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Tingley's problem for a Banach space of Lipschitz functions on the closed unit interval

By

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Abstract

We prove that every surjective isometry on the unit sphere of Lip(I) of all Lipschitz continuous functions on the closed unit interval I is extended to a surjective real linear isometry on Lip(I) with the norm $\|f\|_\sigma = |f(0)| + \|f'\|_{L^\infty}$.

§ 1. Introduction and main results

Let E and F be Banach spaces whose unit spheres are S_E and S_F, respectively. In 1987, Tingley [32] asks whether each surjective isometry $\Delta\colon S_E \to S_F$ is extended to a surjective, real linear isometry from E onto F. Since then, many mathematicians have given affirmative answers to the Tingley's problem for particular Banach spaces. There is a huge list of the research of the problem, here we show only some of them. Tingley's problem is treated for function spaces in [4, 15, 17, 18, 33, 34], and for operator spaces in [7, 8, 9, 10, 11, 12, 22, 23, 24, 29, 30, 31]. Besides the Tingley's problem, the Mazur–Ulam property for Banach spaces has been studying actively; a Banach space E has the Mazur–Ulam property if F is any Banach space, every surjective isometry from S_E onto S_F admits a unique extension to a surjective real linear isometry from E onto F. See, for example, [1, 5, 14, 21, 26, 27].

Let Lip(I) be the complex linear space of all Lipschitz continuous complex valued functions on the closed unit interval $I = [0, 1]$. For each Banach space E, we denote by

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S_E the unit sphere of E. We define $\|f\|_\sigma$ for $f \in \mathrm{Lip}(I)$ by

$$\|f\|_\sigma = |f(0)| + \|f'\|_{L^\infty},$$

where $\|\cdot\|_{L^\infty}$ denotes the essential supremum norm on I. It is well known that each $f \in \mathrm{Lip}(I)$ has essentially bounded derivative f' almost everywhere. Hence, f' belongs to $L^\infty(I)$, the commutative Banach algebra of all essentially bounded measurable functions on I with the essential supremum norm $\|\cdot\|_{L^\infty}$. Consequently, $\|\cdot\|_\sigma$ is a well defined norm on $\mathrm{Lip}(I)$. The purpose of this paper is to prove that every surjective isometry on $S_{\mathrm{Lip}(I)}$ admits a surjective real linear extension to $\mathrm{Lip}(I)$, which gives a solution to Tingley's problem for $\mathrm{Lip}(I)$. The followings are the main results of this paper.

Theorem 1.1. *Let $\Delta\colon S_{\mathrm{Lip}(I)} \to S_{\mathrm{Lip}(I)}$ be a surjective isometry with $\|\cdot\|_\sigma$. Then Δ is extended to a surjective, real linear isometry on* $\mathrm{Lip}(I)$.

Corollary 1.2. *For each surjective isometry $\Delta_1\colon \mathrm{Lip}(I) \to \mathrm{Lip}(I)$ with $\|\cdot\|_\sigma$, there exist a constant α of modulus 1, $h_0 \in S_{L^\infty(I)}$ and a real algebra automorphism Ψ on $L^\infty(I)$ such that*

$$\Delta_1(f)(t) = \Delta_1(0)(t) + \alpha f(0) + \int_0^t h_0\Psi(f')\,dm \qquad (t \in I,\ f \in \mathrm{Lip}(I)), \quad \textit{or}$$

$$\Delta_1(f)(t) = \Delta_1(0)(t) + \alpha\overline{f(0)} + \int_0^t h_0\Psi(f')\,dm \qquad (t \in I,\ f \in \mathrm{Lip}(I)),$$

where m denotes the Lebesgue measure on I.

Remark 1. We should note that Theorem 1.1 is deduced from [34, Theorem 3.5]. In fact, $\mathrm{Lip}(I)$ equipped with $\|\cdot\|_\sigma$ is identified with the ℓ^1-sum of $\mathbb{R}^2$ and $C(X, \mathbb{R}^2)$ for some compact Hausdorff space X. Here, $C(X, \mathbb{R}^2)$ is the Banach space of all continuous $\mathbb{R}^2$ valued maps on X with the supremum norm. In this paper, we will give a different proof from that of [34] of Tingley's problem for $\mathrm{Lip}(I)$.

Koshimizu [16, Theorem 1.2] gave the characterization of surjective complex linear isometries on $\mathrm{Lip}(I)$ with $\|\cdot\|_\sigma$. We will characterize surjective isometries on $\mathrm{Lip}(I)$ in Corollary 1.2.

§ 2. Preliminaries and auxiliary lemmas

We denote by $\mathbb{T}$ the unit circle in the complex number field $\mathbb{C}$. Let $\mathcal{M}$ be the maximal ideal space of $L^\infty(I)$: Then $\mathcal{M}$ is a compact Hausdorff space so that the Gelfand transform, defined by $\widehat{h}(\eta) = \eta(h)$ for $h \in L^\infty(I)$ and $\eta \in \mathcal{M}$, is a continuous function from $\mathcal{M}$ to $\mathbb{C}$. Let $C(X)$ be the commutative Banach algebra of all continuous complex valued functions on a compact Hausdorff space X with the supremum norm $\|\cdot\|_\infty$ on X. The Gelfand–Naimark theorem states that the Gelfand transformation $\Gamma\colon L^\infty(I) \to C(\mathcal{M})$, defined by $\Gamma(h) = \widehat{h}$ for $h \in L^\infty(I)$, is an isometric isomorphism. Thus, $\|h\|_{L^\infty} = \sup_{\eta\in\mathcal{M}} |\widehat{h}(\eta)| = \|\widehat{h}\|_\infty$ for $h \in L^\infty(I)$. We define

$$\widetilde{f}(\eta, z) = f(0) + \widehat{f'}(\eta)z \tag{2.1}$$

for $f \in \mathrm{Lip}(I)$ and $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. Then the function $\widetilde{f}$ is continuous on $\mathcal{M} \times \mathbb{T}$ with the product topology. We set

$$B = \{\widetilde{f} \in C(\mathcal{M} \times \mathbb{T}) : f \in \mathrm{Lip}(I)\}.$$

Then B is a normed linear subspace of $C(\mathcal{M} \times \mathbb{T})$ equipped with the supremum norm $\|\cdot\|_\infty$ on $\mathcal{M} \times \mathbb{T}$.

We define a mapping $U\colon (\mathrm{Lip}(I), \|\cdot\|_\sigma) \to (B, \|\cdot\|_\infty)$ by $U(f) = \widetilde{f}$ for $f \in \mathrm{Lip}(I)$. We see that U is a surjective complex linear map from $\mathrm{Lip}(I)$ onto B. In addition, $\|U(f)\|_\infty = \|f\|_\sigma$ holds for all $f \in \mathrm{Lip}(I)$: In fact, for each $f \in \mathrm{Lip}(I)$, there exist $z_0, z_1 \in \mathbb{T}$ and $\eta_0 \in \mathcal{M}$ such that $f(0) = |f(0)|z_0$ and $\widehat{f'}(\eta_0) = \|\widehat{f'}\|_\infty z_1$. Then

$$\begin{aligned}|U(f)(\eta_0, z_0\overline{z_1})| &= |f(0) + \widehat{f'}(\eta_0)z_0\overline{z_1}| = |(|f(0)| + \|\widehat{f'}\|_\infty)z_0| \\ &= |f(0)| + \|\widehat{f'}\|_\infty = |f(0)| + \|f'\|_{L^\infty} = \|f\|_\sigma.\end{aligned}$$

We thus obtain $\|f\|_\sigma \le \|U(f)\|_\infty$. For each $(\eta, z) \in \mathcal{M} \times \mathbb{T}$, we have

$$|U(f)(\eta, z)| = |f(0) + \widehat{f'}(\eta)z| \le |f(0)| + |\widehat{f'}(\eta)| \le |f(0)| + \|\widehat{f'}\|_\infty = \|f\|_\sigma,$$

which yields $\|U(f)\|_\infty \le \|f\|_\sigma$. Consequently,

$$\|\widetilde{f}\|_\infty = \|U(f)\|_\infty = \|f\|_\sigma \qquad (f \in \mathrm{Lip}(I)).$$

Therefore, the map U is a surjective complex linear isometry from $(\mathrm{Lip}(I), \|\cdot\|_\sigma)$ onto $(B, \|\cdot\|_\infty)$. In particular, $U(S_{\mathrm{Lip}(I)}) \subset S_B$. Since U^{-1} has the same property as U, we obtain $U^{-1}(S_B) \subset S_{\mathrm{Lip}(I)}$, and hence, $U(S_{\mathrm{Lip}(I)}) = S_B$.

For each $f \in \mathrm{Lip}(I)$, we observe that f is absolutely continuous on I. Thus, the following identity holds:

$$f(t) - f(0) = \int_0^t f'\,dm \qquad (t \in I), \tag{2.2}$$

where m denotes the Lebesgue measure on I (see, for example, [25, Theorem 7.20]). Having in mind $\{\widehat{h} : h \in L^\infty(I)\} = C(\mathcal{M})$, for each $u \in C(\mathcal{M})$ there exists a unique $h \in L^\infty(I)$ such that $u = \widehat{h}$. We define $\mathcal{I}(u)$ by

$$\mathcal{I}(u)(t) = \int_0^t h\,dm \qquad (t \in I).$$

We observe that $\mathcal{I}(u)$ is a Lipschitz function on I with

$$\mathcal{I}(u)(0) = 0 \quad \text{and} \quad \mathcal{I}(u)' = h \quad \text{a.e.}$$

In particular, we obtain

$$\widehat{\mathcal{I}(u)'} = u. \tag{2.3}$$

Here, we note that $\mathcal{I}(u) \in S_{\mathrm{Lip}(I)}$ for $u \in S_{C(\mathcal{M})}$: In fact,

$$\|\mathcal{I}(u)\|_\sigma = |\mathcal{I}(u)(0)| + \|\mathcal{I}(u)'\|_{L^\infty} = \|\widehat{\mathcal{I}(u)'}\|_\infty = \|u\|_\infty = 1,$$

which yields $\mathcal{I}(u) \in S_{\mathrm{Lip}(I)}$. Hence, $\mathcal{I}(S_{C(\mathcal{M})}) \subset S_{\mathrm{Lip}(I)}$.

Let $\Delta\colon (S_{\mathrm{Lip}(I)}, \|\cdot\|_\sigma) \to (S_{\mathrm{Lip}(I)}, \|\cdot\|_\sigma)$ be a surjective isometry. We define $T = U\Delta U^{-1}$; we see that T is a well defined surjective isometry from $(S_B, \|\cdot\|_\infty)$ onto itself, since U is a surjective complex linear isometry from $(\mathrm{Lip}(I), \|\cdot\|_\sigma)$ onto $(B, \|\cdot\|_\infty)$ with $U(S_{\mathrm{Lip}(I)}) = S_B$.

$$\begin{array}{ccc} S_{\mathrm{Lip}(I)} & \xrightarrow{\ \Delta\ } & S_{\mathrm{Lip}(I)} \\ {\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle U} \\ S_B & \xrightarrow[\ T\ ]{} & S_B \end{array}$$

The identity $TU = U\Delta$ implies that

$$T(\widetilde{f}) = \widetilde{\Delta(f)} \qquad (f \in S_{\mathrm{Lip}(I)}). \tag{2.4}$$

For each $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$, we define

$$\lambda V_x = \{\widetilde{f} \in S_B : \widetilde{f}(x) = \lambda\},$$

which plays an important role in our arguments. In the rest of this paper, we denote $\mathbf{1}_I$ and $\mathbf{1}_\mathcal{M}$ by the constant functions taking the value only 1 defined on I and $\mathcal{M}$, respectively.

Lemma 2.1. *If $\lambda_1 V_{x_1} \subset \lambda_2 V_{x_2}$ for some $(\lambda_1, x_1), (\lambda_2, x_2) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, then $(\lambda_1, x_1) = (\lambda_2, x_2)$.*

Proof. We first note that $\widetilde{\mathbf{1}_I}$ is a constant function on $\mathcal{M} \times \mathbb{T}$ by (2.1). Then $\lambda_1 \widetilde{\mathbf{1}_I} \in \lambda_1 V_{x_1} \subset \lambda_2 V_{x_2}$, which yields $\lambda_1 = \lambda_1 \widetilde{\mathbf{1}_I}(x_1) = \lambda_1 \widetilde{\mathbf{1}_I}(x_2) = \lambda_2$. This implies $\lambda_1 = \lambda_2$.

Setting $x_j = (\eta_j, z_j)$ for $j = 1, 2$, we first prove $\eta_1 = \eta_2$. Suppose, on the contrary, that $\eta_1 \neq \eta_2$. There exists $u \in S_{C(\mathcal{M})}$ such that $u(\eta_1) = 1$ and $u(\eta_2) = 0$. We set $f = \mathcal{I}(\lambda_1 \overline{z_1} u) \in S_{\mathrm{Lip}(I)}$, and then $\widetilde{f}(\eta_1, z_1) = \lambda_1$ and $\widetilde{f}(\eta_2, z_2) = 0$ by (2.3). This shows that $\widetilde{f} \in \lambda_1 V_{x_1} \setminus \lambda_2 V_{x_2}$, which contradicts the assumption that $\lambda_1 V_{x_1} \subset \lambda_2 V_{x_2}$. Consequently, we have $\eta_1 = \eta_2$.

Finally, we shall prove $z_1 = z_2$. By (2.3), we see that $g = \mathcal{I}(\lambda_1 \overline{z_1}\, \mathbf{1}_{\mathcal{M}})$ satisfies $\widetilde{g} \in S_B$ and $\widetilde{g}(\eta_1, z_1) = \lambda_1$. We thus obtain $\widetilde{g} \in \lambda_1 V_{x_1} \subset \lambda_2 V_{x_2}$, and hence $\lambda_2 = \widetilde{g}(\eta_2, z_2) = \lambda_1 \overline{z_1} z_2$ by the choice of g. This implies $z_1 = z_2$, since $\lambda_1 = \lambda_2$. We have proven that $(\lambda_1, x_1) = (\lambda_2, x_2)$. □

We denote by $\mathcal{F}_B$ the set of all maximal convex subsets of S_B. Let $\mathrm{ext}(B_1^*)$ be the set of all extreme points of the closed unit ball B_1^* of the dual space of B. It is proved in [15, Lemma 3.1] that for each $F \in \mathcal{F}_B$ there exists $\xi \in \mathrm{ext}(B_1^*)$ such that $F = \xi^{-1}(1) \cap S_B$, where $\xi^{-1}(1) = \{\widetilde{f} \in B : \xi(\widetilde{f}) = 1\}$. Let $\mathrm{Ch}(B)$ be the Choquet boundary for B, that is, $\mathrm{Ch}(B)$ is the set of all $x \in \mathcal{M} \times \mathbb{T}$ such that the point evaluation $\delta_x \colon B \to \mathbb{C}$ at x is in $\mathrm{ext}(B_1^*)$. By the Arens–Kelley theorem (cf. [13, Corollary 2.3.6]), we see that $\mathrm{ext}(B_1^*) = \{\lambda \delta_x \in B_1^* : \lambda \in \mathbb{T},\ x \in \mathrm{Ch}(B)\}$.

Lemma 2.2. *For each $x_0 = (\eta_0, z_0) \in \mathcal{M} \times \mathbb{T}$, the Dirac measure concentrated at x_0 is unique representing measure for δ_{x_0}.*

Proof. Fix an arbitrary open set O in $\mathcal{M}$ with $\eta_0 \in O$. By Urysohn's lemma, we can find $u \in S_{C(\mathcal{M})}$ such that $u(\eta_0) = 1$ and $u = 0$ on $\mathcal{M} \setminus O$. Take any representing measure σ for δ_{x_0}, that is, σ is a regular Borel measure on $\mathcal{M} \times \mathbb{T}$ satisfying $\delta_{x_0}(\widetilde{g}) = \int_{\mathcal{M}\times\mathbb{T}} \widetilde{g}\, d\sigma$ for all $\widetilde{g} \in B$ and $\|\sigma\| = 1$, where $\|\sigma\|$ is the total variation of σ. Having in mind that the operator norm $\|\delta_{x_0}\|$ of δ_{x_0} satisfies $\|\delta_{x_0}\| = 1 = \delta_{x_0}(\widetilde{\mathbf{1}_I})$, we observe that σ is a positive measure (see, for example, [2, p.81]). Setting $f = \mathcal{I}(u) \in S_{\mathrm{Lip}(I)}$, we obtain $\widetilde{f}(\eta, z) = u(\eta) z$ for $(\eta, z) \in \mathcal{M} \times \mathbb{T}$ by (2.1) and (2.3). Since $u = 0$ on $\mathcal{M} \setminus O$, we get

$$
\begin{aligned}
1 = |z_0| = |\delta_{x_0}(\widetilde{f})| &= \left| \int_{\mathcal{M}\times\mathbb{T}} \widetilde{f}\, d\sigma \right| \leq \left| \int_{O\times\mathbb{T}} \widetilde{f}\, d\sigma \right| + \left| \int_{(\mathcal{M}\times\mathbb{T})\setminus(O\times\mathbb{T})} \widetilde{f}\, d\sigma \right| \\
&\leq \int_{O\times\mathbb{T}} |\widetilde{f}|\, d\sigma \leq \|\widetilde{f}\|_\infty \sigma(O\times\mathbb{T}) = \sigma(O\times\mathbb{T}) \leq \|\sigma\| = 1.
\end{aligned}
$$

Consequently, $\sigma(O \times \mathbb{T}) = 1$ for all open sets O in $\mathcal{M}$ with $\eta_0 \in O$, and therefore, we observe that $\sigma(\{\eta_0\} \times \mathbb{T}) = 1$ by the regularity of σ. We thus obtain

$$
z_0 = \delta_{x_0}(\widetilde{f}) = \int_{\{\eta_0\}\times\mathbb{T}} \widetilde{f}\, d\sigma = \int_{\{\eta_0\}\times\mathbb{T}} u(\eta) z\, \delta\sigma = \int_{\{\eta_0\}\times\mathbb{T}} z\, \delta\sigma.
$$

We derive from $\sigma(\{\eta_0\}\times\mathbb{T}) = 1$ that $\int_{\{\eta_0\}\times\mathbb{T}} (z_0 - z)\, d\sigma = 0$. Setting $Z = \{\eta_0\} \times (\mathbb{T} \setminus \{z_0\})$, we obtain $\int_Z (1 - \overline{z_0} z)\, d\sigma = -\overline{z_0} \int_Z (z - z_0)\, d\sigma = 0$, which yields $\int_Z \mathrm{Re}(1 - \overline{z_0} z)\, d\sigma = 0$. As $\mathrm{Re}(1 - \overline{z_0} z) > 0$ on Z, we conclude $\sigma(Z) = 0$, and thus $\sigma(\{\eta_0\} \times \{z_0\}) = 1$. This proves that any representing measure for δ_{x_0} is the Dirac measure concentrated at x_0. □

Lemma 2.3. *For each $x_0 = (\eta_0, z_0) \in \mathcal{M} \times \mathbb{T}$, we have $x_0 \in \mathrm{Ch}(B)$, that is, $\mathrm{Ch}(B) = \mathcal{M} \times \mathbb{T}$.*

Proof. We shall prove that δ_{x_0} belongs to $\mathrm{ext}(B_1^*)$. Suppose that $\delta_{x_0} = (\xi_1 + \xi_2)/2$ for $\xi_1, \xi_2 \in B_1^*$. For $j = 1, 2$, there exists a representing measure σ_j for ξ_j by the Hahn–Banach theorem and the Riesz representation theorem (see, for example, [25, Theorems 5.16 and 2.14]). Since $\xi_1(\widetilde{\mathbf{1}_I}) + \xi_2(\widetilde{\mathbf{1}_I}) = 2\delta_{x_0}(\widetilde{\mathbf{1}_I}) = 2$ with $|\xi_j(\widetilde{\mathbf{1}_I})| \leq 1$, we have $\xi_j(\widetilde{\mathbf{1}_I}) = 1 = \|\xi_j\|$ for $j = 1, 2$. Applying the same argument in [2, p.81] to σ_j, we see that σ_j is a positive measure. We put $\sigma = (\sigma_1 + \sigma_2)/2$, and then σ is a positive measure.

First, we prove that σ is a representing measure for δ_{x_0}. Because σ_j is a representing measure for ξ_j, we get

$$\int_{\mathcal{M}\times\mathbb{T}} \widetilde{f} d\sigma = \int_{\mathcal{M}\times\mathbb{T}} \widetilde{f} d(\frac{\sigma_1 + \sigma_2}{2}) = \frac{\xi_1(\widetilde{f}) + \xi_2(\widetilde{f})}{2} = \delta_{x_0}(\widetilde{f}) \quad (\widetilde{f} \in B).$$

Entering $\widetilde{f} = \widetilde{\mathbf{1}_I}$ into the above equality, we have $\sigma(\mathcal{M} \times \mathbb{T}) = \int_{\mathcal{M}\times\mathbb{T}} \widetilde{\mathbf{1}_I} d\sigma = 1$, which shows that $\|\sigma\| = 1 = \|\delta_{x_0}\|$. Therefore, σ is a representing measure for δ_{x_0}. By Lemma 2.2, $\sigma = (\sigma_1 + \sigma_2)/2$ is the Dirac measure, τ_{x_0}, concentrated at x_0.

We note that σ_j is a positive measure with $j = 1, 2$. For each Borel set D with $x_0 \notin D$, we obtain $(\sigma_1(D) + \sigma_2(D))/2 = \sigma(D) = 0$, and thus, $\sigma_j(D) = 0$. Having in mind that $\|\sigma_j\| = \|\xi_j\| = 1$, we conclude that $\sigma_j = \tau_{x_0}$ for $j = 1, 2$. Hence, $\xi_j(\widetilde{f}) = \int_{\mathcal{M}\times\mathbb{T}} \widetilde{f} d\sigma_j = \widetilde{f}(x_0) = \delta_{x_0}(\widetilde{f})$ for any $\widetilde{f} \in B$, which implies that $\xi_1 = \delta_{x_0} = \xi_2$. This proves $\delta_{x_0} \in \mathrm{ext}(B_1^*)$, which yields $x_0 \in \mathrm{Ch}(B)$. □

We now characterize the set of all maximal convex subsets $\mathcal{F}_B$ of S_B. The following result is proved by Hatori, Oi and Shindo Togashi in [15] for uniform algebras. The proof below of the next proposition is quite similar to that of [15, Lemma 3.2].

Proposition 2.4. *Let F be a subset of S_B. Then $F \in \mathcal{F}_B$ if and only if there exist $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$ such that $F = \lambda V_x$.*

Proof. Suppose that F is a maximal convex subset of S_B. By [15, Lemma 3.1], $F = \xi^{-1}(1) \cap S_B$ for some $\xi \in \mathrm{ext}(B_1^*) = \{\lambda\delta_x \in B_1^* : \lambda \in \mathbb{T}, x \in \mathcal{M} \times \mathbb{T}\}$, where we have used Lemma 2.3. There exist $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$ such that $\xi = \lambda\delta_x$. Now we can write

$$F = (\lambda\delta_x)^{-1}(1) \cap S_B = \{\widetilde{f} \in S_B : \lambda\widetilde{f}(x) = 1\} = \overline{\lambda}V_x.$$

We thus obtain $F = \overline{\lambda}V_x$ with $\overline{\lambda} \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$.

Conversely, suppose that $F = \lambda V_x$ for some $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. It is routine to check that F is a convex subset of S_B. Using Zorn's lemma, we can prove that there exists a maximal convex subset K of S_B with $F \subset K$. By the above paragraph, we see that $K = \mu V_y$ for some $\mu \in \mathbb{T}$ and $y \in \mathcal{M} \times \mathbb{T}$. Then $\lambda V_x = F \subset K = \mu V_y$. Lemma 2.1 shows that $(\lambda, x) = (\mu, y)$, which implies that $F = K$. Consequently, F is a maximal convex subset of S_B. □

Tanaka [28, Lemma 3.5] proved that every surjective isometry between the unit spheres of two Banach spaces preserves maximal convex subsets of the spheres (see also [3, Lemma 5.1]). By these results, we can prove the following lemma.

Lemma 2.5. *There exist maps $\alpha\colon \mathbb{T}\times(\mathcal{M}\times\mathbb{T}) \to \mathbb{T}$ and $\phi\colon \mathbb{T}\times(\mathcal{M}\times\mathbb{T}) \to \mathcal{M}\times\mathbb{T}$ such that*

$$T(\lambda V_x) = \alpha(\lambda, x)V_{\phi(\lambda,x)} \tag{2.5}$$

for all $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$.

Proof. For each $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, λV_x is a maximal convex subset of S_B by Proposition 2.4. By [28, Lemma 3.5], surjective isometry $T\colon S_B \to S_B$ preserves maximal convex subsets of S_B, that is, there exists $(\mu, y) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$ such that $T(\lambda V_x) = \mu V_y$. If, in addition, $T(\lambda V_x) = \mu' V_{y'}$ for some $(\mu', y') \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, then we obtain $(\mu, y) = (\mu', y')$ by Lemma 2.1. Therefore, if we define $\alpha(\lambda, x) = \mu$ and $\phi(\lambda, x) = y$, then $\alpha\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathbb{T}$ and $\phi\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathcal{M} \times \mathbb{T}$ are well defined maps with $T(\lambda V_x) = \alpha(\lambda, x)V_{\phi(\lambda,x)}$. □

Lemma 2.6. *The maps α and ϕ from Lemma 2.5 are both surjective maps satisfying*

$$\alpha(-\lambda, x) = -\alpha(\lambda, x) \quad \text{and} \quad \phi(-\lambda, x) = \phi(\lambda, x)$$

for all $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$.

Proof. Take any $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, and then λV_x is a maximal convex subset of S_B by Proposition 2.4. We get $T(-\lambda V_x) = -T(\lambda V_x)$, which was proved by Mori [20, Proposition 2.3] in a general setting. Lemma 2.5 shows that $\alpha(-\lambda, x)V_{\phi(-\lambda,x)} = T(-\lambda V_x) = -T(\lambda V_x) = -\alpha(\lambda, x)V_{\phi(\lambda,x)}$. Applying Lemma 2.1, we obtain $\alpha(-\lambda, x) = -\alpha(\lambda, x)$ and $\phi(-\lambda, x) = \phi(\lambda, x)$.

There exist well defined maps $\beta\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathbb{T}$ and $\psi\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathcal{M} \times \mathbb{T}$ such that

$$T^{-1}(\mu V_y) = \beta(\mu, y)V_{\psi(\mu,y)} \qquad ((\mu, y) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})),$$

since T^{-1} has the same property as T. For each $(\mu, y) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, we have, by (2.5),

$$\mu V_y = T(T^{-1}(\mu V_y)) = T(\beta(\mu, y)V_{\psi(\mu,y)}) = \alpha(\beta(\mu, y), \psi(\mu, y))V_{\phi(\beta(\mu,y),\psi(\mu,y))}.$$

We derive from Lemma 2.1 that $\mu = \alpha(\beta(\mu, y), \psi(\mu, y))$ and $y = \phi(\beta(\mu, y), \psi(\mu, y))$. These prove that both α and ϕ are surjective. $\square$

By definition, $\phi(\lambda, x) \in \mathcal{M} \times \mathbb{T}$ for each $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$. There exist $\phi_1(\lambda, x) \in \mathcal{M}$ and $\phi_2(\lambda, x) \in \mathbb{T}$ such that

$$\phi(\lambda, x) = (\phi_1(\lambda, x), \phi_2(\lambda, x)).$$

We shall regard ϕ_1 and ϕ_2 as maps defined on $\mathbb{T} \times (\mathcal{M} \times \mathbb{T})$ to $\mathcal{M}$ and $\mathbb{T}$, respectively. By Lemma 2.6, both ϕ_1 and ϕ_2 are surjective maps with

$$\phi_j(-\lambda, x) = \phi_j(\lambda, x) \qquad ((\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T}),\ j = 1, 2). \tag{2.6}$$

Lemma 2.7. *Let $\lambda_j \in \mathbb{T}$ and $(\eta_j, z_j) \in \mathcal{M} \times \mathbb{T}$ for $j = 1, 2$. If $\eta_1 \neq \eta_2$, then there exist $\widetilde{f_j} \in S_B$ such that $\widetilde{f_j} \in \lambda_j V_{(\eta_j, z_j)}$ for $j = 1, 2$ and $\|\widetilde{f_1} - \widetilde{f_2}\|_\infty = 1$.*

Proof. Take $j \in \{1, 2\}$ and open sets O_j in $\mathcal{M}$ with $\eta_j \in O_j$ and $O_1 \cap O_2 = \emptyset$. By Urysohn's lemma, there exists $u_j \in S_{C(\mathcal{M})}$ such that $u_j(\eta_j) = 1$ and $u_j = 0$ on $\mathcal{M} \setminus O_j$. Let $f_j = \mathcal{I}(\lambda_j \overline{z_j} u_j)$, and then we see that $\widetilde{f_j}(\eta, z) = \lambda_j \overline{z_j}\, u_j(\eta) z$ for all $(\eta, z) \in \mathcal{M} \times \mathbb{T}$ by (2.1) and (2.3). It follows from $\widetilde{f_j} \in \lambda_j V_{(\eta_j, z_j)}$ for $j = 1, 2$ that $1 = |\widetilde{f_1}(\eta_1, z_1) - \widetilde{f_2}(\eta_1, z_1)| \leq \|\widetilde{f_1} - \widetilde{f_2}\|_\infty$. Hence, it is enough to prove that $\|\widetilde{f_1} - \widetilde{f_2}\|_\infty \leq 1$. We shall prove $|\widetilde{f_1}(\eta, z) - \widetilde{f_2}(\eta, z)| \leq 1$ for all $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. Fix an arbitrary $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. If $\eta \in O_1$, then $u_2(\eta) = 0$, since $O_1 \cap O_2 = \emptyset$, and thus

$$|\widetilde{f_1}(\eta, z) - \widetilde{f_2}(\eta, z)| = |\lambda_1 \overline{z_1} u_1(\eta) - \lambda_2 \overline{z_2} u_2(\eta)| \leq |u_1(\eta)| + |u_2(\eta)| \leq 1.$$

If $\eta \in \mathcal{M} \setminus O_1$, then $|\widetilde{f_1}(\eta, z) - \widetilde{f_2}(\eta, z)| \leq 1$ by the choice of u_1. We conclude that $|\widetilde{f_1}(\eta, z) - \widetilde{f_2}(\eta, z)| \leq 1$ for all $(\eta, z) \in \mathcal{M} \times \mathbb{T}$, which yields $\|\widetilde{f_1} - \widetilde{f_2}\|_\infty \leq 1$. $\square$

Lemma 2.8. *If $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$, then $\phi_1(\lambda, x) = \phi_1(1, x)$; we shall write $\phi_1(\lambda, x) = \phi_1(x)$ for simplicity.*

Proof. Take any $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. Then $T(V_x) = \alpha(1, x) V_{\phi(1,x)}$ and $T(\lambda V_x) = \alpha(\lambda, x) V_{\phi(\lambda,x)}$ by (2.5). Suppose, on the contrary, that $\phi_1(\lambda, x) \neq \phi_1(1, x)$. There exist $\widetilde{f_1} \in \alpha(1, x) V_{\phi(1,x)} = T(V_x)$ and $\widetilde{f_2} \in \alpha(\lambda, x) V_{\phi(\lambda,x)} = T(\lambda V_x)$ such that $\|\widetilde{f_1} - \widetilde{f_2}\|_\infty = 1$ by Lemma 2.7. We infer from the choice of $\widetilde{f_1}$ and $\widetilde{f_2}$ that $T^{-1}(\widetilde{f_1}) \in V_x$ and $T^{-1}(\widetilde{f_2}) \in \lambda V_x$, which implies that $T^{-1}(\widetilde{f_1})(x) = 1$ and $T^{-1}(\widetilde{f_2})(x) = \lambda$. If $\operatorname{Re} \lambda \leq 0$, then $|1 - \lambda| \geq \sqrt{2}$, and thus

$$\begin{aligned} \sqrt{2} &\leq |1 - \lambda| = |T^{-1}(\widetilde{f_1})(x) - T^{-1}(\widetilde{f_2})(x)| \\ &\leq \|T^{-1}(\widetilde{f_1}) - T^{-1}(\widetilde{f_2})\|_\infty = \|\widetilde{f_1} - \widetilde{f_2}\|_\infty = 1, \end{aligned}$$

where we have used that T is an isometry on S_B. We arrive at a contradiction, which shows $\phi_1(\lambda, x) = \phi_1(1, x)$, provided that $\operatorname{Re}\lambda \leq 0$. Now we consider the case when $\operatorname{Re}\lambda > 0$. Then $\phi_1(-\lambda, x) = \phi_1(1, x)$, since $\operatorname{Re}(-\lambda) < 0$. By (2.6), $\phi_1(\lambda, x) = \phi_1(-\lambda, x) = \phi_1(1, x)$, even if $\operatorname{Re}\lambda > 0$. □

Lemma 2.9. *For each $\lambda_1, \lambda_2 \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$, the following inequality holds:*

$$|\lambda_1 - \lambda_2| \leq |1 - \overline{\alpha(\lambda_1, x)}\alpha(\lambda_2, x)|. \tag{2.7}$$

Proof. Fix $\lambda_1, \lambda_2 \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. We set $f_j = \alpha(\lambda_j, x)\mathbf{1}_I \in S_{\mathrm{Lip}(I)}$ for each $j \in \{1, 2\}$. We see that $\widetilde{f_j} \in \alpha(\lambda_j, x)V_{\phi(\lambda_j, x)} = T(\lambda_j V_x)$ by (2.5). Then $T^{-1}(\widetilde{f_j}) \in \lambda_j V_x$, and hence $T^{-1}(\widetilde{f_j})(x) = \lambda_j$. We obtain

$$\begin{aligned}|\lambda_1 - \lambda_2| &= |T^{-1}(\widetilde{f_1})(x) - T^{-1}(\widetilde{f_2})(x)| \leq \|T^{-1}(\widetilde{f_1}) - T^{-1}(\widetilde{f_2})\|_\infty = \|\widetilde{f_1} - \widetilde{f_2}\|_\infty \\ &= |\alpha(\lambda_1, x) - \alpha(\lambda_2, x)|\,\|\widetilde{\mathbf{1}_I}\|_\infty = |1 - \overline{\alpha(\lambda_1, x)}\alpha(\lambda_2, x)|.\end{aligned}$$

Thus, $|\lambda_1 - \lambda_2| \leq |1 - \overline{\alpha(\lambda_1, x)}\alpha(\lambda_2, x)|$ holds for all $\lambda_1, \lambda_2 \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. □

Lemma 2.10. *For each $x \in \mathcal{M} \times \mathbb{T}$, there exists $\varepsilon_0(x) \in \{\pm 1\}$ such that $\alpha(\lambda, x) = \lambda^{\varepsilon_0(x)}\alpha(1, x)$ for all $\lambda \in \mathbb{T}$; for simplicity, we shall write $\alpha(1, x) = \alpha(x)$.*

Proof. Let $\lambda \in \mathbb{T} \setminus \{\pm 1\}$ and $x \in \mathcal{M} \times \mathbb{T}$. Taking $\lambda_1 = 1$ and $\lambda_2 = \pm\lambda$ in (2.7), we obtain

$$|1 - \lambda| \leq |1 - \overline{\alpha(1, x)}\alpha(\lambda, x)| \quad \text{and} \quad |1 + \lambda| \leq |1 + \overline{\alpha(1, x)}\alpha(\lambda, x)|,$$

where we have used Lemma 2.6. Since $\overline{\alpha(1, x)}\alpha(\lambda, x) \in \mathbb{T}$, we conclude that

$$\overline{\alpha(1, x)}\alpha(\lambda, x) \in \{\lambda, \overline{\lambda}\}.$$

If we consider the case when $\lambda = i$, then we have $\overline{\alpha(1, x)}\alpha(i, x) \in \{\pm i\}$. This implies that $\alpha(i, x) = i\varepsilon_0(x)\alpha(1, x)$ for some $\varepsilon_0(x) \in \{\pm 1\}$. Entering $\lambda_1 = i$ and $\lambda_2 = \lambda$ into (2.7) to get

$$|i - \lambda| \leq |1 - \overline{\alpha(i, x)}\alpha(\lambda, x)| = |1 + i\varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x)| = |i - \varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x)|,$$

and thus $|i - \lambda| \leq |i - \varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x)|$. Because $\alpha(-\lambda, x) = -\alpha(\lambda, x)$ by Lemma 2.6, we get $|i + \lambda| \leq |i + \varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x)|$. These inequalities imply $\varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x) \in \{\lambda, -\overline{\lambda}\}$, since $\varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x) \in \mathbb{T}$. Then

$$\overline{\alpha(1, x)}\alpha(\lambda, x) \in \{\lambda, \overline{\lambda}\} \cap \{\varepsilon_0(x)\lambda, -\varepsilon_0(x)\overline{\lambda}\}.$$

We have two possible cases to consider. If $\varepsilon_0(x) = 1$, then we obtain $\overline{\alpha(1, x)}\alpha(\lambda, x) \in \{\lambda, \overline{\lambda}\} \cap \{\lambda, -\overline{\lambda}\}$. Since $\lambda \neq \pm 1$, we conclude that $\overline{\alpha(1, x)}\alpha(\lambda, x) = \lambda$, and hence

$\alpha(\lambda, x) = \lambda^{\varepsilon_0(x)}\alpha(1, x)$. If $\varepsilon_0(x) = -1$, then $\overline{\alpha(1,x)}\alpha(\lambda, x) \in \{\lambda, \overline{\lambda}\} \cap \{-\lambda, \overline{\lambda}\}$, which yields $\overline{\alpha(1,x)}\alpha(\lambda, x) = \overline{\lambda}$. Thus, $\alpha(\lambda, x) = \lambda^{\varepsilon_0(x)}\alpha(1, x)$. These identities are valid even for $\lambda = \pm 1$. By the liberty of the choice of $\lambda \in \mathbb{T}$, we conclude that $\alpha(\lambda, x) = \lambda^{\varepsilon_0(x)}\alpha(1, x)$ for all $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. □

By Lemmas 2.8 and 2.10, we can rewrite (2.5) as

$$T(\lambda V_x) = \lambda^{\varepsilon_0(x)}\alpha(x)V_{(\phi_1(x),\phi_2(\lambda,x))} \tag{2.8}$$

for all $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$.

Definition 1. Let λV_x and μV_y be maximal convex subsets of S_B, where $\lambda, \mu \in \mathbb{T}$ and $x, y \in \mathcal{M} \times \mathbb{T}$. We denote by $d_H(\lambda V_x, \mu V_y)$ the Hausdorff distance of λV_x and μV_y, that is,

$$d_H(\lambda V_x, \mu V_y) = \max\left\{\sup_{\widetilde{f}\in\lambda V_x} d(\widetilde{f}, \mu V_y), \sup_{\widetilde{g}\in\mu V_y} d(\lambda V_x, \widetilde{g})\right\}, \tag{2.9}$$

where $d(\widetilde{f}, \mu V_y) = \inf_{\widetilde{h}\in\mu V_y} \|\widetilde{f} - \widetilde{h}\|_\infty$ and $d(\lambda V_x, \widetilde{g}) = \inf_{\widetilde{h}\in\lambda V_x} \|\widetilde{h} - \widetilde{g}\|_\infty$.

Since T is a surjective isometry on S_B, we obtain

$$d(T(\widetilde{f}), T(\mu V_y)) = \inf_{\widetilde{h}\in T(\mu V_y)} \|T(\widetilde{f}) - \widetilde{h}\|_\infty = \inf_{T^{-1}(\widetilde{h})\in\mu V_y} \|\widetilde{f} - T^{-1}(\widetilde{h})\|_\infty = d(\widetilde{f}, \mu V_y)$$

for every $\widetilde{f} \in \lambda V_x$. Hence, $\sup_{T(\widetilde{f})\in T(\lambda V_x)} d(T(\widetilde{f}), T(\mu V_y)) = \sup_{\widetilde{f}\in\lambda V_x} d(\widetilde{f}, \mu V_y)$. By the same reasoning, we get $\sup_{T(\widetilde{g})\in T(\mu V_y)} d(T(\lambda V_x), T(\widetilde{g})) = \sup_{\widetilde{g}\in\mu V_y} d(\lambda V_x, \widetilde{g})$, and thus

$$d_H(T(\lambda V_x), T(\mu V_y)) = d_H(\lambda V_x, \mu V_y) \qquad (\lambda, \mu \in \mathbb{T}, x, y \in \mathcal{M} \times \mathbb{T}). \tag{2.10}$$

Remark 2. Let $\lambda \in \mathbb{T}$ and $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. For each $\widetilde{f} \in \lambda V_{(\eta,z)}$, we observe that

$$\overline{\lambda}f(0) \in [0, 1] \quad \text{and} \quad \widehat{f'}(\eta)\overline{\lambda}z = \|\widehat{f'}\|_\infty.$$

In fact, $f(0) + \widehat{f'}(\eta)z = \lambda$ by the definition of $\lambda V_{(\eta,z)}$. Then

$$1 = \overline{\lambda}\{f(0) + \widehat{f'}(\eta)z\} = |\overline{\lambda}\{f(0) + \widehat{f'}(\eta)z\}| \leq |\overline{\lambda}f(0)| + |\widehat{f'}(\eta)\overline{\lambda}z| \leq \|f\|_\sigma = 1,$$

and thus, $|\overline{\lambda}f(0) + \widehat{f'}(\eta)\overline{\lambda}z| = |\overline{\lambda}f(0)| + |\widehat{f'}(\eta)\overline{\lambda}z|$. This implies that $\overline{\lambda}f(0) = t\widehat{f'}(\eta)\overline{\lambda}z$ for some $t \geq 0$, provided $\widehat{f'}(\eta) \neq 0$. Since $\overline{\lambda}\{f(0) + \widehat{f'}(\eta)z\} = 1$, we have $\widehat{f'}(\eta)\overline{\lambda}z = 1/(1+t)$ and $\overline{\lambda}f(0) = t/(1+t) \in [0, 1]$. If $\widehat{f'}(\eta) = 0$, then $\overline{\lambda}f(0) = 1$, and hence $\overline{\lambda}f(0) \in [0, 1]$ as well. In particular, $\overline{\lambda}f(0) = |f(0)|$. We infer from $\widehat{f'}(\eta)\overline{\lambda}z = 1 - \overline{\lambda}f(0)$ and $\|\widehat{f'}\|_\infty = 1 - |f(0)|$ that $\widehat{f'}(\eta)\overline{\lambda}z = \|\widehat{f'}\|_\infty$.

Lemma 2.11. *For each $\eta \in \mathcal{M}$, $z \in \mathbb{T}$ and $k \in \{\pm 1\}$, the following equalities hold:*

$$\sup_{\widetilde{f} \in kV_{(\eta,k)}} d(\widetilde{f}, kV_{(\eta,z)}) = \sup_{\widetilde{g} \in kV_{(\eta,z)}} d(kV_{(\eta,k)}, \widetilde{g}) = |1 - kz|. \tag{2.11}$$

In particular, $d_H(kV_{(\eta,k)}, kV_{(\eta,z)}) = |1 - kz|$ for all $\eta \in \mathcal{M}$, $z \in \mathbb{T}$ and $k = \pm 1$.

Proof. Fix an arbitrary $\widetilde{f} \in kV_{(\eta,k)}$ and $\widetilde{g} \in kV_{(\eta,z)}$, and then

$$f(0) + \widehat{f'}(\eta)k = k \quad \text{and} \quad g(0) + \widehat{g'}(\eta)z = k. \tag{2.12}$$

We notice that $kf(0), kg(0) \in [0, 1]$, $\widehat{f'}(\eta) = \|\widehat{f'}\|_\infty$ and $\widehat{g'}(\eta)kz = \|\widehat{g'}\|_\infty$ by *Remark* 2. We deduce from the choice of $\widetilde{f}$ and $\widetilde{g}$ that

$$\begin{aligned}
|(1 - kz)(kf(0) - 1)| &\leq |kf(0) - kg(0)| + |kg(0) - 1 - kz(kf(0) - 1)| \\
&= |f(0) - g(0)| + |\overline{z}(g(0) - k) - (kf(0) - 1)| \\
&= |f(0) - g(0)| + |\widehat{g'}(\eta) - \widehat{f'}(\eta)| && \text{by (2.12)} \\
&\leq |f(0) - g(0)| + \|\widehat{f'} - \widehat{g'}\|_\infty = \|f - g\|_\sigma = \|\widetilde{f} - \widetilde{g}\|_\infty.
\end{aligned}$$

That is, $|1 - kz|(1 - kf(0)) \leq \|\widetilde{f} - \widetilde{g}\|_\infty$. We also have $|(1 - k\overline{z})(kg(0) - 1)| \leq \|\widetilde{f} - \widetilde{g}\|_\infty$ by a similar calculation, and thus, $|1 - kz|(1 - kg(0)) \leq \|\widetilde{f} - \widetilde{g}\|_\infty$. By the liberty of the choice of $\widetilde{f} \in kV_{(\eta,k)}$ and $\widetilde{g} \in kV_{(\eta,z)}$, we obtain

$$|1 - kz|(1 - kf(0)) \leq d(\widetilde{f}, kV_{(\eta,z)}) \quad \text{and} \quad |1 - kz|(1 - kg(0)) \leq d(kV_{(\eta,k)}, \widetilde{g}).$$

Setting $f_1 = f(0) + \mathcal{I}(k\overline{z}\widehat{f'})$ and $g_1 = g(0) + \mathcal{I}(kz\widehat{g'})$, we see that $\widetilde{f_1}(\eta, z) = f(0) + k\widehat{f'}(\eta) = k$ and $\widetilde{g_1}(\eta, k) = g(0) + z\widehat{g'}(\eta) = k$ by (2.12), where we have used that $\mathcal{I}(u)(0) = 0$ for $u \in C(\mathcal{M})$. Consequently, $\widetilde{f_1} \in kV_{(\eta,z)}$ and $\widetilde{g_1} \in kV_{(\eta,k)}$. By the choice of f_1, we have

$$\begin{aligned}
\|\widetilde{f} - \widetilde{f_1}\|_\infty &= \sup_{(\zeta,\nu) \in \mathcal{M} \times \mathbb{T}} |\widetilde{f}(\zeta, \nu) - \widetilde{f_1}(\zeta, \nu)| = \sup_{(\zeta,\nu) \in \mathcal{M} \times \mathbb{T}} |(1 - k\overline{z})\widehat{f'}(\zeta)\nu| \\
&= |1 - k\overline{z}| \, \|\widehat{f'}\|_\infty = |1 - kz| \, \widehat{f'}(\eta) = |1 - kz|(1 - kf(0))
\end{aligned}$$

by (2.12). In the same way, we get

$$\|\widetilde{g_1} - \widetilde{g}\|_\infty = \sup_{(\zeta,\nu) \in \mathcal{M} \times \mathbb{T}} |(kz - 1)\widehat{g'}(\zeta)\nu| = |kz - 1| \, \|\widehat{g'}\|_\infty = |1 - kz|(1 - kg(0)),$$

which yields $d(\widetilde{f}, kV_{(\eta,z)}) = |1 - kz|(1 - kf(0))$ and $d(kV_{(\eta,k)}, \widetilde{g}) = |1 - kz|(1 - kg(0))$. Having in mind that $kf(0), kg(0) \in [0, 1]$, we conclude that $\sup_{\widetilde{f} \in kV_{(\eta,k)}} d(\widetilde{f}, kV_{(\eta,z)}) = |1 - kz| = \sup_{\widetilde{g} \in kV_{(\eta,z)}} d(kV_{(\eta,k)}, \widetilde{g})$. □

Lemma 2.12. *The identity $\phi_1(\eta, z) = \phi_1(\eta, 1)$ holds for all $\eta \in \mathcal{M}$ and $z \in \mathbb{T}$; we shall write $\phi_1(\eta, z) = \phi_1(\eta)$ for the sake of simplicity of notation.*

Proof. Fix arbitrary $k \in \{\pm 1\}$, $\eta \in \mathcal{M}$ and $z \in \mathbb{T} \setminus \{\pm 1\}$. We assume that $\phi_1(\eta, z) \neq \phi_1(\eta, k)$. There exists $u_k \in S_{C(\mathcal{M})}$ such that

$$u_k(\phi_1(\eta, z)) = k\alpha(\eta, z)\overline{\phi_2(k, (\eta, z))} \quad \text{and} \quad u_k(\phi_1(\eta, k)) = -k\alpha(\eta, k)\overline{\phi_2(k, (\eta, k))}.$$

Setting $g_k = \mathcal{I}(u_k)$, we see that $\widetilde{g_k} \in k\alpha(\eta, z)V_{\phi(k,(\eta,z))} \cap (-k\alpha(\eta, k))V_{\phi(k,(\eta,k))}$, where we have used $\phi_1(\lambda, x) = \phi_1(x)$ by Lemma 2.8. For any $\widetilde{f} \in k\alpha(\eta, k)V_{\phi(k,(\eta,k))}$, we obtain

$$2 = |k\alpha(\eta, k) + k\alpha(\eta, k)| = |\widetilde{f}(\phi(k, (\eta, k))) - \widetilde{g_k}(\phi(k, (\eta, k)))| \leq \|\widetilde{f} - \widetilde{g_k}\|_\infty \leq 2,$$

which shows $d(k\alpha(\eta, k)V_{\phi(k,(\eta,k))}, \widetilde{g_k}) = 2$. Combining (2.8), (2.9), (2.10) and (2.11), we get

$$\begin{aligned} 2 &\leq \sup_{\widetilde{g} \in k\alpha(\eta,z)V_{\phi(k,(\eta,z))}} d(k\alpha(\eta, k)V_{\phi(k,(\eta,k))}, \widetilde{g}) \\ &\leq d_H(k\alpha(\eta, k)V_{\phi(k,(\eta,k))}, k\alpha(\eta, z)V_{\phi(k,(\eta,z))}) = d_H(T(kV_{(\eta,k)}), T(kV_{(\eta,z)})) \\ &= d_H(kV_{(\eta,k)}, kV_{(\eta,z)}) = |1 - kz|, \end{aligned}$$

which implies $z = -k$. This contradicts $z \neq \pm 1$, and thus $\phi_1(\eta, z) = \phi_1(\eta, k)$ for $z \neq \pm 1$. Entering $z = i$ and $k = \pm 1$ into the last equality, we get $\phi_1(\eta, 1) = \phi_1(\eta, i) = \phi_1(\eta, -1)$. Therefore, we conclude $\phi_1(\eta, z) = \phi_1(\eta, 1)$ for all $\eta \in \mathcal{M}$ and $z \in \mathbb{T}$. □

Lemma 2.13. *The following inequalities hold for all $\lambda, \mu \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$;*

$$(2.13) \quad |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(\mu, x) - \mu^{\varepsilon_0(x)}| \leq |\lambda - \mu|, \quad \textit{and} \quad |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(\mu, x) + \mu^{\varepsilon_0(x)}| \leq |\lambda + \mu|.$$

Proof. Take any $\lambda, \mu \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. For each $\widetilde{f} \in \lambda V_x$ and $\widetilde{g} \in \mu V_x$, we obtain $|\lambda - \mu| = |\widetilde{f}(x) - \widetilde{g}(x)| \leq \|\widetilde{f} - \widetilde{g}\|_\infty$, which yields $|\lambda - \mu| \leq d(\widetilde{f}, \mu V_x)$. Set $f_0 = \overline{\lambda}\mu f$, and then we see that $\widetilde{f_0} \in \mu V_x$ with $\|\widetilde{f} - \widetilde{f_0}\|_\infty = \|(1 - \overline{\lambda}\mu)\widetilde{f}\|_\infty = |\lambda - \mu|$. This implies $d(\widetilde{f}, \mu V_x) = |\lambda - \mu|$. By the same argument, we see that $d(\lambda V_x, \widetilde{g}) = |\lambda - \mu|$. Consequently, $d_H(\lambda V_x, \mu V_x) = |\lambda - \mu|$ by (2.9).

Let us define $f_1 = \alpha(\lambda, x)\overline{\phi_2(\lambda, x)}\mathcal{I}(\mathbf{1}_{\mathcal{M}})$, and then we see that $\widetilde{f_1} \in \alpha(\lambda, x)V_{\phi(\lambda,x)} = T(\lambda V_x)$ by (2.3) and (2.5). Set $\widetilde{g_1} = T(\widetilde{g})$ for each $\widetilde{g} \in \mu V_x$. Then $\widetilde{g_1} \in T(\mu V_x) = \alpha(\mu, x)V_{\phi(\mu,x)}$. By the definition of the set νV_y, we have $\widehat{f_1'}(\phi_1(x))\phi_2(\lambda, x) = \lambda^{\varepsilon_0(x)}\alpha(x)$ and $g_1(0) + \widehat{g_1'}(\phi_1(x))\phi_2(\mu, x) = \mu^{\varepsilon_0(x)}\alpha(x)$, where we have used (2.8). We deduce from $\alpha(x), \phi_2(\lambda, x), \phi_2(\mu, x) \in \mathbb{T}$ that

$$\begin{aligned} |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)} - \mu^{\varepsilon_0(x)}\overline{\phi_2(\mu, x)}| &\leq |\widehat{f_1'}(\phi_1(x)) - \widehat{g_1'}(\phi_1(x))| + |g_1(0)| \\ &\leq |f_1(0) - g_1(0)| + \|\widehat{f_1'} - \widehat{g_1'}\|_\infty = \|f_1 - g_1\|_\sigma = \|\widetilde{f_1} - \widetilde{g_1}\|_\infty, \end{aligned}$$

which shows $|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)} - \mu^{\varepsilon_0(x)}\overline{\phi_2(\mu,x)}| \le d(\widetilde{f}_1, T(\mu V_x))$. We infer from (2.9) and (2.10) that

$$\begin{aligned}|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)} - \mu^{\varepsilon_0(x)}\overline{\phi_2(\mu,x)}| &\le \sup_{T(\widetilde{f})\in T(\lambda V_x)} d(T(\widetilde{f}), T(\mu V_x)) \\ &\le d_H(T(\lambda V_x), T(\mu V_x)) = d_H(\lambda V_x, \mu V_x) = |\lambda - \mu|.\end{aligned}$$

Thus, $|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)}\phi_2(\mu,x) - \mu^{\varepsilon_0(x)}| \le |\lambda-\mu|$. Noting that $\phi_2(-\mu,x) = \phi_2(\mu,x)$ by (2.6), we obtain $|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)}\phi_2(\mu,x) + \mu^{\varepsilon_0(x)}| \le |\lambda+\mu|$. □

Lemma 2.14. *For each $x \in \mathcal{M}\times\mathbb{T}$, there exists $\varepsilon_1(x) \in \{\pm 1\}$ such that $\phi_2(\lambda,x) = \lambda^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(1,x)$ for all $\lambda\in\mathbb{T}$.*

Proof. Fix arbitrary $x \in \mathcal{M}\times\mathbb{T}$ and $\lambda \in \mathbb{T}\setminus\{\pm1\}$. We obtain

$$|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)}\phi_2(1,x) \pm 1| \le |\lambda \pm 1|$$

by (2.13) with $\mu = 1$, which implies $\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)}\phi_2(1,x) \in \{\lambda, \overline{\lambda}\}$. Hence,

$$\overline{\phi_2(\lambda,x)}\phi_2(1,x) \in \{\lambda^{1-\varepsilon_0(x)}, \lambda^{-1-\varepsilon_0(x)}\}.$$

In particular, $\overline{\phi_2(i,x)}\phi_2(1,x) \in \{\pm\varepsilon_0(x)\}$, and thus $\phi_2(i,x) = \varepsilon_1(x)\varepsilon_0(x)\phi_2(1,x)$ for some $\varepsilon_1(x) \in \{\pm1\}$. Entering $\mu = i$ into (2.13) to get

$$|\lambda - i| \ge |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)}\phi_2(i,x) - \varepsilon_0(x)i| = |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)}\varepsilon_1(x)\phi_2(1,x) - i|.$$

By the same reasoning, we have $|\lambda+i| \ge |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)}\varepsilon_1(x)\phi_2(1,x) + i|$. Then we derive from these two inequalities that $\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda,x)}\varepsilon_1(x)\phi_2(1,x) \in \{\lambda, -\overline{\lambda}\}$. Thus, $\varepsilon_1(x)\overline{\phi_2(\lambda,x)}\phi_2(1,x) \in \{\lambda^{1-\varepsilon_0(x)}, -\lambda^{-1-\varepsilon_0(x)}\}$. Now we obtain

$$\overline{\phi_2(\lambda,x)}\phi_2(1,x) \in \{\lambda^{1-\varepsilon_0(x)}, \lambda^{-1-\varepsilon_0(x)}\} \cap \{\varepsilon_1(x)\lambda^{1-\varepsilon_0(x)}, -\varepsilon_1(x)\lambda^{-1-\varepsilon_0(x)}\}.$$

Note that $\lambda \ne \pm1$. If $\varepsilon_1(x) = 1$, then we get $\overline{\phi_2(\lambda,x)}\phi_2(1,x) = \lambda^{1-\varepsilon_0(x)}$, and if $\varepsilon_1(x) = -1$, then $\overline{\phi_2(\lambda,x)}\phi_2(1,x) = \lambda^{-1-\varepsilon_0(x)}$. These imply that $\overline{\phi_2(\lambda,x)}\phi_2(1,x) = \lambda^{\varepsilon_1(x)-\varepsilon_0(x)}$ for $\lambda \in \mathbb{T}\setminus\{\pm1\}$. The last identity is valid even for $\lambda\in\{\pm1\}$ by (2.6). Therefore, we conclude that $\phi_2(\lambda,x) = \lambda^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(1,x)$ for all $\lambda\in\mathbb{T}$. □

We shall write $\phi_2(1,x) = \phi_2(x)$ for $x\in\mathcal{M}\times\mathbb{T}$. Let $\lambda\in\mathbb{T}$ and $x \in \mathcal{M}\times\mathbb{T}$. By (2.8), $T(\widetilde{f})(\phi_1(x), \phi_2(\lambda,x)) = \lambda^{\varepsilon_0(x)}\alpha(x) = \alpha(\lambda,x)$ for $f \in S_{\mathrm{Lip}(I)}$ with $\widetilde{f}\in\lambda V_x$. Noting that $T(\widetilde{f}) = \widetilde{\Delta(f)}$ by (2.4), we infer from Lemma 2.12 that

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda,x) = \alpha(\lambda,x) \tag{2.14}$$

for all $\lambda \in \mathbb{T}$, $x = (\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S_{\mathrm{Lip}(I)}$ with $\widetilde{f} \in \lambda V_x$. If we apply Lemma 2.14, then we can rewrite the last equality as

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(x) = \lambda^{\varepsilon_0(x)}\alpha(x) \tag{2.15}$$

for $\lambda \in \mathbb{T}$, $x = (\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S_{\mathrm{Lip}(I)}$ satisfying $\widetilde{f} \in \lambda V_x$.

Lemma 2.15. *Suppose that $\Delta(\lambda_0 \mathbf{1}_I)(0) = 0$ for some $\lambda_0 \in \mathbb{T}$. Then $\widehat{\Delta(\lambda_0 \,\mathrm{id})'} = 0$ on $\mathcal{M}$ for the identity function* id *on I.*

Proof. Fix arbitrary $\eta \in \mathcal{M}$ and $z \in \mathbb{T}$, and we set $x = (\eta, z)$. We note $\lambda_0\widetilde{\mathbf{1}_I} \in \lambda_0 V_x$, and then equality (2.15) shows that $\widehat{\Delta(\lambda_0\mathbf{1}_I)'}(\phi_1(\eta))\lambda_0^{-\varepsilon_1(x)}\phi_2(x) = \alpha(x)$. We set $e(\eta) = \widehat{\Delta(\lambda_0\mathbf{1}_I)'}(\phi_1(\eta))$ for the sake of simplicity of notation. Then we can rewrite the above equality as

$$e(\eta)\lambda_0^{-\varepsilon_1(x)}\phi_2(x) = \alpha(x). \tag{2.16}$$

Since $\lambda_0 \,\mathrm{id} \in \lambda_0 z V_{(\eta,z)}$, we get, by (2.15),

$$\Delta(\lambda_0 \,\mathrm{id})(0) + \widehat{\Delta(\lambda_0 \,\mathrm{id})'}(\phi_1(\eta))(\lambda_0 z)^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(x) = (\lambda_0 z)^{\varepsilon_0(x)}\alpha(x).$$

Combining (2.16) with the last equality, we obtain

$$\Delta(\lambda_0 \,\mathrm{id})(0) + \widehat{\Delta(\lambda_0 \,\mathrm{id})'}(\phi_1(\eta))(\lambda_0 z)^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(x) = (\lambda_0 z)^{\varepsilon_0(x)}e(\eta)\lambda_0^{-\varepsilon_1(x)}\phi_2(x),$$

which leads to

$$\Delta(\lambda_0 \,\mathrm{id})(0) = (\lambda_0 z)^{\varepsilon_0(x)}\left\{e(\eta)z^{\varepsilon_1(x)} - \widehat{\Delta(\lambda_0 \,\mathrm{id})'}(\phi_1(\eta))\right\}(\lambda_0 z)^{-\varepsilon_1(x)}\phi_2(x).$$

Note that $|e(\eta)| = 1$ by (2.16). Taking the modulus of the above equality, we get $|\Delta(\lambda_0 \,\mathrm{id})(0)| = |z^{\varepsilon_1(x)} - \overline{e(\eta)}\,\widehat{\Delta(\lambda_0 \,\mathrm{id})'}(\phi_1(\eta))|$. Since $z \in \mathbb{T}$ is arbitrary, the last equality holds for $z = \pm 1, i$. Then we have $\widehat{\Delta(\lambda_0 \,\mathrm{id})'}(\phi_1(\eta)) = 0$. Having in mind that $\eta \in \mathcal{M}$ is arbitrarily fixed, we obtain $\widehat{\Delta(\lambda_0 \,\mathrm{id})'} = 0$ on $\mathcal{M}$, where we have used $\phi_1(\mathcal{M}) = \mathcal{M}$ by Lemmas 2.6, 2.8 and 2.12. □

Lemma 2.16. *For each $\lambda \in \mathbb{T}$, the value $\Delta(\lambda\mathbf{1}_I)(0)$ is nonzero.*

Proof. Suppose, on the contrary, that $\Delta(\lambda_0\mathbf{1}_I)(0) = 0$ for some $\lambda_0 \in \mathbb{T}$. Then $\widehat{\Delta(\lambda_0 \,\mathrm{id})'} = 0$ on $\mathcal{M}$ by Lemma 2.15. We define a function $f_0 \in S_{\mathrm{Lip}(I)}$ by $f_0 = \lambda_0(2\,\mathrm{id} + \mathrm{id}^2)/4$. We shall prove that $\widehat{f_0'}(\eta_0) = \lambda_0$ for some $\eta_0 \in \mathcal{M}$. Let $\mathcal{R}(\mathrm{id})$ be the *essential range* of $\mathrm{id} \in \mathrm{Lip}(I)$, that is, $\mathcal{R}(\mathrm{id})$ is the set of all $\zeta \in \mathbb{C}$ for which $\{t \in I : |\,\mathrm{id}(t) - \zeta| < \epsilon\}$ has positive measure for all $\epsilon > 0$. By definition, we see that $\mathcal{R}(\mathrm{id}) = \mathrm{id}(I) = I$. For the spectrum $\sigma(\mathrm{id})$ of id, we observe that $\mathcal{R}(\mathrm{id}) = \sigma(\mathrm{id}) = \widehat{\mathrm{id}}(\mathcal{M})$

(see, for example, [6, Lemma 2.63]). Thus, there exists $\eta_0 \in \mathcal{M}$ such that $\widehat{\mathrm{id}}(\eta_0) = 1$, which yields $\widehat{f_0'}(\eta_0) = \lambda_0(2 + 2\widehat{\mathrm{id}}(\eta_0))/4 = \lambda_0$ as is claimed. Fix an arbitrary $z \in \mathbb{T}$, and then we see that $\lambda_0\widetilde{\mathrm{id}} \in \lambda_0 z V_{(\eta_0,z)}$ with $\widehat{\Delta(\lambda_0\,\mathrm{id})'} = 0$ on $\mathcal{M}$. Applying (2.14) to $f = \lambda_0\,\mathrm{id}$, we have $\Delta(\lambda_0\,\mathrm{id})(0) = \alpha(\lambda_0 z, (\eta_0, z))$. Having in mind that $z \in \mathbb{T}$ is arbitrary, we may enter $z = \pm 1$ into the last equality. Then we get

$$\alpha(\lambda_0, (\eta_0, 1)) = \alpha(-\lambda_0, (\eta_0, -1)). \tag{2.17}$$

Note also that $\widetilde{f_0} \in \lambda_0 z V_{(\eta_0,z)}$, and thus

$$\Delta(f_0)(0) + \widehat{\Delta(f_0)'}(\phi_1(\eta_0))\phi_2(\lambda_0 z, (\eta_0, z)) = \alpha(\lambda_0 z, (\eta_0, z))$$

by (2.14). Since $\Delta(\lambda_0\,\mathrm{id})(0) = \alpha(\lambda_0 z, (\eta_0, z))$, we can rewrite the above equality as

$$\Delta(f_0)(0) + \widehat{\Delta(f_0)'}(\phi_1(\eta_0))\phi_2(\lambda_0 z, (\eta_0, z)) = \Delta(\lambda_0\,\mathrm{id})(0), \tag{2.18}$$

which yields $|\Delta(\lambda_0\,\mathrm{id})(0) - \Delta(f_0)(0)| = |\widehat{\Delta(f_0)'}(\phi_1(\eta_0))| \le \|\widehat{\Delta(f_0)'}\|_\infty$. We thus obtain

$$\begin{aligned} 2\|\widehat{\Delta(f_0)'}\|_\infty &\ge |\Delta(\lambda_0\,\mathrm{id})(0) - \Delta(f_0)(0)| + \|\widehat{\Delta(f_0)'}\|_\infty \\ &= |\Delta(\lambda_0\,\mathrm{id})(0) - \Delta(f_0)(0)| + \|\widehat{\Delta(\lambda_0\,\mathrm{id})'} - \widehat{\Delta(f_0)'}\|_\infty \\ &= \|\Delta(\lambda_0\,\mathrm{id}) - \Delta(f_0)\|_\sigma = \|\lambda_0\,\mathrm{id} - f_0\|_\sigma = \frac{1}{2}\|\widehat{\mathbf{1}_I} - \widehat{\mathrm{id}}\|_\infty = \frac{1}{2}. \end{aligned}$$

Hence, we have $\|\widehat{\Delta(f_0)'}\|_\infty \ge 1/4$, which implies $|\Delta(f_0)(0)| \le 3/4$, since $\|\Delta(f_0)\|_\sigma = 1$. It follows from (2.18) that

$$1 = |\alpha(\lambda_0 z, (\eta_0, z))| = |\Delta(\lambda_0\,\mathrm{id})(0)| = |\Delta(f_0)(0) + \widehat{\Delta(f_0)'}(\phi_1(\eta_0))\phi_2(\lambda_0 z, (\eta_0, z))|.$$

Since $|\Delta(f_0)(0)| \le 3/4$, we see that $\widehat{\Delta(f_0)'}(\phi_1(\eta_0)) \neq 0$. By the liberty of the choice of $z \in \mathbb{T}$, we deduce from (2.18) that $\phi_2(\lambda_0 z, (\eta_0, z))$ is invariant with respect to $z \in \mathbb{T}$. Entering $z = \pm 1$ into $\phi_2(\lambda_0 z, (\eta_0, z))$, we get

$$\phi_2(\lambda_0, (\eta_0, 1)) = \phi_2(-\lambda_0, (\eta_0, -1)). \tag{2.19}$$

Set $f_1 = \lambda_0(2 + \mathrm{id}^2)/4 \in S_{\mathrm{Lip}(I)}$, and then we have $\widetilde{f_1} \in \lambda_0 V_{(\eta_0,1)}$, because $\widehat{\mathrm{id}}(\eta_0) = 1$. We deduce from (2.14) that

$$\Delta(f_1)(0) + \widehat{\Delta(f_1)'}(\phi_1(\eta_0))\phi_2(\lambda_0, (\eta_0, 1)) = \alpha(\lambda_0, (\eta_0, 1)). \tag{2.20}$$

Combining (2.17) and (2.19) with (2.20), we have

$$\Delta(f_1)(0) + \widehat{\Delta(f_1)'}(\phi_1(\eta_0))\phi_2(-\lambda_0, (\eta_0, -1)) = \alpha(-\lambda_0, (\eta_0, -1)).$$

Here, we recall that $T(\widetilde{f_1}) = \widetilde{\Delta(f_1)}$ by (2.4). Then the above equality with (2.5) and (2.14) implies that $T(\widetilde{f_1}) \in \alpha(-\lambda_0, (\eta_0, -1))V_{\phi(-\lambda_0,(\eta_0,-1))} = T(-\lambda_0 V_{(\eta_0,-1)})$, which shows $\widetilde{f_1} \in (-\lambda_0)V_{(\eta_0,-1)}$. Consequently, $\widetilde{f_1} \in (-\lambda_0)V_{(\eta_0,-1)} \cap \lambda_0 V_{(\eta_0,1)}$, and therefore, we obtain

$$f_1(0) - \widehat{f_1'}(\eta_0) = -\lambda_0 = -\{f_1(0) + \widehat{f_1'}(\eta_0)\}.$$

This leads to $f_1(0) = -f_1(0)$, which yields $f_1(0) = 0$. On the other hand, $f_1(0) = \lambda_0(2 + \mathrm{id}^2(0))/4 = \lambda_0/2 \neq 0$. This is a contradiction. We conclude that $\Delta(\lambda \mathbf{1}_I)(0) \neq 0$ for all $\lambda \in \mathbb{T}$. □

Lemma 2.17. *The values $\alpha(x)$ and $\varepsilon_0(x)$ are both independent from the variable $x \in \mathcal{M} \times \mathbb{T}$; we shall write $\alpha(x) = \alpha$ and $\varepsilon_0(x) = \varepsilon_0$.*

Proof. Take any $\lambda \in \mathbb{T}$ and $x = (\eta, z) \in \mathcal{M} \times \mathbb{T}$. According to (2.14), applied to $f = \lambda \mathbf{1}_I$, we have

$$\begin{aligned} 1 = |\lambda^{\varepsilon_0(x)}\alpha(x)| &= |\Delta(\lambda \mathbf{1}_I)(0) + \widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))\phi_2(\lambda, x)| \\ &\leq |\Delta(\lambda \mathbf{1}_I)(0)| + |\widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))| \leq \|\Delta(\lambda \mathbf{1}_I)\|_\sigma = 1. \end{aligned}$$

The above inequalities show that

$$|\Delta(\lambda \mathbf{1}_I)(0) + \widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))\phi_2(\lambda, x)| = 1 = |\Delta(\lambda \mathbf{1}_I)(0)| + |\widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))|.$$

Note that $\Delta(\lambda \mathbf{1}_I)(0) \neq 0$ by Lemma 2.16. By the above equality, there exists $t \geq 0$ such that $\widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))\phi_2(\lambda, x) = t\Delta(\lambda \mathbf{1}_I)(0)$. We thus obtain

$$|t\Delta(\lambda \mathbf{1}_I)(0)| = |\widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))| = 1 - |\Delta(\lambda \mathbf{1}_I)(0)|,$$

which yields $(1 + t)|\Delta(\lambda \mathbf{1}_I)(0)| = 1$. Consequently,

$$\lambda^{\varepsilon_0(x)}\alpha(x) = \Delta(\lambda \mathbf{1}_I)(0) + \widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))\phi_2(\lambda, x) = (1 + t)\Delta(\lambda \mathbf{1}_I)(0) = \frac{\Delta(\lambda \mathbf{1}_I)(0)}{|\Delta(\lambda \mathbf{1}_I)(0)|}$$

by (2.14). Then $\alpha(x) = \Delta(\mathbf{1}_I)(0)/|\Delta(\mathbf{1}_I)(0)|$ is independent from $x \in \mathcal{M} \times \mathbb{T}$. Letting $\lambda = i$ in the above equality, we get $i\varepsilon_0(x)\alpha(x) = \Delta(i\mathbf{1}_I)(0)/|\Delta(i\mathbf{1}_I)(0)|$. Thus, ε_0 is constant on $\mathcal{M} \times \mathbb{T}$. □

By Lemma 2.17, we can rewrite (2.15) as

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0 - \varepsilon_1(x)}\phi_2(x) = \lambda^{\varepsilon_0}\alpha \tag{2.21}$$

for all $\lambda \in \mathbb{T}$, $x = (\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S_{\mathrm{Lip}(I)}$ with $\widetilde{f} \in \lambda V_x$.

Lemma 2.18. *Let $\eta \in \mathcal{M}$, $\lambda \in \mathbb{T}$ and $f \in S_{\mathrm{Lip}(I)}$ be such that $\widehat{f'}(\eta) = \lambda$. Then $\Delta(f)$ satisfies $\Delta(f)(0) = 0$ and*

$$\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0}\alpha \tag{2.22}$$

for all $z \in \mathbb{T}$.

Proof. Fix an arbitrary $z \in \mathbb{T}$. By the choice of f, we have $\widetilde{f} \in \lambda z V_{(\eta,z)}$. By (2.21) with $\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0 - \varepsilon_1(\eta,z)}\phi_2(\eta, z)$, we obtain

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0}\alpha. \tag{2.23}$$

We observe that $\|\widehat{\Delta(f)'}\|_\infty \neq 0$; for if $\|\widehat{\Delta(f)'}\|_\infty = 0$, then we would have $\Delta(f)(0) = (\lambda z)^{\varepsilon_0}\alpha$ for all $z \in \mathbb{T}$, which is impossible. Equality (2.23) shows that

$$\begin{aligned} 1 &= |\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z))| \\ &\leq |\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta))| \leq \|\Delta(f)\|_\sigma = 1, \end{aligned}$$

and hence, $|\widehat{\Delta(f)'}(\phi_1(\eta))| = \|\widehat{\Delta(f)'}\|_\infty \neq 0$. Then there exists $s \geq 0$ such that

$$\Delta(f)(0) = s\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)). \tag{2.24}$$

It follows from (2.23) that

$$(1 + s)\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0}\alpha,$$

which yields $(1 + s)\|\widehat{\Delta(f)'}\|_\infty = 1$, or equivalently, $s\|\widehat{\Delta(f)'}\|_\infty = 1 - \|\widehat{\Delta(f)'}\|_\infty$. These equalities show that

$$\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = \|\widehat{\Delta(f)'}\|_\infty(\lambda z)^{\varepsilon_0}\alpha.$$

We deduce from the last equality with (2.24) that $\Delta(f)(0) = s\|\widehat{\Delta(f)'}\|_\infty(\lambda z)^{\varepsilon_0}\alpha = (1 - \|\widehat{\Delta(f)'}\|_\infty)(\lambda z)^{\varepsilon_0}\alpha$, that is,

$$\Delta(f)(0) = (1 - \|\widehat{\Delta(f)'}\|_\infty)(\lambda z)^{\varepsilon_0}\alpha.$$

By the liberty of the choice of $z \in \mathbb{T}$, we get $1 - \|\widehat{\Delta(f)'}\|_\infty = 0 = \Delta(f)(0)$. Thus, by (2.23), $\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0}\alpha$ for all $z \in \mathbb{T}$. □

Lemma 2.19. *For each $\lambda, z \in \mathbb{T}$ and $\eta \in \mathcal{M}$,*

$$\phi_2(\lambda, (\eta, z)) = \lambda^{\varepsilon_0 - \varepsilon_1(\eta)}\phi_2(1, (\eta, 1))z^{\varepsilon_1(\eta)},$$

where $\varepsilon_1(\eta) = \varepsilon_1(\eta, 1)$.

Proof. Fix arbitrary $\lambda, z \in \mathbb{T}$ and $\eta \in \mathcal{M}$. Setting $\mu = \lambda\overline{z}$ and $v = \mu\mathbf{1}_{\mathcal{M}} \in S_{C(\mathcal{M})}$, we see that $\mathcal{I}(v) \in S_{\mathrm{Lip}(I)}$ satisfies $\widehat{\mathcal{I}(v)'}(\eta) = \mu$ by (2.3). We may apply (2.22) to $f = \mathcal{I}(v)$, and we get $\widehat{\Delta(\mathcal{I}(v))'}(\phi_1(\eta))\phi_2(\mu z, (\eta, z)) = (\mu z)^{\varepsilon_0}\alpha$. Therefore, we obtain

$$\widehat{\Delta(\mathcal{I}(v))'}(\phi_1(\eta))\phi_2(\mu z, (\eta, z)) = \mu^{\varepsilon_0}\alpha \cdot z^{\varepsilon_0} = \widehat{\Delta(\mathcal{I}(v)'}(\phi_1(\eta))\phi_2(\mu, (\eta, 1))z^{\varepsilon_0}.$$

Then $\widehat{\Delta(\mathcal{I}(v))}'(\phi_1(\eta)) \neq 0$, and hence $\phi_2(\mu z, (\eta, z)) = \phi_2(\mu, (\eta, 1))z^{\varepsilon_0}$. This implies

$$\phi_2(\lambda, (\eta, z)) = \phi_2(\lambda\overline{z}, (\eta, 1))z^{\varepsilon_0}.$$

Applying Lemmas 2.14 and 2.17 to the last equality, we now get

$$\begin{aligned}\phi_2(\lambda, (\eta, z)) = \phi_2(\lambda\overline{z}, (\eta, 1))z^{\varepsilon_0} &= (\lambda\overline{z})^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(1, (\eta, 1))z^{\varepsilon_0} \\ &= \lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(1, (\eta, 1))z^{\varepsilon_1(\eta)}.\end{aligned}$$

Consequently, $\phi_2(\lambda, (\eta, z)) = \lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(1, (\eta, 1))z^{\varepsilon_1(\eta)}$. □

We shall write $\phi_2(1, (\eta, 1)) = \phi_2(\eta)$ for simplicity. According to Lemma 2.19, we can write

$$\phi_2(\lambda, (\eta, z)) = \lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)} \tag{2.25}$$

for all $\lambda \in \mathbb{T}$ and $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. Combining (2.21) and (2.25), with $\phi_2(\lambda, x) = \lambda^{\varepsilon_0-\varepsilon_1(x)}\phi_2(x)$, we obtain

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)} = \lambda^{\varepsilon_0}\alpha \tag{2.26}$$

for all $\lambda \in \mathbb{T}$, $(\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S_{\mathrm{Lip}(I)}$ with $\widetilde{f} \in \lambda V_{(\eta,z)}$.

Lemma 2.20. *Let $\lambda \in \mathbb{T}$, $(\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S_{\mathrm{Lip}(I)}$ be such that $\widetilde{f} \in \lambda V_{(\eta,z)}$. Then*

$$\Delta(f)(0) = |\Delta(f)(0)|\lambda^{\varepsilon_0}\alpha \quad \text{and} \quad \widehat{\Delta(f)'}(\phi_1(\eta)) = \|\widehat{\Delta(f)'}\|_\infty\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}z^{-\varepsilon_1(\eta)}.$$

In particular,

$$|\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta))| = |f(0)| + |\widehat{f}'(\eta)| \tag{2.27}$$

for all $f \in S_{\mathrm{Lip}(I)}$ with $\widetilde{f} \in \lambda V_{(\eta,z)}$.

Proof. By assumption, (2.26) holds. Taking the modulus of (2.26) to get

$$\begin{aligned}1 &\leq |\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)}| \\ &\leq |\Delta(f)(0)| + \|\widehat{\Delta(f)'}\|_\infty = \|\Delta(f)\|_\sigma = 1.\end{aligned} \tag{2.28}$$

We derive from the last inequalities that $|\widehat{\Delta(f)'}(\phi_1(\eta))| = \|\widehat{\Delta(f)'}\|_\infty$.

If $\Delta(f)(0) = 0$, then the identity $\Delta(f)(0) = |\Delta(f)(0)|\lambda^{\varepsilon_0}\alpha$ is obvious; in addition, $\|\widehat{\Delta(f)'}\|_\infty = \|\Delta(f)\|_\sigma = 1$, and hence $\widehat{\Delta(f)'}(\phi_1(\eta)) = \|\widehat{\Delta(f)'}\|_\infty\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}z^{-\varepsilon_1(\eta)}$ by (2.26). We next consider the case when $\Delta(f)(0) \neq 0$. There exists $s \geq 0$ such that $\widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)} = s\Delta(f)(0)$ by (2.28). Entering the last equality into (2.26) to get $(1+s)\Delta(f)(0) = \lambda^{\varepsilon_0}\alpha$. We thus obtain $(1+s)|\Delta(f)(0)| = 1$, and consequently, $\Delta(f)(0) = |\Delta(f)(0)|\lambda^{\varepsilon_0}\alpha$ holds even if $\Delta(f)(0) \neq 0$. Having in mind that $|\Delta(f)(0)| + \|\widehat{\Delta(f)'}\|_\infty = 1$, we infer from (2.26) that

$$\begin{aligned}\|\widehat{\Delta(f)'}\|_\infty\lambda^{\varepsilon_0}\alpha &= (1-|\Delta(f)(0)|)\lambda^{\varepsilon_0}\alpha = \lambda^{\varepsilon_0}\alpha - \Delta(f)(0)\\ &= \widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)}.\end{aligned}$$

This shows that $\widehat{\Delta(f)'}(\phi_1(\eta)) = \|\widehat{\Delta(f)'}\|_\infty\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}z^{-\varepsilon_1(\eta)}$. Since $\widetilde{f} \in \lambda V_{(\eta,z)}$, we get

$$1 = |\lambda| = |f(0) + \widehat{f'}(\eta)z| \le |f(0)| + |\widehat{f'}(\eta)| \le \|f\|_\sigma = 1,$$

and hence $|\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta))| = 1 = |f(0)| + |\widehat{f'}(\eta)|$. □

For each $\lambda \in \mathbb{T}$ and $\eta \in \mathcal{M}$, we define λP_η by

$$\lambda P_\eta = \{u \in S_{C(\mathcal{M})} : u(\eta) = \lambda\}.$$

Lemma 2.21. *Let $\eta_0 \in \mathcal{M}$ and $f \in S_{\mathrm{Lip}(I)}$. We set $\lambda = \widehat{f'}(\eta_0)/|\widehat{f'}(\eta_0)|$ if $\widehat{f'}(\eta_0) \neq 0$, and $\lambda = 1$ if $\widehat{f'}(\eta_0) = 0$. For each $t \in \mathbb{R}$ with $0 < t < 1$, there exists $u_t \in P_{\eta_0}$ such that*

$$|tf(0)|\lambda + t\widehat{f'} + \left\{1 - |tf(0)| - |t\widehat{f'}(\eta_0)|\right\}\lambda u_t \in \lambda P_{\eta_0}.$$

Proof. Note first that $1-|tf(0)|-|t\widehat{f'}(\eta_0)| > 0$, since $|tf(0)|+|t\widehat{f'}(\eta_0)| \le \|tf\|_\sigma < 1$. We set $r = 1 - |tf(0)| - |t\widehat{f'}(\eta_0)|$,

$$G_0 = \left\{\eta \in \mathcal{M} : |t\widehat{f'}(\eta) - t\widehat{f'}(\eta_0)| \ge \frac{r}{4}\right\},$$
$$\text{and}\quad G_m = \left\{\eta \in \mathcal{M} : \frac{r}{2^{m+2}} \le |t\widehat{f'}(\eta) - t\widehat{f'}(\eta_0)| \le \frac{r}{2^{m+1}}\right\}$$

for each $m \in \mathbb{N}$. We see that G_n is a closed subset of $\mathcal{M}$ with $\eta_0 \notin G_n$ for all $n \in \mathbb{N}\cup\{0\}$. For each $n \in \mathbb{N}\cup\{0\}$, there exists $v_n \in P_{\eta_0}$ such that

$$v_n = 0 \quad \text{on } G_n \tag{2.29}$$

by Urysohn's lemma. Setting $u_t = v_0\sum_{n=1}^\infty v_n/2^n$, we see that u_t converges in $C(\mathcal{M})$, since $\|v_n\|_\infty = 1$ for all $n \in \mathbb{N}$. We observe that

$$1 = u_t(\eta_0) \le \|u_t\|_\infty \le \|v_0\|_\infty\sum_{n=1}^\infty \frac{\|v_n\|_\infty}{2^n} = 1,$$

and hence $u_t \in P_{\eta_0}$. Here, we define

$$w_t = |tf(0)|\lambda + t\widehat{f'} + r\lambda u_t \in C(\mathcal{M}).$$

We shall prove that $w_t \in \lambda P_{\eta_0}$. Since $u_t(\eta_0) = 1$ and $t\widehat{f'}(\eta_0) = |t\widehat{f'}(\eta_0)|\lambda$, we have

$$w_t(\eta_0) = |tf(0)|\lambda + t\widehat{f'}(\eta_0) + \left\{1 - |tf(0)| - |t\widehat{f'}(\eta_0)|\right\}\lambda = \lambda.$$

Fix an arbitrary $\eta \in \mathcal{M}$. To prove that $|w_t(\eta)| \leq 1$, we shall consider three cases. First, we consider the case when $\eta \in G_0$. Then $v_0(\eta) = 0$ by (2.29), and hence $u_t(\eta) = 0$ by definition. We thus obtain $|w_t(\eta)| \leq ||tf(0)|\lambda + t\widehat{f'}(\eta)| \leq \|tf\|_\sigma < 1$, and consequently, $|w_t(\eta)| < 1$ if $\eta \in G_0$.

We next consider the case when $\eta \in \cup_{n=1}^\infty G_n$, and then $\eta \in G_m$ for some $m \in \mathbb{N}$. By the choice of G_m, we get $|t\widehat{f'}(\eta) - t\widehat{f'}(\eta_0)| \leq r/2^{m+1}$. Thus, $|t\widehat{f'}(\eta)| \leq |t\widehat{f'}(\eta_0)| + r/2^{m+1}$. We derive from (2.29) that $|r\lambda u_t(\eta)| \leq r|v_0(\eta)|\sum_{n\neq m}|v_n(\eta)|/2^n \leq r(1-2^{-m})$. Since $|tf(0)| + |\widehat{tf'}(\eta_0)| = 1 - r$, we obtain

$$\begin{aligned}|w_t(\eta)| &\leq |tf(0)| + |t\widehat{f'}(\eta)| + |r\lambda u_t(\eta)| \leq |tf(0)| + |t\widehat{f'}(\eta_0)| + \frac{r}{2^{m+1}} + r\left(1 - \frac{1}{2^m}\right) \\ &= (1-r) - \frac{r}{2^{m+1}} + r = 1 - \frac{r}{2^{m+1}} < 1.\end{aligned}$$

Hence, $|w_t(\eta)| < 1$ for $\eta \in \cup_{n=1}^\infty G_n$.

Finally we consider the case when $\eta \notin \cup_{n=0}^\infty G_n$. Then $\widehat{f'}(\eta) = \widehat{f'}(\eta_0)$, and hence $|w_t(\eta)| \leq |tf(0)| + |t\widehat{f'}(\eta_0)| + r = 1$. We thus conclude that $|w_t(\eta)| \leq 1$ for all $\eta \in \mathcal{M}$, and consequently, $w_t \in \lambda P_{\eta_0}$. □

§ 3. Proof of Main results

Proof of Theorem 1.1. Fix arbitrary $f \in S_{\mathrm{Lip}(I)}$ and $\eta \in \mathcal{M}$. Set $\zeta = \phi_1(\eta)$ and $\lambda = \widehat{f'}(\eta)/|\widehat{f'}(\eta)|$ if $\widehat{f'}(\eta) \neq 0$, and $\lambda = 1$ if $\widehat{f'}(\eta) = 0$. Thus, $\widehat{f'}(\eta) = |\widehat{f'}(\eta)|\lambda$. For each $t \in \mathbb{R}$ with $0 < t < 1$, we define $r = 1 - |tf(0)| - |t\widehat{f'}(\eta)|$, and then $r > 0$. By Lemma 2.21, there exists $u_t \in P_\eta$ such that $w_t = |tf(0)|\lambda + t\widehat{f'} + r\lambda u_t \in \lambda P_\eta$. We obtain

$$\begin{aligned}\|w_t - \widehat{f'}\|_\infty &= \||tf(0)|\lambda + (t-1)\widehat{f'} + r\lambda u_t\|_\infty \\ &\leq |tf(0)| + (1-t)\|\widehat{f'}\|_\infty + 1 - |tf(0)| - |t\widehat{f'}(\eta)| \\ &= (1-t)\|\widehat{f'}\|_\infty + 1 - |t\widehat{f'}(\eta)|.\end{aligned}$$

Since $w_t \in \lambda P_\eta$, we see that $\widehat{\mathcal{I}(w_t)'}(\eta) = w_t(\eta) = \lambda$, that is, $\widetilde{\mathcal{I}(w_t)} \in \lambda V_{(\eta,1)}$. Then $\Delta(\mathcal{I}(w_t))(0) = 0$ and $\Delta(\widehat{\mathcal{I}(w_t)})'(\zeta) = \Delta(\widehat{\mathcal{I}(w_t)})'(\phi_1(\eta)) = \lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}$ by Lemma 2.20.

We get

$$\begin{aligned}1-|\widehat{\Delta(f)'}(\zeta)| &= |\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}| - |\widehat{\Delta(f)'}(\zeta)| \le |\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)} - \widehat{\Delta(f)'}(\zeta)| \\ &= |\Delta(\widehat{\mathcal{I}(w_t)})'(\zeta) - \widehat{\Delta(f)'}(\zeta)| \le \|\Delta(\widehat{\mathcal{I}(w_t)})' - \widehat{\Delta(f)'}\|_\infty \\ &= \|\Delta(\mathcal{I}(w_t)) - \Delta(f)\|_\sigma - |\Delta(f)(0)| \\ &= \|\mathcal{I}(w_t) - f\|_\sigma - |\Delta(f)(0)| = |f(0)| + \|w_t - \widehat{f'}\|_\infty - |\Delta(f)(0)| \\ &\le |f(0)| + (1-t)\|\widehat{f'}\|_\infty + 1 - |t\widehat{f'}(\eta)| - |\Delta(f)(0)|,\end{aligned}$$

where we have used that $\Delta(\mathcal{I}(w_t))(0) = 0 = \mathcal{I}(w_t)(0)$ and Δ is an isometry. Letting $t \nearrow 1$ in the above inequalities, we have

$$1-|\widehat{\Delta(f)'}(\zeta)| \le |\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)} - \widehat{\Delta(f)'}(\zeta)| \le |f(0)| + 1 - |\widehat{f'}(\eta)| - |\Delta(f)(0)|. \tag{3.1}$$

In particular, we obtain $|\Delta(f)(0)| - |\widehat{\Delta(f)'}(\zeta)| \le |f(0)| - |\widehat{f'}(\eta)|$, that is,

$$|\Delta(f)(0)| - |\widehat{\Delta(f)'}(\phi_1(\eta))| \le |f(0)| - |\widehat{f'}(\eta)|. \tag{3.2}$$

Let $\eta_0 \in \mathcal{M}$ be such that $|\widehat{f'}(\eta_0)| = \|\widehat{f'}\|_\infty$. There exist $\mu, z \in \mathbb{T}$ such that $f(0) = |f(0)|\mu$ and $\widehat{f'}(\eta_0) = |\widehat{f'}(\eta_0)|z = \|\widehat{f'}\|_\infty z$. Thus,

$$f(0) + \widehat{f'}(\eta_0)\overline{z}\mu = (|f(0)| + \|\widehat{f'}\|_\infty)\mu = \|f\|_\sigma \mu = \mu,$$

and hence $\widetilde{f} \in \mu V_{(\eta_0, \overline{z}\mu)}$. Equality (2.27) shows that

$$|\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta_0))| = |f(0)| + |\widehat{f'}(\eta_0)|. \tag{3.3}$$

Note that $|\Delta(f)(0)| - |\widehat{\Delta(f)'}(\phi_1(\eta_0))| \le |f(0)| - |\widehat{f'}(\eta_0)|$ holds by (3.2). If we add the last inequality to (3.3), we get $|\Delta(f)(0)| \le |f(0)|$. We may apply the above arguments to Δ^{-1}, then we obtain $|\Delta^{-1}(g)(0)| \le |g(0)|$ for all $g \in S_{\mathrm{Lip}(I)}$. Entering $g = \Delta(f)$ into the last inequality to get $|f(0)| \le |\Delta(f)(0)|$, and thus

$$|\Delta(f)(0)| = |f(0)|.$$

It follows from (3.2) that $|\widehat{f'}(\eta)| \le |\widehat{\Delta(f)'}(\phi_1(\eta))|$. Having in mind that $\widetilde{f} \in \mu V_{(\eta_0, \overline{z}\mu)}$ and $f(0) = |f(0)|\mu$, we derive from Lemma 2.20 that

$$\Delta(f)(0) = |\Delta(f)(0)|\mu^{\varepsilon_0}\alpha = |f(0)|\mu^{\varepsilon_0}\alpha = [f(0)]^{\varepsilon_0}\alpha, \tag{3.4}$$

where $[\nu]^{\varepsilon_0} = \nu$ if $\varepsilon_0 = 1$ and $[\nu]^{\varepsilon_0} = \overline{\nu}$ if $\varepsilon_0 = -1$ for $\nu \in \mathbb{C}$.

Now we shall prove that ϕ_1 is injective. Suppose that $\phi_1(\eta_1) = \phi_1(\eta_2)$ for $\eta_1, \eta_2 \in \mathcal{M}$. Set $f_1 = \mathcal{I}(\mathbf{1}_\mathcal{M})$, and thus $\widehat{f_1'}(\eta_j) = 1$ for $j = 1, 2$ by (2.3). Equalities (2.22) and (2.25) show that $\widehat{\Delta(f_1)'}(\phi_1(\eta_j))\phi_2(\eta_j) = \alpha$ for $j = 1, 2$. Since $\phi_1(\eta_1) = \phi_1(\eta_2)$, we have

$\phi_2(\eta_1) = \phi_2(\eta_2)$. Applying Lemmas 2.12, 2.17 and 2.19 to (2.8) with $\lambda = 1$, we obtain $T(V_{(1,(\eta,1))}) = \alpha V_{(\phi_1(\eta),\phi_2(\eta))}$. Therefore, we get $T(V_{(1,(\eta_1,1))}) = T(V_{(1,(\eta_2,1))})$, and consequently, $V_{(1,(\eta_1,1))} = V_{(1,(\eta_2,1))}$. Lemma 2.1 shows that $\eta_1 = \eta_2$, which proves that ϕ_1 is injective. Now, we may apply the arguments in the last paragraph to Δ^{-1} and ϕ_1^{-1}, and then we obtain $|\widehat{\Delta(f)'}(\zeta)| \le |(\Delta^{-1}(\widehat{\Delta(f)}))'(\phi_1^{-1}(\zeta))|$, which shows $|\widehat{\Delta(f)'}(\phi_1(\eta))| \le |\widehat{f'}(\eta)|$. We thus conclude that $|\widehat{\Delta(f)'}(\zeta)| = |\widehat{\Delta(f)'}(\phi_1(\eta))| = |\widehat{f'}(\eta)|$. By inequalities (3.1) and $|\Delta(f)(0)| = |f(0)|$, we obtain

$$|\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)} - \widehat{\Delta(f)'}(\zeta)| + |\widehat{\Delta(f)'}(\zeta)| = 1.$$

The above equality implies that $\widehat{\Delta(f)'}(\zeta) = s\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}$ for some $s \ge 0$. Then $s = |s\lambda^{\varepsilon_1(z)}\alpha\overline{\phi_2(\eta)}| = |\widehat{\Delta(f)'}(\zeta)| = |\widehat{f'}(\eta)|$, which shows $\widehat{\Delta(f)'}(\zeta) = |\widehat{f'}(\eta)|\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)} = [\widehat{f'}(\eta)]^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}$, where we have used $\widehat{f'}(\eta) = |\widehat{f'}(\eta)|\lambda$. Thus,

$$\widehat{\Delta(f)'}(\phi_1(\eta)) = \alpha\overline{\phi_2(\eta)}\,[\widehat{f'}(\eta)]^{\varepsilon_1(\eta)} \tag{3.5}$$

for all $f \in S_{\mathrm{Lip}(I)}$ and $\eta \in \mathcal{M}$.

We now define $\Delta_0\colon \mathrm{Lip}(I) \to \mathrm{Lip}(I)$ by

$$\Delta_0(g) = \begin{cases} \|g\|_\sigma \Delta\left(\dfrac{g}{\|g\|_\sigma}\right) & \text{if} \quad g \in \mathrm{Lip}(I) \setminus \{0\}, \\ 0 & \text{if} \quad g = 0. \end{cases}$$

By the definition of Δ_0 with (3.4) and (3.5), we observe that

$$\Delta_0(g)(0) = \alpha[g(0)]^{\varepsilon_0} \quad \text{and} \quad \widehat{\Delta_0(g)'}(\phi_1(\eta)) = \alpha\overline{\phi_2(\eta)}[\widehat{g'}(\eta)]^{\varepsilon_1(\eta)} \tag{3.6}$$

for all $g \in \mathrm{Lip}(I)$ and $\eta \in \mathcal{M}$. We thus obtain

$$\begin{aligned} \|\Delta_0(g_1) - \Delta_0(g_2)\|_\sigma &= |\Delta_0(g_1)(0) - \Delta_0(g_2)(0)| + \sup_{\eta\in\mathcal{M}} |\widehat{\Delta_0(g_1)'}(\phi_1(\eta)) - \widehat{\Delta_0(g_2)'}(\phi_1(\eta))| \\ &= |g_1(0) - g_2(0)| + \sup_{\eta\in\mathcal{M}} |\widehat{g_1'}(\eta) - \widehat{g_2'}(\eta)| = \|g_1 - g_2\|_\sigma \end{aligned}$$

for all $g_1, g_2 \in \mathrm{Lip}(I)$, where we have used $\phi_1(\mathcal{M}) = \mathcal{M}$. Hence Δ_0 is an isometry on $\mathrm{Lip}(I)$. We infer from (3.6) that Δ_0 is real linear. We deduce that Δ_0 is surjective, since so is Δ. Therefore, Δ_0 is a surjective, real linear isometry on $\mathrm{Lip}(I)$ that extends Δ to $\mathrm{Lip}(I)$. □

Proof of Corollary 1.2. Let Δ_1 be a surjective isometry on $\mathrm{Lip}(I)$. By the Mazur–Ulam theorem [19], $\Delta_1 - \Delta_1(0)$ is a surjective, real linear isometry. Without loss of generality, we may and do assume that Δ_1 is a surjective real linear isometry.

Since Δ_1^{-1} has the same property as Δ_1, we see that Δ_1 maps $S_{\mathrm{Lip}(I)}$ onto itself. Now we may apply (3.4) and (3.5) to Δ_1, and then we obtain

$$\Delta_1(f)(0) = \alpha[f(0)]^{\varepsilon_0} \quad \text{and} \quad \widehat{\Delta_1(f)}'(\phi_1(\eta)) = \alpha\overline{\phi_2(\eta)}[\widehat{f'}(\eta)]^{\varepsilon_1(\eta)}$$

for all $f \in \mathrm{Lip}(I)$ and $\eta \in \mathcal{M}$, where $\alpha \in \mathbb{T}$, $\varepsilon_0 \in \{\pm 1\}$, $\phi_1 \colon \mathcal{M} \to \mathcal{M}$, $\phi_2 \colon \mathcal{M} \to \mathbb{T}$ and $\varepsilon_1 \colon \mathcal{M} \to \{\pm 1\}$ are from proof of Theorem 1.1. As we proved in the second paragraph of Proof of Theorem 1.1, we know that ϕ_1 is injective. By Lemma 2.6, $\psi_1 = \phi_1^{-1}$ is well defined, and then we have

$$\widehat{\Delta_1(f)}'(\eta) = \alpha\overline{\phi_2(\psi_1(\eta))}[\widehat{f'}(\psi_1(\eta))]^{\varepsilon_1(\psi_1(\eta))} \tag{3.7}$$

for $f \in \mathrm{Lip}(I)$ and $\eta \in \mathcal{M}$. We shall prove that ψ_1 and ϕ_2 are both continuous. Let $\{\eta_a\}$ be a net in $\mathcal{M}$ converging to $\eta \in \mathcal{M}$. By the continuity of $\widehat{\Delta_1(f)}'$, we see that $|\widehat{\Delta_1(f)}'(\eta_a)|$ converges to $|\widehat{\Delta_1(f)}'(\eta)|$ for each $f \in \mathrm{Lip}(I)$. This implies that $|\widehat{f'}(\psi_1(\eta_a))|$ converges to $|\widehat{f'}(\psi_1(\eta))|$ for every $f \in \mathrm{Lip}(I)$ by (3.7). Since the weak topology of $\mathcal{M}$ induced by the family $\{|\widehat{f'}| : f \in \mathrm{Lip}(I)\}$ is Hausdorff, we observe that the identity map from $\mathcal{M}$ with the original topology onto $\mathcal{M}$ with the weak topology is a homeomorphism. Hence, $\psi_1(\eta_a)$ converges to $\psi_1(\eta)$ with respect to the original topology of $\mathcal{M}$, and thus ψ_1 is continuous on $\mathcal{M}$. Since ψ_1 is a bijective continuous map on the compact Hausdorff space $\mathcal{M}$, it must be a homeomorphism. Let id be the identity function on I. Then we have $\widehat{\Delta_1(\mathrm{id})}' = \alpha\overline{\phi_2 \circ \psi_1}$ by (3.7), which implies the continuity of ϕ_2 on $\mathcal{M}$. Moreover, the identity $\widehat{\Delta_1(i\,\mathrm{id})}' = \alpha\overline{\phi_2 \circ \psi_1}\, i(\varepsilon_1 \circ \psi_1)$ shows that $\varepsilon_1 \circ \psi_1$ is continuous on $\mathcal{M}$. Since ψ_1 is a homeomorphism, we have $\varepsilon_1 = (\varepsilon_1 \circ \psi_1) \circ \psi_1^{-1}$ is continuous on $\mathcal{M}$ as well. Then $\mathcal{M}_1 = \{\eta \in \mathcal{M} : \varepsilon_1(\psi_1(\eta)) = 1\}$ is a closed and open subset of $\mathcal{M}$ with $\varepsilon_1(\psi_1(\eta)) = -1$ for all $\eta \in \mathcal{M} \setminus \mathcal{M}_1$.

We define a map $\Phi \colon C(\mathcal{M}) \to C(\mathcal{M})$ by $\Phi(u)(\eta) = [u(\psi_1(\eta))]^{\varepsilon_1(\psi_1(\eta))}$ for $u \in C(\mathcal{M})$ and $\eta \in \mathcal{M}$. We see that Φ is a well defined real linear map on $C(\mathcal{M})$. For each $v_0 \in C(\mathcal{M})$, we set $u_0(\eta) = [v_0(\psi_1^{-1}(\eta))]^{\varepsilon_1(\eta)}$ for $\eta \in \mathcal{M}$. Then we have $\Phi(u_0)(\eta) = [u_0(\psi_1(\eta))]^{\varepsilon_1(\psi_1(\eta))} = [v_0(\eta)]^{\varepsilon_1(\psi_1(\eta))\varepsilon_1(\psi_1(\eta))} = v_0(\eta)$, which shows that Φ is surjective. It is routine to check that Φ is an injective homomorphism, and consequently, Φ is a real algebra automorphism on $C(\mathcal{M})$. Let Γ be the Gelfand transformation from $L^\infty(I)$ onto $C(\mathcal{M})$, that is, $\Gamma(h) = \widehat{h}$ for $h \in L^\infty(I)$. We define a real algebra automorphism $\Psi = \Gamma^{-1} \circ \Phi \circ \Gamma$ on $L^\infty(I)$. For each $f \in \mathrm{Lip}(I)$ and $\eta \in \mathcal{M}$, we obtain

$$[\widehat{f'}(\psi_1(\eta))]^{\varepsilon_1(\psi_1(\eta))} = \Phi(\widehat{f'})(\eta) = (\Phi \circ \Gamma)(f')(\eta) = (\Gamma \circ \Psi)(f')(\eta) = \Gamma(\Psi(f'))(\eta).$$

By the continuity of ϕ_2 and ψ_1, we may set $h_0 = \Gamma^{-1}(\alpha\overline{\phi_2 \circ \psi_1}) \in L^\infty(I)$. We derive from (3.7) that

$$\widehat{\Delta_1(f)}'(\eta) = \Gamma(h_0)(\eta)\Gamma(\Psi(f'))(\eta) = \Gamma(h_0\Psi(f'))(\eta) = \widehat{h_0\Psi(f')}(\eta)$$

for all $\eta \in \mathcal{M}$. Therefore, we conclude $\Delta_1(f)' = h_0\Psi(f')$ for every $f \in \mathrm{Lip}(I)$. According to (2.2), we have

$$\Delta_1(f)(t) = \Delta_1(f)(0) + \int_0^t \Delta_1(f)'\,dm = \alpha[f(0)]^{\varepsilon_0} + \int_0^t h_0\Psi(f')\,dm$$

for every $t \in I$ and $f \in \mathrm{Lip}(I)$. □

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