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Isometries, Jordan $*$-isomorphisms and order isomorphisms on spaces of a unital C^*-algebra-valued continuous maps

By

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Abstract

This article presents a survey of the paper [14] with applications. In this paper, we study Jordan $*$-isomorphisms, surjective linear isometries and order isomorphisms on the spaces of continuous maps taking values in unital C^*-algebras.

§ 1. Introduction

Let X be a compact Hausdorff space and $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ a unital C^*-algebra. We denote the unit of $\mathcal{A}$ by $1_{\mathcal{A}}$. In this paper, every C^*-algebra is assumed to be unital. An element a in a C^*-algebra is positive if a is self-adjoint and the spectrum $\sigma(a) \subset \{r \in \mathbb{R} \,|r \geq 0\}$. We denote by $\mathcal{A}^+$ the collection of positive elements in $\mathcal{A}$. We denote the set of all pure states on $\mathcal{A}$ by $\mathrm{PS}(\mathcal{A})$. For any $\rho \in \mathrm{PS}(\mathcal{A})$, there exists an irreducible representation $\pi_\rho : \mathcal{A} \to B(H_\rho)$, where $B(H_\rho)$ is the Banach space of all bounded linear operators on a Hilbert space H_ρ. We denote $C(X, \mathcal{A})$ by the space of all $\mathcal{A}$-valued continuous maps on X with the supremum norm $\|\cdot\|_\infty$, that is, $\|F\|_\infty = \sup\{\|F(x)\|_{\mathcal{A}} : x \in X\}$. When $\mathcal{A} = \mathbb{C}$, we denote $C(X, \mathbb{C})$ by $C(X)$. For any $F \in C(X, \mathcal{A})$, we define $F^* \in C(X, \mathcal{A})$ by $F^*(x) = [F(x)]^*$ for any $x \in X$. Then $C(X, \mathcal{A})$ is a unital C^*-algebra. The unit of $C(X, \mathcal{A})$ is a constant map $1 : X \to \mathcal{A}$ which satisfies $1(x) = 1_{\mathcal{A}}$ for any $x \in X$.

Throughout the paper let X_1, X_2 be compact Hausdorff spaces and $\mathcal{A}_1$, $\mathcal{A}_2$ be unital C^*-algebras.

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Definition 1.1. Let $T : \mathcal{A}_1 \to \mathcal{A}_2$ be a bijective linear map.

- If the map T satisfies that $\|Ta - Tb\|_{\mathcal{A}_2} = \|a - b\|_{\mathcal{A}_1}$ for any $a, b \in \mathcal{A}_1$, T is called an isometry.
- If both T and T^{-1} preserve the order structure, i.e., $a \in \mathcal{A}_1^+$ if and only if $Ta \in \mathcal{A}_2^+$, then T is called an order isomorphism.
- If the map T holds $T(a^*) = (Ta)^*$ and $T(a^2) = (Ta)^2$ for any $a \in \mathcal{A}_1$, then T is called a Jordan $*$-isomorphism.

In this paper we study Jordan $*$-isomorphisms, surjective linear isometries and order isomorphisms from $C(X_1, \mathcal{A}_1)$ onto $C(X_2, \mathcal{A}_2)$.

In the classical period three results stand out, namely the theorems of Banach and Stone, Gelfand and Kolmogorov, and Kaplansky.

Theorem 1.2 (Banach and Stone [1, 16]). *Let X_1 and X_2 be compact Hausdorff spaces. Let $T : C(X_1) \to C(X_2)$ be a map. Then T is a surjective linear isometry if and only if there is a homeomorphism $\varphi : X_2 \to X_1$ and a continuous function $u \in C(X_2)$ with $|u| = 1$ such that*

$$Tf(y) = u(y)(f(\varphi(y))), \quad f \in C(X_1), \quad y \in X_2.$$

Remark. Banach proved in [1] that if T is a surjective linear isometry from $C_{\mathbb{R}}(X_1)$ onto $C_{\mathbb{R}}(X_2)$, where $C_{\mathbb{R}}(X_j)$ is the space of all real-valued continuous functions on a compact metric space X_j, then T is a weighted composition operator. In the case when X_j are compact Hausdorff spaces, Stone proved that every surjective linear isometry between $C_{\mathbb{R}}(X_j)$ is also a weighted composition operator in [16]. The statement of Theorem 1.2 is considered as the modern version of Banach's and Stone's contributions and is called the Banach-Stone theorem.

Theorem 1.3 (Kaplansky [10]). *Let X_1 and X_2 be compact Hausdorff spaces. Let $T : C(X_1) \to C(X_2)$ be a map. Then T is an order isomorphism if and only if there is a homeomorphism $\varphi : X_2 \to X_1$ and a positive invertible element $u \in C(X_2)$ such that*

$$Tf(y) = u(y)(f(\varphi(y))), \quad f \in C(X_1), \quad y \in X_2.$$

Theorem 1.4 (Gelfand and Kolmogorov [4]). *Let X_1 and X_2 be compact Hausdorff spaces. Let $T : C(X_1) \to C(X_2)$ be a map. Then T is an algebra isomorphism if and only if there is a homeomorphism $\varphi : X_2 \to X_1$ such that*

$$Tf(y) = f(\varphi(y)), \quad f \in C(X_1), \quad y \in X_2.$$

These imply that a compact Hausdorff space X is determined by the metric structure, the order structure and the algebraic structure respectively. How about the case of the vector-valued maps? Jerison in [7] obtained the first vector-valued version of the Banach-Stone Theorem: Suppose that E is a strictly convex Banach space. Let T be a surjective linear isometry from $C(X_1, E)$ onto $C(X_2, E)$. Then there is a homeomorphism $\varphi : X_2 \to X_1$ and a continuous map U with the strong operator topology from X_2 into the space of surjective linear isometries on E, where $U_y : E \to E$ is a surjective linear isometry for any $y \in X_2$, such that

$$TF(y) = U_y(F(\varphi(y))), \quad F \in C(X_1, E), \quad y \in X_2.$$

Cambern, in [2], defined the Banach-Stone property. A Banach space E is said to have the Banach-Stone property if every surjective linear isometry $T : C(X_1, E) \to C(X_2, E)$ admits a homeomorphism $\varphi : X_2 \to X_1$ and a strongly continuous family $\{V_y\}_{y \in X_2}$ of surjective linear isometries on E such that

$$TF(y) = V_y(F(\varphi(y))), \quad F \in C(X_1, E), \quad y \in X_2.$$

Later many results of surjective linear isometries are exhibited including in [3, 5, 6, 11]. They studied Banach spaces which have the Banach-Stone property. In particular, in [5], the authors studied unital surjective linear isometries between the injective tensor product of a uniform algebra and a unital factor C^*-algebra. A unital C^*-algebra $\mathcal{A}$ is called a factor if its center is trivial. As a corollary of their main theorems, they show that unital factor C^*-algebras have the Banach-Stone property as the following:

Theorem 1.5 (Corollary 5 in [5]). *Let $\mathcal{A}_i$ be a unital factor C^*-algebra for $i = 1, 2$. Then a bijective linear map $T : C(X_1, \mathcal{A}_1) \to C(X_2, \mathcal{A}_2)$ is a surjective linear isometry if and only if there exists a homeomorphism $\varphi : X_2 \to X_1$, a strongly continuous family $\{V_y\}_{y \in X_2}$ of Jordan $*$-isomorphisms from $\mathcal{A}_1$ onto $\mathcal{A}_2$ and a unitary element $U \in C(X_2, \mathcal{A}_2)$ such that*

$$TF(y) = U(y)V_y(F(\varphi(y)))$$

for any $F \in C(X_1, \mathcal{A}_1)$ and $y \in X_2$.

The authors studied hermitian operators on the injective tensor product of a uniform algebra and a unital factor C^*-algebra. They applied the notion of hermitian operators and the technique, which was introduced by Lumer in [12, 13], to characterize surjective linear isometries on the spaces.

In this paper, we study surjective linear isometries by applying studies of Jordan $*$-isomorphisms. In Theorem 3.3 we prove that a primitive C^*-algebra $\mathcal{A}$ has the Banach-Stone property. Since a primitive C^*-algebra is factor, Theorem 3.3 follows from Theorem 1.5. However the proof is quite different and simple. We do not need studying hermitian operators on $C(X,\mathcal{A})$.

§ 2. Jordan $*$-isomorphisms

We introduce the studies of Jordan $*$-isomorphisms on $C(X,\mathcal{A})$ in [14]. In addition, we will proceed to sketch the main ideas of the proof for completeness. We refer the reader to the paper [14] for more details.

In the proof, we consider the algebraic tensor product space. The algebraic tensor product space of $C(X)$ and $\mathcal{A}$ over $\mathbb{C}$ is denoted by $C(X)\otimes\mathcal{A}$.

Theorem 2.1 (Theorem 2.4. in [14]). *Let $J : C(X_1,\mathcal{A}_1) \to C(X_2,\mathcal{A}_2)$ be a Jordan $*$-isomorphism. Then there exist a continuous map $\varphi_\rho : X_2 \to X_1$ for every $\rho \in \mathrm{PS}(\mathcal{A}_2)$ and a Jordan $*$-homomorphism $V_y : \mathcal{A}_1 \to \mathcal{A}_2$ for each $y \in X_2$ such that*

$$\pi_\rho(JF(y)) = \pi_\rho(V_y(F(\varphi_\rho(y))))$$

for all $F \in C(X_1,\mathcal{A}_1)$ and all $y \in X_2$.

Outline of the proof of Theorem 2.1.
We sketch the outline of the proof of Theorem 2.1. We have

$$\mathrm{PS}(C(X_2,\mathcal{A}_2)) = \{\rho\circ\delta_y|\ \rho\in\mathrm{PS}(\mathcal{A}_2), y\in X_2\},$$

where δ_y is a complex linear operator from $C(X_2,\mathcal{A}_2)$ into $\mathcal{A}_2$ such that $\delta_y(F) = F(y)$ for any $F \in C(X_2,\mathcal{A}_2)$. We denote the commutant of $\pi_{\rho\circ\delta_y}(C(X_2,\mathcal{A}_2))$ by $\pi_{\rho\circ\delta_y}(C(X_2,\mathcal{A}_2))'$. We denote the identity operator on $H_{\rho\circ\delta_y}$ by $I_{H_{\rho\circ\delta_y}}$. Since a space $\pi_{\rho\circ\delta_y}(C(X_2,\mathcal{A}_2))$ acts irreducibly on $H_{\rho\circ\delta_y}$, we get $\pi_{\rho\circ\delta_y}(C(X_2,\mathcal{A}_2))' = \mathbb{C}I_{H_{\rho\circ\delta_y}}$. Corollary 3.4 in [15] implies that $\pi_{\rho\circ\delta_y}\circ J : C(X_1,\mathcal{A}_1) \to B(H_{\rho\circ\delta_y})$ is either a $*$-homomorphism or an anti $*$-homomorphism. Thus we get for any $f \in C(X_1)$, $\pi_{\rho\circ\delta_y}\circ J(f\otimes 1_{\mathcal{A}_1}) \in \mathbb{C}I_{H_{\rho\circ\delta_y}}$. We define $\lambda_f \in \mathbb{C}$ by

$$\pi_{\rho\circ\delta_y}\circ J(f\otimes 1_{\mathcal{A}_1}) = \lambda_f\cdot I_{H_{\rho\circ\delta_y}}. \tag{2.1}$$

Since $J^*(\rho\circ\delta_y) \in \mathrm{PS}(C(X_1,\mathcal{A}_1))$, there exist $\phi_{\rho,y} \in \mathrm{PS}(\mathcal{A}_1)$ and $x \in X_1$ such that $J^*(\rho\circ\delta_y) = \phi_{\rho,y}\circ\delta_x$. We define $\varphi_\rho : X_2 \to X_1$ by $J^*(\rho\circ\delta_y) = \phi_{\rho,y}\circ\delta_{\varphi_\rho(y)}$. Since $J(1) = 1$, we get $f(\varphi_\rho(y)) = \lambda_f$ for any $f \in C(X_1)$. Moreover (2.1) shows that $\pi_\rho(J(f\otimes 1_{\mathcal{A}_1})(y)) = f(\varphi_\rho(y))I_{H_\rho}$. Fix $y \in Y_2$. We define a Jordan $*$-homomorphism $V_y : \mathcal{A}_1 \to \mathcal{A}_2$ by

$$V_y(a) = J(1\otimes a)(y)$$

for any $a \in \mathcal{A}_1$. As $\rho \in \mathrm{PS}(\mathcal{A}_2)$, π_ρ is an irreducible representation of $\mathcal{A}_2$. It follows that

$$\begin{aligned}\pi_\rho(J(f\otimes a)(y)) &= \pi_\rho(J(f\otimes 1)(y))\pi_\rho(J(1\otimes a)(y))\\ &= \pi_\rho(V_y(f(\varphi_\rho(y))a)) = \pi_\rho(V_y(f\otimes a(\varphi_\rho(y)))).\end{aligned}$$

Since $C(X_1)\otimes\mathcal{A}_1$ is dense in $C(X_1,\mathcal{A}_1)$ and J is a bounded operator with $\|\cdot\|_\infty$, we have $\pi_\rho(JF(y)) = \pi_\rho(V_y(F(\varphi_\rho(y))))$ for all $F \in C(X_1,\mathcal{A}_1)$ and all $y \in X_2$. □

Remark. Since there is a Jordan $*$-isomorphism from $C(\{a\},\mathbb{C}^2)$ onto $C(\{x,y\},\mathbb{C})$, $C(\{a\},\mathbb{C}^2)$ and $C(\{x,y\},\mathbb{C})$ is isometric $*$-isomorphic. On the other hand, $\{a\}$ is not homeomorphic to $\{x,y\}$ and $\mathbb{C}^2$ is not $*$-isomorphic to $\mathbb{C}$. We can not expect to get that V_y is an isomorphism and φ_ρ is a homeomorphism.

Theorem 2.2 (Theorem 2.7. in [14]). *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are primitive. Then $J : C(X_1,\mathcal{A}_1)\to C(X_2,\mathcal{A}_2)$ is a Jordan $*$-isomorphism if and only if there exist a homeomorphism φ from X_2 onto X_1 and a Jordan $*$-isomorphism $V_y : \mathcal{A}_1\to\mathcal{A}_2$ for each $y\in X_2$ so that the map $y\mapsto V_y$ is continuous with respect to the strong operator topology such that*

$$JF(y) = V_y(F(\varphi(y))) \tag{2.2}$$

for all $F\in C(X_1,\mathcal{A}_1)$ and $y\in X_2$.

Outline of the proof of Theorem 2.2. We omit the proof of the statement that if J is of the form described as (2.2) in the statement then J is a Jordan $*$-isomorphism from $C(X_1,\mathcal{A}_1)$ onto $C(X_2,\mathcal{A}_2)$.

We only mention that if $J : C(X_1,\mathcal{A}_1)\to C(X_2,\mathcal{A}_2)$ is a Jordan $*$- isomorphism, then J has the form as (2.2) with the desired properties for φ and V_y. Since $\mathcal{A}_2$ is a primitive C^*-algebra, there is a $\rho\in\mathrm{PS}(\mathcal{A}_2)$ such that π_ρ is faithful. We define $\varphi : X_2\to X_1$ by $\varphi=\varphi_\rho$. By Theorem 2.1, we have

$$JF(y) = V_y(F(\varphi(y))).$$

We prove that φ is a homeomorphism. Let $y_1,y_2\in X_2$ such that $\varphi(y_1)=\varphi(y_2)$. Since $\mathcal{A}_1$ is a primitive C^*- algebra, there exist a continuous map $\psi : X_1\to X_2$ and a Jordan $*$-homomorphism $S_x : \mathcal{A}_2\to\mathcal{A}_1$ for each $x\in X_1$ such that $J^{-1}F(x) = S_x(F(\psi(x)))$, for all $F\in C(X_2,\mathcal{A}_2)$. This implies that for all $F\in C(X_1,\mathcal{A}_1)$, we have $F(x) = S_x(V_{\psi(x)}F(\varphi(\psi(x))))$. As $C(X_1,\mathcal{A}_1)$ separates the points of X_1, we get $\varphi(\psi(x)) = x$ for any $x\in X_1$. Applying a similar argument, we get $F(y) = V_y(S_{\varphi(y)}(F(\psi(\varphi(y)))))$ for all $F\in C(X_2,\mathcal{A}_2)$ and $\psi(\varphi(y)) = y$ for any $y\in X_2$. As X_1 and X_2 are compact

Hausdorff spaces and φ is a continuous map, φ is a homeomorphism. By $\varphi^{-1} = \psi$, we get $S_{\varphi(y)} \circ V_y = I_{\mathcal{A}_1}$ and $V_y \circ S_{\varphi(y)} = I_{\mathcal{A}_2}$, where $I_{\mathcal{A}_i}$ is the identity operator on $\mathcal{A}_i$ for $i = 1, 2$. Thus V_y is a bijection and V_y is a Jordan $*$-isomorphism such that $V_y^{-1} = S_{\varphi(y)}$.

Let $\{y_\lambda\} \subset X_2$ be a net with $y_\lambda \to y_0$. For any $a \in \mathcal{A}_1$, $\|V_{y_\lambda}(a) - V_{y_0}(a)\|_{\mathcal{A}_2} = \|J(1 \otimes a)(y_\lambda) - J(1 \otimes a)(y_0)\|_{\mathcal{A}_2} \to 0$ as $J(1 \otimes a) \in C(X_2, \mathcal{A}_2)$. This implies that the map $y \mapsto V_y$ is continuous with respect to the strong operator topology. □

§ 3. Applications of Theorems 2.1 and 2.2

§ 3.1. Surjective linear isometries

Kadison in 1951 obtained the following characterization for surjective complex linear isometries between unital C^*-algebras.

Theorem 3.1 (Kadison [8]). *Let $\mathcal{A}_i$ be a unital C^*-algebra for $i = 1, 2$. Then $T : \mathcal{A}_1 \to \mathcal{A}_2$ is a surjective linear isometry if and only if there is a unitary element $u \in \mathcal{A}_2$ and a Jordan $*$-isomorphism $J : \mathcal{A}_1 \to \mathcal{A}_2$ such that*

$$T(a) = uJ(a), \quad a \in \mathcal{A}_1.$$

Applying Theorem 3.1, we obtain the following theorems.

Theorem 3.2. *Let $T : C(X_1, \mathcal{A}_1) \to C(X_2, \mathcal{A}_2)$ be a surjective linear isometry. Then there exist a unitary element $U \in C(X_2, \mathcal{A}_2)$, a continuous map $\varphi_\rho : X_2 \to X_1$ for every $\rho \in \mathrm{PS}(\mathcal{A}_2)$ and a Jordan $*$-homomorphism $V_y : \mathcal{A}_1 \to \mathcal{A}_2$ for each $y \in X_2$ such that*

$$\pi_\rho(TF(y)) = \pi_\rho(U(y)V_y(F(\varphi_\rho(y))))$$

for all $\rho \in \mathrm{PS}(\mathcal{A}_2)$, $F \in C(X_1, \mathcal{A}_1)$ and $y \in X_2$.

Proof. By theorem 3.1, there is a unitary element $U \in C(X_2, \mathcal{A}_2)$ and a Jordan $*$-isomorphism $J : C(X_1, \mathcal{A}_1) \to C(X_2, \mathcal{A}_2)$ such that $T(F) = UJ(F)$ for any $F \in C(X_1, \mathcal{A}_1)$. By Theorem 2.1, there exist a continuous map $\varphi_\rho : X_2 \to X_1$ for every $\rho \in \mathrm{PS}(\mathcal{A}_2)$ and a Jordan $*$-homomorphism $V_y : \mathcal{A}_1 \to \mathcal{A}_2$ for each $y \in X_2$ such that

$$\pi_\rho(JF(y)) = \pi_\rho(V_y(F(\varphi_\rho(y))))$$

for all $F \in C(X_1, \mathcal{A}_1)$ and all $y \in X_2$. Since $\pi_\rho : \mathcal{A}_2 \to B(H_\rho)$ is an irreducible representation for any $\rho \in \mathrm{PS}(\mathcal{A}_2)$, we have

$$\begin{aligned}\pi_\rho(TF(y)) = \pi_\rho(U(y)J(F)(y)) &= \pi_\rho(U(y))\pi_\rho(J(F)(y)) \\ &= \pi_\rho(U(y))\pi_\rho(V_y(F(\varphi_\rho(y)))) = \pi_\rho(U(y)V_y(F(\varphi_\rho(y))))\end{aligned}$$

for any $F \in C(X_1, \mathcal{A}_1)$, $y \in X_2$. $\square$

In particular when $\mathcal{A}_1$ and $\mathcal{A}_2$ are primitive C^*-algebras, we have the following characterization.

Theorem 3.3. *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are primitive. Then $T : C(X_1, \mathcal{A}_1) \to C(X_2, \mathcal{A}_2)$ is a surjective linear isometry if and only if there exist a unitary element $U \in C(X_2, \mathcal{A}_2)$, a homeomorphism φ from X_2 onto X_1 and a Jordan $*$-isomorphism $V_y : \mathcal{A}_1 \to \mathcal{A}_2$ for each $y \in X_2$ so that the map $y \mapsto V_y$ is continuous with respect to the strong operator topology such that*

$$TF(y) = U(y)V_y(F(\varphi(y))) \tag{3.1}$$

for all $F \in C(X_1, \mathcal{A}_1)$ and $y \in X_2$.

Proof. Firstly suppose that T is of the form described as (3.1). Then by Theorem 2.2, the map $F \mapsto V_\cdot(F(\varphi(\cdot)))$ from $C(X_1, \mathcal{A}_1)$ onto $C(X_2, \mathcal{A}_2)$ is a Jordan $*$ isomorphism. Thus Theorem 3.1 implies that T is a surjective linear isometry. Now the converse statement is clear by Theorem 3.1 and Theorem 2.2. $\square$

Remark 3.4. Let $\mathcal{A}$ be a unital C^*-algebra. If $\mathcal{A}$ is primitive then it is factor. But the converse is not true. Theorem 3.3 is a corollary of Theorem 1.5. Thus, this result was already obtained but our proof is much simpler than that presented in [5].

§ 3.2. Order isomorphisms

In [9], Kadison proved that every order isomorphism which carries the identity into identity between unital C^*-algebras is a Jordan $*$-isomorphism. We obtain the following characterization of order isomorphisms between unital C^*-algebras as a corollary of the theorem of Kadison. Although this is a well-known fact, we give a proof by applying [9, Corollary 5] for completeness.

Theorem 3.5 (Kadison [9]). *Let $\mathcal{A}_i$ be a unital C^*-algebra for $i = 1, 2$. Then $T : \mathcal{A}_1 \to \mathcal{A}_2$ is an order isomorphism if and only if there is a positive invertible element $u \in \mathcal{A}_2$ and a Jordan $*$-isomorphism $J : \mathcal{A}_1 \to \mathcal{A}_2$ such that*

$$T(a) = uJ(a)u, \quad a \in \mathcal{A}.$$

Proof. Let $T : \mathcal{A}_1 \to \mathcal{A}_2$ be an order isomorphism. Then there is a positive element $a \in \mathcal{A}_1$ with $a \neq 0$ such that $T(a) = 1_{\mathcal{A}_2}$. Since $a \leq \|a\| 1_{\mathcal{A}_1}$, we get

$$1_{\mathcal{A}_2} = T(a) \leq \|a\| T(1_{\mathcal{A}_1}).$$

As $\|a\| \neq 0$, we get $\frac{1_{\mathcal{A}_2}}{\|a\|} \leq T(1_{\mathcal{A}_1})$. By the Gelfand representation applied to the C^*-algebra generated by $T(1_{\mathcal{A}_1})$ and $1_{\mathcal{A}_2}$, we get $0 \notin \sigma(T1_{\mathcal{A}_1})$ and $T1_{\mathcal{A}_1}$ is an invertible element. Hence we put $u = (T1_{\mathcal{A}_1})^{\frac{1}{2}}$. Since u is also invertible and positive in $\mathcal{A}_2$, we define a map $J : \mathcal{A}_1 \to \mathcal{A}_2$ by $J(a) = u^{-1}T(a)u^{-1}$ for any $a \in \mathcal{A}_1$. It is clear that J is an order isomorphism which satisfies that $J(1_{\mathcal{A}_1}) = 1_{\mathcal{A}_2}$. By [9, Corollary 5], J is a Jordan $*$-isomorphism. We conclude that $T(a) = uJ(a)u$ for all $a \in \mathcal{A}_1$.

Conversely we assume that $T(a) = uJ(a)u$ for any $a \in \mathcal{A}_1$, where J is a Jordan $*$-isomorphism. Then T is a linear bijective map. Since J is an order isomorphism, for any $a \in \mathcal{A}_1$ there is $b \in \mathcal{A}_2$ such that $J(a^*a) = b^*b$. Hence we obtain that

$$T(a^*a) = uJ(a^*a)u = ub^*bu = (bu)^*(bu) \geq 0.$$

In addition, $T^{-1}(b) = J^{-1}(u^{-1}bu^{-1})$ for any $b \in \mathcal{A}_2$. We have

$$T^{-1}(b^*b) = J^{-1}(u^{-1}b^*bu^{-1}) = J^{-1}((bu^{-1})^*bu^{-1}) \geq 0$$

for all $b \in \mathcal{A}_2$. Thus T is an order isomorphism. □

By applying Theorem 3.5 and similar arguments as in the case of surjective linear isometries, we obtain the following theorems.

Theorem 3.6. *Let $T : C(X_1, \mathcal{A}_1) \to C(X_2, \mathcal{A}_2)$ be an order isomorphism. Then there exist a positive invertible element $U \in C(X_2, \mathcal{A}_2)$, a continuous map $\varphi_\rho : X_2 \to X_1$ for any $\rho \in \mathrm{PS}(\mathcal{A}_2)$ and a Jordan $*$-homomorphism $V_y : \mathcal{A}_1 \to \mathcal{A}_2$ for each $y \in X_2$ such that*

$$\pi_\rho(TF(y)) = \pi_\rho(U(y)V_y(F(\varphi_\rho(y)))U(y))$$

for all $\rho \in \mathrm{PS}(\mathcal{A}_2)$, $F \in C(X_1, \mathcal{A}_1)$ and $y \in X_2$.

Theorem 3.7. *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are primitive. Then $T : C(X_1, \mathcal{A}_1) \to C(X_2, \mathcal{A}_2)$ is an order isomorphism if and only if there exist a positive invertible element $U \in C(X_2, \mathcal{A}_2)$, a homeomorphism φ from X_2 onto X_1 and a Jordan $*$-isomorphism $V_y : \mathcal{A}_1 \to \mathcal{A}_2$ for each $y \in X_2$ where the map $y \mapsto V_y$ is continuous with respect to the strong operator topology such that*

$$TF(y) = U(y)V_y(F(\varphi(y)))U(y) \tag{3.2}$$

for all $F \in C(X_1, \mathcal{A}_1)$ and $y \in X_2$.

Remark 3.8. We obtain that for any unital C^*-algebra $\mathcal{A}$, Jordan $*$-isomorphisms, surjective linear isomorphisms and order isomorphisms on $C(X, \mathcal{A})$ are represented by weighted composition operators by using the irreducible representations on $\mathcal{A}$. Moreover when $\mathcal{A}$ is a primitive C^*-algebra, we obtain complete representations of these operators. These are one of the vector-valued versions of the classical theorems by Banach and Stone, Gelfand and Kolmogorov, and Kaplansky. It is not clear to the author whether complete representations of Jordan $*$-isomorphisms, surjective linear isomorphisms and order isomorphisms on $C(X, \mathcal{A})$ are obtained for other classes of C^*-algebras $\mathcal{A}$.

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