = 0.$$

Moreover, $f_\alpha \xrightarrow{o} f_0$ in $C^1[0,1]$ if and only if $f_\alpha(0) \to f_0(0)$ and $f'_\alpha \xrightarrow{o} f'_0$ in $C[0,1]$.

In this paper, we will give the representation of ε-disjointness preserving functional on $C^1[0,1]$ in Theorem 2.5. For the basic nations about Banach lattices, we refer the reader to [11].

§ 2. Almost disjointness preserving functionals

Let X be a compact Hausdorff space and μ a regular finite Borel measure. Then μ induces a bounded linear functional Δ on $C(X)$ by

$$\Delta(f) = \int_X f d\mu, \quad \forall f \in C(X). \tag{2.1}$$

Moreover, $\|\Delta\| = |\mu|(X)$. The Riesz Representation Theorem tells us that the converse is also true.

Theorem 2.1. *Suposee that X be a compact Hausdorff space and μ is a regular Borel finite measure, then the bounded linear functional $\Delta : C(X) \to \mathbb{R}$ defined by (2.1) is not ε-disjointness preserving if and only if there exist two disjoint compact subsets K_1 and K_2 of X such that $|\mu|(K_1) > \varepsilon$ and $|\mu|(K_2) > \varepsilon$.*

Proof. Suppose that K_1, K_2 are disjoint compact sets in X with $|\mu|$ measure greater than ϵ. Let U_1, U_2 be disjoint open sets in X containing K_1, K_2, respectively. It follows from the Hahn decomposition theorem (for $|\mu|$ restricting to K_1) (see, e.g., [15, Theorem 6.14]), we can assume that

$$|\mu|(K_1) = \mu(A) - \mu(B) > \varepsilon$$

where $K_1 = A \cup B$ is a partition into the positive and negative subsets of K_1 with respect to the real measure μ. By the regularity of μ, we can choose compact subsets A_1, B_1 of A, B such that

$$\mu(A \setminus A_1) - \mu(B \setminus B_1) < \varepsilon - \mu(A) + \mu(B).$$

By the Urysohn Lemma, we can choose a function f_1 in $C(X)$ such that $f_1 = 1$ on A_1, $f_1 = -1$ on B_1, $-1 \leq f_1 \leq 1$ and $f_1 = 0$ outside U_1. Then

$$\Delta(f_1) = \int_X f_1 d\mu > \mu(A_1) - \mu(B_1) - \mu(A \setminus A_1) + \mu(B \setminus B_1) > \varepsilon.$$

In a similar manner, we have $f_2 \in C(X)$ such that $f_1 \perp f_2$ and $\Delta(f_2) > \varepsilon$. This disjoint pair f_1, f_2 verifies that Δ is not ε-disjointness preserving.

Conversely, suppose that, for every two disjoint compact sets K_1 and K_2, we have that $|\mu|(K_1) \leq \varepsilon$ or $|\mu|(K_2) \leq \varepsilon$. Let $f \perp g$ with $|f| \leq 1$ and $|g| \leq 1$. Without loss of generality, one can assume that there is a compact subset $L_1 \subset \text{coz}(f)$ such that $|\mu|(L_1) > \varepsilon$, this forces that, for every compact subset $L_2 \subset \text{coz}(g)$, $|\mu|(L_2) \leq \varepsilon$. It follows from the regularity of μ that $|\mu|(\text{coz}(g)) \leq \varepsilon$, and then

$$\begin{aligned} |\Delta(g)| = |\int_X g d\mu| &\leq \int_X |g| d|u| \\ &= \int_{\text{coz}(g)} |g| d|u| \leq |\mu|(\text{coz}(g)) \leq \varepsilon. \end{aligned}$$

This implies that $|\Delta(f)| \wedge |\Delta(g)| \leq \varepsilon$, and then Δ is a bounded ε-disjointness preserving linear functional. □

Recall that $\mathbb{R} \oplus_1 C(X)$ is the Banach lattice with the canonical order and norm, where X is a compact Hausdorff space and $C(X)$ is the spaces of continuous functions on X.

Suppose that ψ is a linear functional on $\mathbb{R} \oplus_1 C(X)$. Let

$$\psi_l : r \in \mathbb{R} \mapsto \psi(r, 0) \in \mathbb{R}$$

and

$$\psi_r : f \in C(X) \mapsto \psi(0, f) \in \mathbb{R},$$

we have that $\psi(r, f) = \psi_l(r) + \psi_r(f)$ for all $(r, f) \in \mathbb{R} \oplus_1 C(X)$.

Theorem 2.2. *Let ψ be a continuous linear functional on $\mathbb{R} \oplus_1 C(X)$. Then ψ is of the form*

$$\psi(r, f) = kr + \int_X f d\mu,$$

where k is a real number and μ is a regular Borel finite measure.

Proof. Since ψ is continuous, ψ_1 and ψ_r both are continuous, then there exists a regular Borel finite measure μ such that

$$\psi_r(f) = \int_X f d\mu, \quad \forall f \in C(X),$$

and there exists a real number k such that $\psi_l(r) = kr$ for all $r \in \mathbb{R}$. $\square$

It is easy to show that

Theorem 2.3. *Let ψ be a bounded ε-disjointness preserving linear functional on $\mathbb{R} \oplus_1 C(X)$. Then ψ_l and ψ_r are both bounded ε-disjointness preserving.*

Theorem 2.4. *Suppose that ψ is a continuous linear functional on $\mathbb{R} \oplus_1 C(X)$, then ψ is ε-disjointness preserving if and only if following two conditions holds.*

(i) $\|\psi_l\| \leq \varepsilon$ or $\|\psi_r\| \leq \varepsilon$.

(ii) ψ_l and ψ_r are bounded ε-disjointness preserving linear functional.

Proof. Since ψ is a bounded ε-disjointness preserving linear functional, it follows from Theorem 2.3 that ψ_l and ψ_r are bounded ε-disjointness preserving. Suppose on the contrary that $\|\psi_l\| > \varepsilon$ and $\|\psi_r\| > \varepsilon$, then there exists a $f \in C(X)$ with $\|f\| = 1$ such that $|\psi_r(f)| > \varepsilon$. Let $\xi_1 = (0, f)$ and $\xi_2 = (1, 0)$, we can derive that $\xi_1 \perp \xi_2$, $|\psi(\xi_1)| = |\psi_r(f)| > \varepsilon$ and $|\psi(\xi_2)| = |\psi_l(1)| = \|\psi_l\| > \varepsilon$, which implies ψ is not ε-disjointness preserving.

Conversely, suppose that $\eta_1 = (r_1, f_1)$ and $\eta_2 = (r_2, f_2)$ be in $\mathbb{R} \oplus_1 C(X)$ with $\eta_1 \perp \eta_2$ and $\|\eta_1\| \leq 1$ and $\|\eta_2\| \leq 1$, then one can derive that $r_1 r_2 = 0$ and $f_1 f_2 = 0$. Without loss of generality, we can assume that r_1=0 and $r_2 \neq 0$. It follows from Theorem 2.3 that ψ_r is ε-disjointness preserving, which implies that

$$|\psi_r(f_1)| \wedge |\psi_r(f_2)| \leq \varepsilon.$$

Since $\|\eta_2\| = |r_2| + \|f_2\|_\infty \leq 1$, then we have that

$$\frac{\|f_2\|_\infty}{1 - |r_2|} \leq 1.$$

On the one hand, if $|\psi(\eta_1)| = |\psi_r(f_1)| \leq \varepsilon$, then $|\psi(\eta_1)| \wedge |\psi(\eta_2)| \leq \varepsilon$. On the other hand, if $|\psi(\eta_1)| = |\psi_r(f_1)| > \varepsilon$, since ψ_r is ε-disjointness preserving, then one can derive that

$$|\psi_r(\frac{f_2}{1 - |r_2|})| \leq \varepsilon. \tag{2.2}$$

In case of $\|\psi_l\| \leq \varepsilon$, we have that

$$\begin{aligned} |\psi(\eta_2)| &= |\psi_l(r_2) + \psi_r(f_2)| \\ &\leq \varepsilon|r_2| + (1 - |r_2|)\psi_r(\frac{f_2}{1 - |r_2|}) \\ &\leq \varepsilon|r_2| + (1 - |r_2|)\varepsilon = \varepsilon, \end{aligned}$$

and then $|\psi(\eta_1)| \wedge |\psi(\eta_2)| \leq \varepsilon$. In case of $\|\psi_r\| \leq \epsilon$, by the similar argument, one can derive that

$$|\psi(\eta_1)| = |\psi_r(f_1)| \leq \|\psi_r\| \leq \varepsilon,$$

and then $|\psi(\eta_1)| \wedge |\psi(\eta_2)| \leq \varepsilon$. □

Theorem 2.5. *Suppose that ψ is a bounded linear functional on $C^1[0,1]$, then ψ is ε-disjointness preserving if and only if*

$$\psi f = kf(0) + \int_{[0,1]} f' d\mu,$$

where k is a real number and μ is a regular Borel finite measure such that $|k| \leq \varepsilon$ or $|\mu|([0,1]) \leq \varepsilon$, and μ satisfies the conditions of Theorem 2.1.

Proof. Since ψ is ε-disjointness preserving if and only if $\psi\pi^{-1}$ is a ε-disjointness preserving on $\mathbb{R} \oplus_1 C[0,1]$, we can complete the proof using Theorem 2.4. □

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