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On isometries of Wasserstein spaces

By

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Abstract

It is known that if $p \geq 1$, then the isometry group of the metric space (X, ϱ) embeds into the isometry group of the Wasserstein space $\mathcal{W}_p(X, \varrho)$. Those isometries that belong to the image of this embedding are called trivial. In many concrete cases, all isometries are trivial, however, this is not always the case. The aim of this survey paper is to provide a structured overview of recent results concerning trivial and different types of nontrivial isometries.

Contents

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References

§ 1. Introduction

Let $\mathcal{D}$ be a subset of all Borel probability measures $\mathcal{P}(X)$ over a complete separable metric space (X, ϱ). In our considerations, (X, ϱ) will be a very concrete object, rather than a general complete separable metric space: the unit interval, the real line or in general a Euclidean space; a high dimensional sphere, a high-dimensional torus, or a nice manifold; a countable discrete metric space, or in general a graph metric space. We will refer to these spaces as *underlying spaces*. Similarly, $\mathcal{D}$ will be a reasonable subset of $\mathcal{P}(X)$, rather than an arbitrary one. Having the set $\mathcal{D}$ at hand, one can endow it with a metric in quite numerous ways depending on the nature of the problem under consideration. For example, to metrize the weak convergence of measures, one can use the Lévy-Prokhorov metric (see e.g. Theorem 6.8 in [3]), or the p-Wasserstein metric (see Theorem 6.9 in [25]). In this paper, we will consider the latter one, assuming that $\mathcal{D}$ is the collection of all probability measures with finite p-th moment. (See the precise definitions later.)

When working in a metric setting, a natural question arises: *can we describe the structure of distance preserving maps?*[1] In recent years, there has been a lot of activity concerning this question, see e.g. [2, 5, 6, 9–15, 18, 19, 21, 24, 26]. It turned out that isometries of these spaces of measures are strongly related to self-maps of the underlying space X. Concerning the Kolmogorov-Smirnov distance, Dolinar and Molnár showed in [6] that there is a one-to-one correspondence between all isometries of $\mathcal{P}(\mathbb{R})$ and all homeomorphisms of the real line. Concerning the Lévy-Prokhorov metric, the first and the second author showed in [10] that $\mathcal{P}(X)$ endowed with the Lévy-Prokhorov distance is more rigid: assuming that X is a real separable Banach space, a self-map of X induces an isometry on $\mathcal{P}(X)$ if and only if it is itself an isometry. For a more detailed overview of similar results we refer the reader to the survey [26], where the case of the Kolmogorov-Smirnov [6] and the Kuiper distances [9] are also discussed. Last but not least, we mention a paper of Dolinar, Kuzma and Mitrović. In [5] they described the structure of all w^*-continuous isometries of $\mathcal{P}(\mathbb{R})$ with respect to the total variation distance: all these isometries are induced by continuous bijections. However, if one considers a smaller domain $\mathcal{D} \subseteq \mathcal{P}(\mathbb{R})$, then the structure of isometries can be

[1] As Hermann Weyl said in [27]: " Whenever you have to do with a structure–endowed entity Σ try to determine its group of automorphisms, the group of those element–wise transformations which leave all structural relations undisturbed. You can expect to gain a deep insight into the constitution of Σ in this way. "

much more involved. The aim of this paper is to show that the case of the p-Wasserstein metric is similarly colourful: the structure of isometries depends in an interesting way both on some characteristics of the underlying space X and on the value of p.

§ 2. Trivial isometries

This section aims to introduce all the necessary notions and to take a closer look at a very important class of isometries, the so-called trivial isometries. Besides that these are the most natural isometries in this setting, it seems that it is a very rare phenomenon that a p-Wasserstein space possesses nontrivial isometries (only a few such examples are known). After the precise definitions, we will give a short overview of all known cases where the isometry group of a p-Wasserstein space consists of only trivial isometries. The cases where nontrivial isometries occur will be discussed in Section 3.

Let us recall first what a p-Wasserstein space is. Let $p \geq 1$ be a fixed real number, and let (X, ϱ) be a complete and separable metric space. We denote by $\mathcal{P}(X)$ the set of all Borel probability measures on X and by $\Delta_1(X)$ the set of all Dirac measures: $\Delta_1(X) = \{\delta_x \,|\, x \in X\}$. A probability measure π on $X \times X$ is called a *coupling* for $\mu, \nu \in \mathcal{P}(X)$ if the marginals of π are μ and ν, that is, $\pi\,(A \times X) = \mu(A)$ and $\pi\,(X \times B) = \nu(B)$ for all Borel sets $A, B \subseteq X$. The set of all couplings is denoted by $C(\mu, \nu)$. By means of couplings, we can define the p-Wasserstein distance and the corresponding p-Wasserstein space as follows: the p-Wasserstein space $\mathcal{W}_p(X, \varrho)$ is the set of all $\mu \in \mathcal{P}(X)$ that satisfy

$$\int_X \varrho(x, \hat{x})^p \, \mathrm{d}\mu(x) < \infty \tag{2.1}$$

for some (and hence all) $\hat{x} \in X$, endowed with the *p-Wasserstein distance*

$$d_p\,(\mu, \nu) := \left(\inf_{\pi \in C(\mu,\nu)} \iint_{X \times X} \varrho(x, y)^p \, \mathrm{d}\pi(x, y) \right)^{1/p}. \tag{2.2}$$

It is known (see e.g. Theorem 1.5 in [1] with $c = \varrho^p$) that the infimum in (2.2) is in fact a minimum in this setting. Those couplings that minimize (2.2) are called *optimal transport plans*. As the terminology suggests, all notions introduced above are strongly related to the theory of optimal transportation. Indeed, for given sets A and B the quantity $\pi(A, B)$ is the weight of mass that is transported from A to B as μ is transported to ν along the transport plan π, while the quantity $\iint_{X \times X} \varrho(x, y)^p \, \mathrm{d}\pi(x, y)$ is the cost of the transport assuming that the cost of moving one unit of mass from x to y is $\varrho(x, y)^p$. In other words, the p-Wasserstein distance measures the minimal effort required to morph one probability mass into another, when the cost of transporting mass is the p-th power of the distance. Due to this connection and many nice geometric features of the p-Wasserstein distance, p-Wasserstein spaces have strong connections to many flourishing areas in pure and applied sciences. We mention here only a few papers [4,8,16,17,20,22], for a comprehensive overview and for more references we refer the reader to Ambrosio's, Figalli's, Santambrogio's, and Villani's textbooks [1,7,23,25].

After a small detour, we continue by introducing the key notions that are needed in the sequel. A self-map $\psi\colon X\to X$ is called an *isometry* if it is surjective and preserves the distance, that is, $\varrho(\psi(x),\psi(y))=\varrho(x,y)$ for all $x,y\in X$. The symbol $\mathrm{Isom}(X)$ will stand for the group of all isometries. Similarly, the group of all distance preserving bijective self-maps of $\mathcal{W}_p(X,\varrho)$ will be denoted by $\mathrm{Isom}(\mathcal{W}_p(X,\varrho))$.

For an isometry $\psi\in\mathrm{Isom}(X)$ the induced *push-forward map* $\psi_\#\colon\mathcal{P}(X)\to\mathcal{P}(X)$ is defined by $\big(\psi_\#(\mu)\big)(A)=\mu(\psi^{-1}[A])$ for all Borel set $A\subseteq X$ and $\mu\in\mathcal{P}(X)$), where $\psi^{-1}[A]=\{x\in X\,|\,\psi(x)\in A\}$. We call $\psi_\#(\mu)$ the *push-forward* of μ with ψ.

A very important feature of p-Wasserstein spaces is that $\mathcal{W}_p(X,\varrho)$ contains an isometric copy of (X,ϱ). Indeed, since $C(\delta_x,\delta_y)$ has only one element (the Dirac measure $\delta_{(x,y)}$), we have that

$$d_p(\delta_x,\delta_y)=\Big(\iint\limits_{X\times X}\varrho(u,v)^p\ \mathrm{d}\delta_{(x,y)}(u,v)\Big)^{1/p}=\varrho(x,y),$$

and thus the embedding

$$\iota\colon X\to\mathcal{W}_p(X,\varrho),\qquad \iota(x):=\delta_x \tag{2.3}$$

is distance preserving. Furthermore, the set of finitely supported measures (in other words, the collection of all finite convex combinations of Dirac measures) is dense in $\mathcal{W}_p(X,\varrho)$ (see e.g. Example 6.3 and Theorem 6.18 in [25]). Another very important observation is that isometries of the underlying space appear in $\mathrm{Isom}(\mathcal{W}_p(X))$ by means of a natural group homomorphism

$$\#\colon\ \mathrm{Isom}(X)\to\mathrm{Isom}\left(\mathcal{W}_p(X,\varrho)\right),\qquad \psi\mapsto\psi_\#. \tag{2.4}$$

To see that $\psi_\#\in\mathrm{Isom}(\mathcal{W}_p(X,\varrho))$ whenever $\psi\in\mathrm{Isom}(X)$, let us introduce the map $\widehat{\psi}:X\times X\to X\times X$ as $\widehat{\psi}(x,y):=(\psi(x),\psi(y))$. Then observe that if $\pi\in C(\mu,\nu)$ is a coupling, then $\widehat{\psi}_\#\pi$ belongs to $C(\psi_\#(\mu),\psi_\#(\nu))$. Now using that ψ^{-1} is an isometry as well, change of variables ($\psi(x) := u$ and $\psi(y) := v$) gives

$$\begin{aligned} d_p^p(\mu,\nu) &= \inf_{\pi \in C(\mu,\nu)} \iint\limits_{X\times X} \varrho^p(x,y)\ \mathrm{d}\pi(x,y) \\ &= \inf_{\pi \in C(\mu,\nu)} \iint\limits_{X\times X} \varrho^p(\psi^{-1}(u),\psi^{-1}(v))\ \mathrm{d}\widehat{\psi}_{\#}\pi(u,v) \\ &= \inf_{\pi \in C(\mu,\nu)} \iint\limits_{X\times X} \varrho^p(u,v)\ \mathrm{d}\widehat{\psi}_{\#}\pi(u,v) \\ &\geq \inf_{\eta \in C(\psi_{\#}(\mu),\psi_{\#}(\nu))} \iint\limits_{X\times X} \varrho^p(u,v)\ \mathrm{d}\eta(u,v) \\ &= d_p^p(\psi_{\#}(\mu),\psi_{\#}(\nu)). \end{aligned}$$

The reverse inequality can be proved along the same lines. Indeed, since ψ^{-1} is an isometry, the above inequality gives

$$d_p^p(\mu',\nu') \geq d_p^p\big((\psi^{-1})_{\#}(\mu'),(\psi^{-1})_{\#}(\nu')\big)$$

for all $\mu',\nu' \in \mathcal{W}_p(X,\varrho)$. Using that $\psi_{\#}$ is surjective and that $(\psi^{-1})_{\#} = (\psi_{\#})^{-1}$, substituting $\mu' := \psi_{\#}(\mu)$ and $\nu' := \psi_{\#}(\nu)$ gives $d_p^p(\psi_{\#}(\mu),\psi_{\#}(\nu)) \geq d_p^p(\mu,\nu)$.

As we have just seen, $\#$ embeds $\mathrm{Isom}(X)$ into $\mathrm{Isom}(\mathcal{W}_p(X,\varrho))$. We call an isometry of $\mathcal{W}_p(X,\varrho)$ a *trivial isometry* if it belongs to the image of this embedding. If the embedding is surjective (i.e., if all isometries are trivial), we call $\mathcal{W}_p(X,\varrho)$ isometrically rigid. We close this section by collecting all the known isometrically rigid p-Wasserstein spaces.[2] Bertrand and Kloeckner showed in [2] that if (X,ϱ) is a negatively curved geodesically complete Hadamard space then $\mathcal{W}_2(X,\varrho)$ is isometrically rigid. In [24], Santos-Rodríguez proved that rigidity holds for 2-Wasserstein spaces over closed Riemannian manifolds with strictly positive sectional curvature. Furthermore, for compact rank one symmetric spaces (CROSSes), he was able to prove isometric rigidity not only for the $p = 2$ case, but for general p-Wasserstein spaces with $1 < p < \infty$. In [15] we showed that p-Wasserstein spheres $\mathcal{W}_p(\mathbb{S}^n,\sphericalangle)$ and p-Wasserstein tori $\mathcal{W}_p(\mathbb{T}^n,r)$ are rigid for all $n \geq 1$ and $p \geq 1$. Here $\mathbb{S}^n$ denotes the unit sphere of $\mathbb{R}^{n+1}$ and $\sphericalangle$ denotes the geodesic distance $\sphericalangle(x,y) = \arccos\langle x,y\rangle$ $(x,y \in \mathbb{S}^n)$, while $\mathbb{T}^n$ is the n-dimensional torus, that is, the set $\mathbb{R}^n/\mathbb{Z}^n \simeq \big([-1/2,1/2)/_\sim\big)^n$ (the equivalence relation $\sim$ denotes

[2] We remark that one can define the p-Wasserstein space for $0 < p < 1$ as well. In that case the p-Wasserstein distance is defined as $d_p\left(\mu,\nu\right) := \inf_{\pi \in C(\mu,\nu)} \iint\limits_{X\times X} \varrho(x,y)^p\ \mathrm{d}\pi(x,y)$. In [14] we proved that if $0 < p < 1$ then the p-Wasserstein space $\mathcal{W}_p(X,\varrho)$ is isometrically rigid no matter what the underlying space (X,ϱ) is.

the identification of $-1/2$ and $1/2$) equipped with the usual metric

$$r(x,y)=\left(\sum_{k=1}^{n}\left|(x_k-y_k)_{\mathrm{mod}\ 1}\right|^2\right)^{\frac{1}{2}},$$

where $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $y=(y_1,\dots,y_n)\in\mathbb{R}^n$. In [18] we proved isometric rigidity in the discrete setting, namely when (X,ϱ) is a graph metric space. This class contains several important metric spaces: any countable set with the discrete metric (see also [11]), the set of natural numbers and the set of integers with the usual $|\cdot|$-distance; d-dimensional lattices endowed with the l_1-metric, finite strings with the Hamming distance (the classical 1-Wasserstein distance with respect to the Hamming metric is called Ornstein ' s distance), just to mention a few.

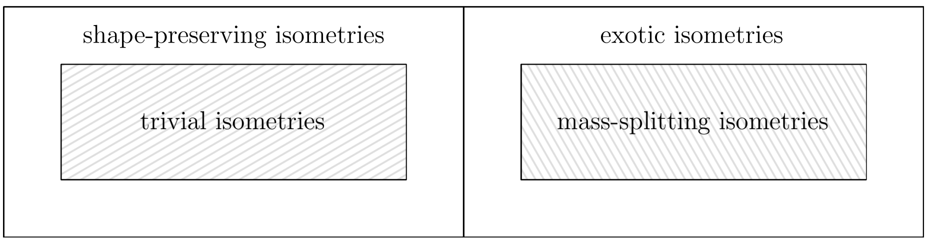
The last special case what we mention is when $(X,\varrho)\in\{([0,1],|\cdot|),(\mathbb{R},|\cdot|),(E,\|\cdot\|)\}$, where E is a real Hilbert space space with $2\le\dim E\le\infty$. Bullets in the table below indicate that the corresponding p-Wasserstein space is isometrically rigid. For more details we refer the reader to [12] and [14].

	$\mathcal{W}_p([0,1],\lvert\cdot\rvert)$	$\mathcal{W}_p(\mathbb{R},\lvert\cdot\rvert)$	$\mathcal{W}_p(E,\lVert\cdot\rVert)$ (dim$E\ge 2$)
$p=1$		•	•
$p=2$	•		
$p\ne 1,2$	•	•	•

We will see in the next section that the missing cases $\mathcal{W}_p([0,1],|\cdot|)$, $\mathcal{W}_2(\mathbb{R},|\cdot|)$ and $\mathcal{W}_2(E,\|\cdot\|)$ with $\dim E\ge 2$ are basically all the known examples where various types of nontrivial isometries appear.

§ 3. Nontrivial isometries

Now we turn to those cases where nontrivial isometries exist. We will discuss three types of nontrivial isometries (see the precise definitions later): 1) shape-preserving nontrivial isometries, 2) exotic isometries, 3) mass-splitting isometries.



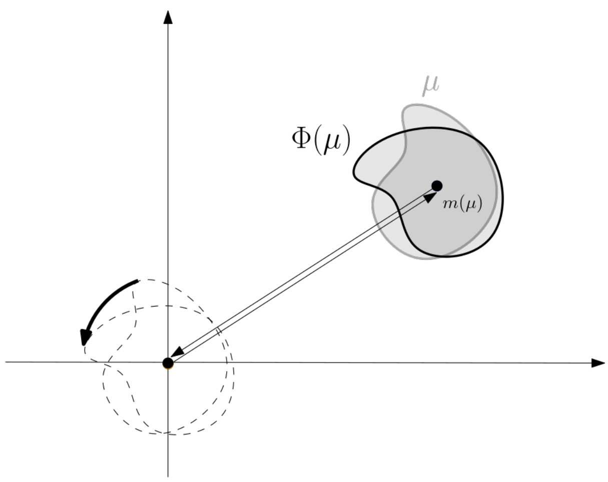
As we will see, the set of trivial isometries is a proper subset of shape-preserving isometries if $p=2$ and the underlying space is a Euclidean space. Isometries that do not preserve shape are termed as exotic isometries. We will see that $\mathcal{W}_2(\mathbb{R},|\cdot|)$ possesses such isometries, however, even these isometries of $\mathcal{W}_2(\mathbb{R},|\cdot|)$ leave the set of Dirac masses invariant. Those exotic isometries which do not leave the set of Dirac measures invariant are called mass-splitting.

§ 3.1. Shape-preserving isometries

Let $\mathcal{W}_p(X,\varrho)$ be a p-Wasserstein space. An isometry $\Phi:\mathcal{W}_p(X,\varrho)\to\mathcal{W}_p(X,\varrho)$ is called shape-preserving if for all measures μ there exists an isometry ψ_μ of X (depending on μ) such that $\Phi(\mu)$ is the push-forward of μ with respect to ψ_μ. Of course, every trivial isometry is shape-preserving. In that case, there exists a $\psi\in\mathrm{Isom}(X)$ such that $\psi_\mu=\psi$ for all $\mu\in\mathcal{W}_p(X,\varrho)$. Kloeckner in [19] proved that $\mathcal{W}_2(\mathbb{R}^n,\|\cdot\|)$ has shape-preserving isometries which are nontrivial. In [14] we extended his result to the infinite dimensional case. Combining these two results, we have the following characterization of isometries.

Let E be a separable real Hilbert space of dimension at least two, and let Φ be an isometry of $\mathcal{W}_2(E,\|\cdot\|)$. Then Φ can be written as the following composition:

$$\Phi(\mu)=\left(\psi\circ t_{m(\mu)}\circ R\circ t^{-1}_{m(\mu)}\right)_{\#}\mu \qquad (\mu\in\mathcal{W}_2(E)), \tag{3.1}$$



where $\psi\colon E\to E$ is an affine isometry, $R\colon E\to E$ is a linear isometry, and $t_{m(\mu)}\colon E\to E$ is the translation on E by the barycenter $m(\mu)$ of μ. Recall that the barycenter of μ is the point $m(\mu)\in E$ such that

$$\langle m(\mu),z\rangle=\int_E\langle x,z\rangle\mathrm{d}\mu(x) \tag{3.2}$$

holds for all $z\in E$. We will see in the next subsection that the assumption $dimE\geq 2$ is essential.

§ 3.2. Exotic isometries

All isometries that we have seen until this point were related to isometries of the underlying space. Now we turn to the case when such a connection does not exist, and an isometry can distort the shape of measures. Following Kloeckner's terminology we call these isometries *exotic*, rather than nonshape-preserving. All known exotic isometries are related to two concrete p-Wasserstein space: $\mathcal{W}_2(\mathbb{R}, |\cdot|)$ and $\mathcal{W}_1([0,1], |\cdot|)$. In case of $(X, \varrho) = (\mathbb{R}, |\cdot|)$, the *cumulative distribution function* of a measure $\mu \in \mathcal{W}_2(\mathbb{R}, |\cdot|)$ is defined as

$$F_\mu \colon \mathbb{R} \to [0,1], \quad x \mapsto F_\mu(x) := \mu\left((-\infty, x]\right). \tag{3.3}$$

The *quantile function* of μ (or the right-continuous generalized inverse of F_μ) is denoted by F_μ^{-1} and is defined as

$$F_\mu^{-1} \colon (0,1) \to \mathbb{R}, \quad y \mapsto F_\mu^{-1}(y) := \sup\left\{x \in \mathbb{R} \,|\, F_\mu(x) \le y\right\}. \tag{3.4}$$

In case of $\big(X, \varrho\big) = \big([0,1], |\cdot|\big)$ we shall handle the cumulative distribution and the quantile functions (with the convention $\sup\{\emptyset\} := 0$) of a $\mu \in \mathcal{P}([0,1])$ as $[0,1] \to [0,1]$ functions. The quantile function in this case is defined by right-continuity at 0 and it takes the value 1 at 1.

A very important feature of p-Wasserstein spaces over the interval and the real line is that they embed isometrically into the corresponding $L^p(0,1)$ space by means of the map $\mu \mapsto F^{-1}(\mu)$. In particular, $d_2(\mu,\nu) = \|F_\mu^{-1} - F_\nu^{-1}\|_2$ for all $\mu, \nu \in \mathcal{W}_2(\mathbb{R}, |\cdot|)$ and $d_1(\mu,\nu) = \|F_\mu^{-1} - F_\nu^{-1}\|_1$ for all $\mu, \nu \in \mathcal{W}_1([0,1], |\cdot|)$.

First we sketch the very surprising result of Kloeckner on the isometry group of the quadratic Wasserstein space over the real line. For the details we refer the reader to [19]. Let us introduce the notations

$$\Delta_1(\mathbb{R}) := \{\delta_x \,|\, x \in \mathbb{R}\}, \qquad \Delta_2(\mathbb{R}) := \{\lambda\delta_x + (1-\lambda)\delta_y \,|\, x, y \in \mathbb{R},\ \lambda \in [0,1]\}. \tag{3.5}$$

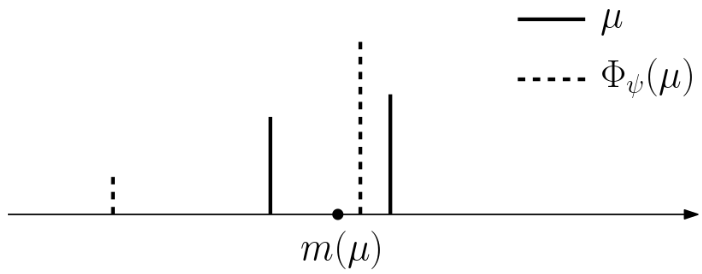
One can prove that if Φ is an isometry of $\mathcal{W}_2(\mathbb{R}, |\cdot|)$, then there exists an isometry $\psi \in \mathrm{Isom}(\mathbb{R})$ such that for all $x \in \mathbb{R}$: $\Phi(\delta_x) = \delta_{\psi(x)}$. Furthermore, Φ maps the set $\Delta_2(\mathbb{R})$ onto itself. The next task is to understand what happens with measures supported on two points. Consider the following parametrization of $\Delta_2'(\mathbb{R}) := \Delta_2(\mathbb{R}) \setminus \Delta_1(\mathbb{R})$: any such measure μ can be written as

$$\mu(m, \sigma, r) := \frac{e^{-r}}{e^{-r} + e^{r}}\delta_{m - \sigma e^{r}} + \frac{e^{r}}{e^{-r} + e^{r}}\delta_{m + \sigma e^{-r}}. \tag{3.6}$$

In probabilistic terms, if μ is the law of a random variable then m is its expected value and σ^2 is its variance. Since

$$d_2^2\Big(\mu(m_1, \sigma_1, r_1), \mu(m_2, \sigma_2, r_2)\Big) = (m_1 - m_2)^2 + \sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2 e^{|r_1 - r_2|}, \tag{3.7}$$

it is obvious that for any $\psi \in \mathrm{Isom}(\mathbb{R})$, the map $\Phi_\psi : \mu(m, \sigma, r) \mapsto \mu(m, \sigma, \psi(r))$ is distance preserving on $\Delta_2'(\mathbb{R})$, and is not a restriction of a shape-preserving isometry of $\mathcal{W}_2(\mathbb{R}, |\cdot|)$, unless $\psi(r) = -r$. Indeed, a push-forward isometry can modify the support, but it does not modify the weights of a $\mu \in \Delta_2'(\mathbb{R})$. But for the map Φ_ψ defined above, μ and $\Phi_\psi(\mu)$ typically have different weights at their supporting points.



Kloeckner proved that Φ_ψ can be uniquely extended into an isometry of $\mathcal{W}_2(\mathbb{R}, |\cdot|)$. The key points of his proof are the following:

(1) The geodesic convex hull of $\Delta_2(\mathbb{R})$ contains the set of all finitely supported measures, and therefore it is dense in $\mathcal{W}_2(\mathbb{R}, |\cdot|)$. Recall that we call a subset S geodesically convex if every geodesic segment whose endpoints are in S lies entirely in S. The geodesic convex hull of a set S is the smallest geodesically convex set that contains S. (See Proposition 3.3 and Proposition 3.4 in [19].)

(2) Any distance preserving map on $\Delta_2(\mathbb{R})$ can be extended into an isometry of the closure of its geodesic convex hull. (See Proposition 4.2 in [19].)

(3) Isometries of $\mathcal{W}_2(\mathbb{R}, |\cdot|)$ are completely determined by their action on $\Delta_2(\mathbb{R})$.

This shows that the image of # is indeed a proper subgroup of $\mathrm{Isom}(\mathcal{W}_2(\mathbb{R}, |\cdot|))$. In fact, Kloeckner showed in [19, Theorem 1.1] that every isometry of $\mathcal{W}_2(\mathbb{R})$ is a composition of some of the following maps:

- a trivial isometry, that is, $\psi_\#$ for some $\psi \in \mathrm{Isom}(\mathbb{R})$;
- the map $\mu \mapsto (r_{m(\mu)})_\#(\mu)$, that is, the isometry that reflects every measure through its center of mass, which is Φ_ψ for $\psi : r \mapsto -r$
- an exotic isometry Φ_ψ for some $\psi \in \mathrm{Isom}(\mathbb{R}) \setminus \{r \mapsto -r\}$.

§ 3.3. Mass-splitting isometries

After showing that $\mathcal{W}_2(\mathbb{R}, |\cdot|)$ admits exotic isometries, Kloeckner posed the following two questions.

- Does there exist a Polish (or Hadamard) space $X \neq \mathbb{R}$ such that $\mathcal{W}_2(X)$ admits exotic isometries?
- Does there exist a Polish space X whose Wasserstein space $\mathcal{W}_2(X)$ possess an isometry that does not preserve the set of Dirac masses?

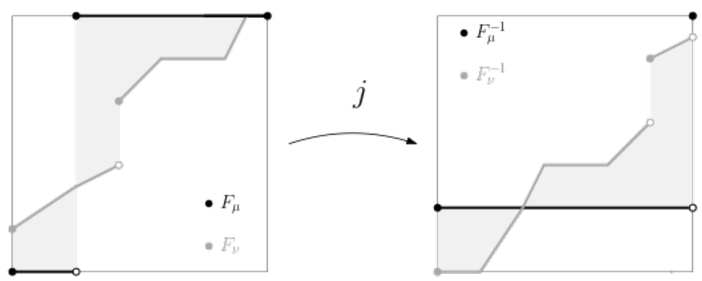
In [12] we showed that the answer is affirmative for both questions if $p = 1$. Namely, the 1-Wasserstein space $\mathcal{W}_1([0,1], |\cdot|)$ admits a mass-splitting isometry (which is of course exotic). As it was already mentioned, a very special feature of $\mathcal{W}_1([0,1], |\cdot|)$ is that it embeds into $L^1(0,1)$ isometrically. In fact,

$$d_1(\mu,\nu) = \int_0^1 |F_\mu^{-1}(t) - F_\nu^{-1}(t)|dt = \int_0^1 |F_\mu(t) - F_\nu(t)|dt \tag{3.8}$$

holds for all $\mu, \nu \in \mathcal{W}_1([0,1], |\cdot|)$. Recall that a cumulative distribution function of a $\mu \in \mathcal{P}([0,1])$ is monotone increasing, continuous from the right and takes the value 1 at the point 1. Conversely, any function $F : [0,1] \to [0,1]$ satisfying the above three conditions is the cumulative distribution function of some Borel probability measure on $[0,1]$. Consequently, for any measure $\mu \in \mathcal{P}([0,1])$, the quantile function F_μ^{-1} is a cumulative distribution function of some measure $\nu \in \mathcal{P}([0,1])$, that is, $F_\nu = F_\mu^{-1}$. It is easy to see that the map $j : \mathcal{W}_1([0,1], |\cdot|) \to \mathcal{W}_1([0,1], |\cdot|)$ defined by the equation

$$F_{j(\mu)} := F_\mu^{-1} \qquad (\mu \in \mathcal{W}_1([0,1]))$$

preserves the distance. As $j \circ j$ is the identity of $\mathcal{W}_1([0,1])$, we see also that j is a bijection, and thus an isometry.



Observe that j does not preserve the set of Dirac masses. Indeed, as it can be seen on the figure above, $j(\delta_t) = t\delta_0 + (1-t)\delta_1$ for all $0 \le t \le 1$. More details about isometries and isometric embeddings of $\mathcal{W}_p([0,1])$ and $\mathcal{W}_p(\mathbb{R})$ spaces can be found in [12].

Finally, we show that the answer to Kloeckner's questions is affirmative for all parameters $p \ge 1$. Let us fix a $1 \le p < \infty$, and let us equip the set $[0,1]$ with the metric $\varrho(x,y) = |x-y|^{1/p}$. Since this metric space has finite diameter, every Borel probability measure on $[0,1]$ is automatically an element of both $\mathcal{W}_p([0,1], |\cdot|^{1/p})$ and $\mathcal{W}_1([0,1], |\cdot|)$.

In order to avoid misunderstanding, below we use a slightly more detailed notation for the Wasserstein distance. We denote the distance of $\mathcal{W}_p([0,1],|\cdot|^{1/p})$ and $\mathcal{W}_1([0,1],|\cdot|)$ by $d_{\mathcal{W}_p([0,1],|\cdot|^{1/p})}$ and $d_{\mathcal{W}_1([0,1],|\cdot|)}$, respectively. Notice that

$$\begin{aligned} d^p_{\mathcal{W}_p([0,1],|\cdot|^{1/p})}(\mu,\nu) &= \inf_{\pi\in C(\mu,\nu)} \iint\limits_{[0,1]\times[0,1]} (|x-y|^{1/p})^p \, \mathrm{d}\pi(x,y) \\ &= \inf_{\pi\in C(\mu,\nu)} \iint\limits_{[0,1]\times[0,1]} \big|x-y\big| \, \mathrm{d}\pi(x,y) = d_{\mathcal{W}_1([0,1],|\cdot|)}(\mu,\nu) \end{aligned}$$

holds for all $\mu,\nu \in \mathcal{P}([0,1])$, and therefore the map j defined by $F_{j(\mu)} = F_\mu^{-1}$ is a mass-splitting isometry of $\mathcal{W}_p(X,|\cdot|^{1/p})$. For more details, see Section 2 in [14].

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