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Optimal Control and Station Relocation of Vehicle-Sharing Systems With Distributed Dynamic Pricing

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ABSTRACT This paper addresses an optimal control problem for one-way car-sharing systems under dynamic pricing. One-way car-sharing service allows customers to return vehicles to any available station. Although this service provides great convenience to customers, it has a serious drawback such that vehicles can be unevenly parked and some stations can be unavailable. To solve this problem, dynamic pricing is promising to distribute car parking, with which customers change their origins and destinations by walking according to prices. First, we develop a model of this system by using differential equations, including stochastic processes which represent uncertain human behavior such as the demand shift. Next, we design a uniform, distributed dynamic pricing policy by solving the optimization problem to minimize the unevenness of the vehicles. Then, we show that the effectiveness of the dynamic pricing depends on the network topology of stations. So, we propose an optimization method to relocate stations for effective dynamic pricing with PSO (Particle Swarm Optimization). Finally, numerical examples demonstrate the effectiveness of the developed methods.

INDEX TERMS Car-sharing service, stochastic process modeling, optimal control, dynamic pricing, demand shift, consensus control.

I. INTRODUCTION

CAR-SHARING services are focused on as a modern transportation form because the cost of owning cars is rising and the reduction of the traffic is expected in urban areas. There are some variants in car-sharing services: round-trip, one-way, and free-floating styles. The free-floating style is the most convenient and is employed in European countries and the U.S., whereas this style does not fit the traffic system of many other countries. For example, Japan has few parking space on street sides. In such countries, the one-way style is the most promising. In one-way car-sharing service, customers can park cars only in predetermined parking stations while they can return the vehicle to any available station. That is, they do not need to return cars to the stations where they rent the cars, which is different from the round-trip style. In Japan, one-way car-sharing service has been legally allowed since 2014, and many companies have been conducting field trials [1]. Especially, Toyota launched and managed a one-way car-sharing service, called Ha:mo Ride Toyota [2], with 110 electric vehicles. The service area of Ha:mo Ride Toyota is shown in Figure 1, consisting of 57 stations. The locations of the stations are depicted by blue circles in this figure.

Despite the convenience of one-way car-sharing service, it has the critical problem that packing vehicles will be unevenly distributed among stations because the usage of customers is usually focused on convenient stations or famous spots. Eventually, no vehicles may be parked in some stations, while no parking slots may be available in other stations. To solve the issue of the uneven distribution, vehicles must be redeployed, which is usually conducted by the service staff. The service staff move to a station with extra parking vehicles and drive a vehicle to another station with insufficient parking vehicles. Because the redeployment requires labor costs, other

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strategies are expected to implement one-way car-sharing service in practice. Due to these motivations, one-way car-sharing service has been investigated [3]. As concrete research, [4] employed a model predictive control method to minimize the redeployment cost, [5] developed a model with the behavior of staff for redeployment, and [6] conducted a simulation-based study for a car-sharing service in Southern California to minimize the number of deployments.

Dynamic pricing can be another solution, which encourages customers to move vehicles to relieve the uneven distribution by adjusting prices dynamically according to the current numbers of the parking vehicles. Although the effect of dynamic pricing is uncertain compared with the direct redeployment, it does not involve staff cost. Hence, it is worth studying the effect of car-sharing services with dynamic pricing. Reference [7] developed a mixed integer non-linear programming model of one-way car-sharing service with dynamic pricing, which sets prices to maximize profit. The model in [8] involved several factors including the prices and the staff assignment and aimed at maximizing profits. Reference [9] solved an optimization problem for pricing and charge scheduling of electric vehicles. References [10] and [11] also investigated pricing policies. These papers have considered the models where the number of the customers in each station increases or decreases according to the price. However, in practice, the number of the customers does not increase or decrease individually because customers will not give up using even if the price is higher than expected, or customers will not be generated even if the price is lower than usual. Instead, customers possibly shift the stations to be used by walking. This scenario is realistic especially in urban areas because a certain number of stations can be located within walking distance for convenience. We can confirm this in Ha:mo Ride Toyota, as shown in Figure 1. The shift of customers was firstly modeled for bike sharing systems [12]. The authors developed a model of one-way car sharing service with dynamic pricing and demand shift [13]. In this model, the customers shift links to cheaper ones, that is, change their origins and destinations by walking. These existing papers have developed models and pricing methods and confirmed the effect of pricing by simulations. To investigate the effect more rigorously, it is worth analyzing the models in a theoretical way. For example, if we understood the connection between the effect of pricing and the locations of stations, we could propose an appropriate locations of stations.

As for the problem of station location, or network layout planning, [14] has investigated the problem for maximizing profits. In some studies, demand uncertainty has been taken into account for this problem [15], [16]. Furthermore, in [17], [18], the strategic problem, say the station location problem, is considered along with the operational problem, e.g., the vehicle redeployment. However, no research considers the effect of demand shift by dynamic pricing because the effect has not been analytically investigated so far.

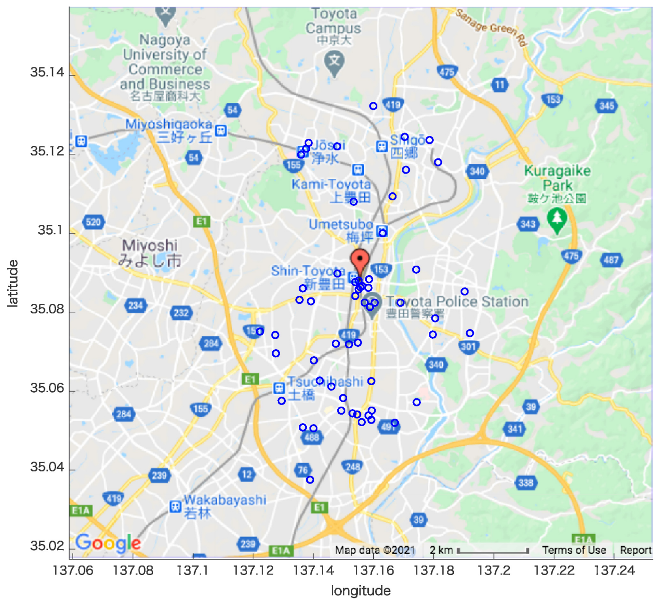

FIGURE 1. Station locations of Ha:mo Ride Toyota (blue circles indicate locations of stations).

In this paper, we analyze a model of one-way car sharing service considering demand shift under dynamic pricing and design the best distributed dynamic pricing policy from a control-theoretic perspective. Furthermore, we propose a method of station relocation for effective dynamic pricing. First, we make a model of this system consisting of the numbers of available parking slots and demand of customers. In particular, we represent the uncertain shifts of customers to other links by using stochastic processes. Then, for the system consisting of the models, we design the best uniform, distributed dynamic pricing policy, which minimizes the unevenness of parking vehicles. Next, we consider a problem to determine the better locations of stations for shifting links by pricing. The PSO (Particle Swarm Optimization) is employed to solve this problem. Finally, we demonstrate the effectiveness of the method through numerical examples.

The main contributions of this paper are threefold. First, this paper develops a model to represent the demand shift and designs a dynamic pricing strategy from a control-theoretic perspective. This approach allows us to disclose the essence of this system and to utilize the various analysis and design tools in the field of control engineering. Actually, this system is shown to be equivalent to a basic consensus control system with external disturbance [19], [20]. Second, this paper takes into account the effect of dynamic pricing on the station relocation problem, which has not been explicitly considered in existing papers. This effect is evaluated in a closed form to use as an objective function of the optimization problem due to the first contribution. This approach enables us to systematically integrate the strategic and operational problems of station location and dynamic pricing. Third, we propose a uniform, distributed dynamic-pricing policy such that its parameters are uniform among all links and the price of each

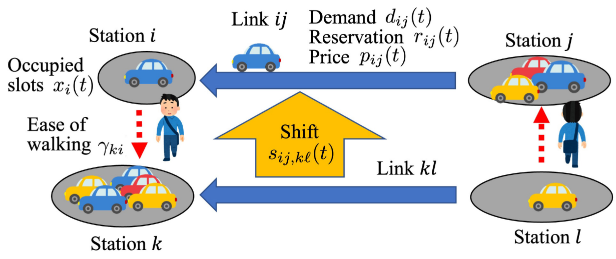

FIGURE 2. Target car-sharing system.

link is computable from local information. This computation can be done on a device of each consumer, which contrasts with a centralized optimization-based method involving a heavy computational burden, employed by many papers.

This paper is based on the authors' conference paper [21], which verified the validity of the traffic model by using a traffic simulator, Simulation On Urban road Network with Dynamic route choice (SOUND) [22]. The difference from it is as follows. First, the proofs of all the theorems and lemmas are provided whereas they were omitted in [21] because of the lack of space. Second, the relocation problem of stations is newly addressed and solved with PSO in this paper.

The rest of this paper is organized as follows. In Section II, the car-sharing service is modeled with stochastic process and the problem setting is provided. Section III gives the main result, i.e., the best distributed dynamic pricing policy is designed. In Section IV, we consider the relocation problem of stations. Section V presents numerical examples under various setting. Finally, Section VI concludes the paper.

Notation is found in Appendix A.

II. PROBLEM FORMULATION

In this section, the car-sharing service is modeled with four differential equations including stochastic process. Then, we formulate the problem tackled in this paper: the design of the dynamic pricing policy to minimize the unevenness of parking vehicles.

A. SYSTEM MODELS

We consider a one-way car-sharing system, which is managed by an operator. Let $n \in \mathbb{Z}_+$ be the number of stations, where customers can rent and return vehicles. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ denote the index set of stations. For the origin $j \in \mathcal{N}$ and destination $i \in \mathcal{N}$, a customer is said to be travel from j to i, or through link ij. Afterwards, dynamic pricing is introduced, and the time interval of the change of prices corresponds to the sampling time in the following models.

Table 1 lists the variables and constants used in this paper. Figure 2 depicts the target car-sharing system. The whole system consists of four models: parking-slot, reservation, demand, and demand-shift models, which are introduced below.

TABLE 1. Terminology and notations in the models.

$x_i(t)$	# of used or reserved parking slots at station i
$x_i^{\max}$	capacity of the parking slots at station i
$x_i^+(t)$	expected next number of $x_i(t)$ for demands $d_{ij}(t)$
$r_{ij}(t)$	reservation # for link ij
$d_{ij}(t)$	actual demand # for link ij
$\hat{d}_{ij}(t)$	original demand # for link ij
$p_{ij}(t)$	price of link ij
$\hat{p}_{ij}$	standard price of link ij
$u_{ij}(t)$	unsatisfied demand # for link ij
δ_{ij}	expectation of original demand number $\hat{d}_{ij}(t)$
γ_{ik}	ease of changing stations from k to i by walking
$s_{ij,k\ell}(t)$	demand shift # from link $k\ell$ to ij
n	# of stations
m	total # of vehicles
$m^{\max}$	sum of all capacities
κ	unit of price in the car-sharing service
$\phi(\cdot)$	sensitivity of customers to prices
$\hat{\phi}$	coefficient of the sensitivity

1) PARKING-SLOT MODEL

In the car-sharing service in question, we assume that each customer reserves a vehicle by selecting both the origin and destination. Hence, the numbers of used or reserved parking slots are available for operation. Let $x_i(t) \in \mathbb{Z}_+$ be the number of used or reserved parking slots at station $i \in \mathcal{N}$. The restriction and transition of $x_i(t)$ are given.

Let $x_i^{\max} \in \mathbb{Z}_+$ be the capacity of the parking slots at station i. Hence, $x_i(t)$ must satisfy

$$0 \leq x_i(t) \leq x_i^{\max} \ \forall t \in \mathbb{Z}_+. \tag{1}$$

Let $m \in \mathbb{Z}_+$ be the total number of vehicles, and the sum of the vehicle numbers is preserved in the following way:

$$\sum_{i \in \mathcal{N}} x_i(t) = m. \tag{2}$$

The sum of capacities of the stations is denoted as

$$m^{\max} = \sum_{i \in \mathcal{N}} x_i^{\max}. \tag{3}$$

We need to assume that $m^{\max}$ is sufficiently larger than m to completely resolve the parking shortage with dynamic pricing. Even though $m^{\max}$ is not sufficiently large, the method proposed below is effective for alleviating the shortage.

Let $r_{ij}(t) \in \mathbb{Z}_+$ be the reservation number for link ij. Then, $x_i(t)$ varies according to the differential equation

$$x_i(t+1) = x_i(t) + \sum_{j \in \mathcal{N}} \big(r_{ij}(t) - r_{ji}(t)\big), \tag{4}$$

which implies that the number of used or reserved parking slots at station i increases (decrease) according to the sum of reservation numbers into station i (out of station i). The operator has to control the number $x_i(t)$, which varies according to (4), to satisfy the restriction in (1).

Remark 1: We assume that the sampling time (price change interval) is long enough for most vehicles to reach the destination within the sampling time. Then, the model (4) without travel time is approximately valid. A model with the travel time proposed by the author [23] shows the validity of this approximation.

2) RESERVATION MODEL

Each customer requests for a vehicle with his/her origin and destination. If there is an available vehicle and parking slot, the request is accepted and the customer can reserve a vehicle. Whether the request is accepted or not is determined by the operator. Let $d_{ij}(t) \in \mathbb{Z}_+$ be the demand (request) number for link ij, and the reservation number $r_{ij}(t)$ is determined under the following strategy: (i) The reservation number is not over the demand number, i.e., $r_{ij}(t) \leq d_{ij}(t)$. (ii) $x_i(t)$ must satisfy the capacity constraint given in (1). (iii) The unsatisfied demand $u_{ij}(t) = d_{ij}(t) - r_{ij}(t) \in \mathbb{Z}_+$ must be minimized when not all demands can be fulfilled.

Here, the first-come-first-served policy is employed for reservation. Then, strategy (iii) cannot be strictly followed. Nevertheless, the reservation number can be assigned as $r_{ij}(t) = d_{ij}(t)$ if the capacity of each station is fulfilled at the next step for the demands, i.e.,

$$0 \leq x_i^+(t) \leq x_i^{\max} \tag{5}$$

holds for

$$x_i^+(t) = x_i(t) + \sum_{j\in\mathcal{N}} \big(d_{ij}(t) - d_{ji}(t)\big). \tag{6}$$

Then, $x_i^+(t)$, equivalent to $x_i(t+1)$ in (4), satisfies (1); thus, (iii) is realized. From this viewpoint, to follow strategy (iii), the operator needs to control $x_i^+(t)$ so as to satisfy (5).

3) DEMAND MODEL

The operator determines the prices of links, and each customer may shift the origin and/or destination of the link to be used by walking according to the prices. We model the demand shifts below.

Let $\hat{d}_{ij}(t) \in \mathbb{Z}_+$ be the original demand number for link ij when the price of the link is standard. In contrast, $d_{ij}(t)$ is the actual demand number for link ij, which can change according to the price. Assume that $\hat{d}_{ij}(t)$ is a random variable with expectation $\delta_{ij} > 0$, that is,

$$\mathrm{E}\Big[\hat{d}_{ij}(t)\Big] = \delta_{ij}. \tag{7}$$

Here, $\hat{d}_{ij}(t)$ are assumed to be independent for any $i, j \in \mathcal{N}$ and $t \in \mathbb{Z}_+$. A typical example of the demand number is the Poisson distribution

$$\Pr\Big(\hat{d}_{ij}(t) = \hat{d}_{ij}\Big) = \frac{\delta_{ij}^{\hat{d}_{ij}} e^{-\delta_{ij}}}{\hat{d}_{ij}!}.$$

Let $s_{ij,k\ell}(t) \in \mathbb{Z}_+$ be the demand shift number, i.e., the number of customers who shift links from $k\ell$ to ij. As shown in Figure 2, this means that the customer changes the origin from ℓ to j and the destination from k to i. Then, the actual demand number $d_{ij}(t)$ is varied by $s_{ij,k\ell}(t) \in \mathbb{Z}_+$ from the original one $\hat{d}_{ij}(t)$ as

$$d_{ij}(t) = \max\left\{\hat{d}_{ij}(t) + \sum_{k\ell\in\mathcal{N}^2} \big(s_{ij,k\ell}(t) - s_{k\ell,ij}(t)\big), 0\right\}, \tag{8}$$

which implies that the demand increases (decreases) by the customers who shift into link ij (who shift away from link ij).

4) DEMAND-SHIFT MODEL

The demand shift is caused by the prices of links, and it occurs more often as changing links by walking is easier. We model the number $s_{ij,k\ell}(t)$ of the demand shifts from this viewpoint.

We set the unit of price in the service (e.g., ten cents or ten yen) at $\kappa \in \mathbb{Z}_+$ ($\kappa > 0$). Let $p_{ij}(t) \in \kappa\mathbb{Z}$ be the price of link ij, which takes only a multiple of κ. Let $\gamma_{ik} (= \gamma_{ki}) \geq 0$ be the degree of the ease of walking between stations i and k. In general, the larger the distance between stations i and k is, the smaller the degree γ_{ik} of the ease of walking is. For example, the distance between stations can be used to represent the ease such that

$$\gamma_{ik} = e^{-\eta\|\rho_i - \rho_k\|} \tag{9}$$

with a parameter $\eta > 0$, where $\rho_i \in \mathbb{R}^2$ is the position of station i. γ_{ij} can reflect other factors on inconvenience, e.g., walking road conditions.

When each customer reserves a vehicle for a desired link, he/she can access two pieces of information from a device (e.g., smart phone). (i) (Benefit received from shifting) the difference $p_{k\ell}(t) - p_{ij}(t)$ of prices of links around the desired link. (ii) (Cost of shifting) the ease $\gamma_{ik}, \gamma_{j\ell}$ of changing stations by walking around the desired link. Then, the shift occurs according the difference between prices, which is modeled by a random variation with expectation $\gamma_{ik}\gamma_{j\ell}\phi(p_{ij}(t) - p_{k\ell}(t))$. Here, the function $\phi : \mathbb{Z} \to \mathbb{R}_+$ represents the sensitivity to prices. This function can be assumed to be a monotonically non-increasing function such that $\phi(p) = 0$ for $p \geq 0$ and $\phi(p) \leq \phi_{\max}$ for some $\phi_{\max} > 0$ because of the two reasons: (i) More customers shift to cheaper links but none of them shift to more expensive links. (ii) The sensitivity tends to be flat as it increases.

Then, the random variable $s_{ij,k\ell}(t)$ can be modeled with the conditional expectation as

$$\mathrm{E}\big[s_{ij,k\ell}(t)|P(t) = P\big] = \gamma_{ik}\gamma_{j\ell}\phi\big(p_{ij} - p_{k\ell}\big), \tag{10}$$

where $P(t) \in \mathbb{Z}^{n\times n}$ is the price matrix, whose (i, j)-entry is $p_{ij}(t)$.

5) SUMMARY OF THE MODELS

The system model consists of the parking-slot model (4) with (2), the demand model (8) with (7), and the demand-shift model (10). The goal of the system is reduced to the restriction (5) in terms of reservation.

B. CONTROL OBJECTIVE

The most important control objective is to reduce the unsatisfied demands $u_{ij}(t)$, which is achieved by controlling $x_i^+(t)$ to satisfy the constraint (5), as discussed in Section II-A2. Let $\bar{x}_i^+(t) = x_i^+(t) - x_i^{\max}/2$ be the discrepancy between the used parking slots and its desired number, which is set at half the capacity. Our goal is to minimize the discrepancies to satisfy (5). The discrepancies are evaluated with the average of the squared discrepancies as follows:

$$V_{\mathrm{x}}(x(t)) = \frac{1}{n}\sum_{i\in\mathcal{N}} \mathrm{E}\big[\bar{x}_i^+(t)\big]^2, \tag{11}$$

where $x(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^\top$.

To realize the control objective, the operator implements dynamic pricing in the car-sharing service. We assume that the operator employs a uniform, distributed policy of dynamic pricing to facilitate the implementation of dynamic pricing. That is, the policy is uniform in all the links and the price of each link is computable based on information of only neighboring stations. This dynamic pricing policy is described by a function $\pi : \mathbb{Z}_+^2 \to \kappa\mathbb{Z}$, with which the price $p_{ij}(t)$ of link ij is adjusted as

$$p_{ij}(t) = \hat{p}_{ij} + \pi\left(\bar{x}_i(t), \bar{x}_j(t)\right) \tag{12}$$

where $\bar{x}_i(t) = x_i(t) - x_i^{\max}/2$ and $\hat{p}_{ij} \in \kappa\mathbb{Z}$ represents the standard price between stations i and j. We assume that $\hat{p}_{ij} = \hat{p}_{ji}$. The dynamic pricing policy π is uniform in links; therefore the operator does not require to design a policy for each link. Moreover, π is distributed, i.e., the function $\pi(\bar{x}_i(t), \bar{x}_j(t))$ depends only on the numbers of the neighboring stations i and j. Hence, when customers refer to the prices of links, these prices are easily computed with the information only on the neighborhoods. This computation can be done on a device of each consumer, and thus a central computer system is unnecessary.

The discrepancy between the price $p_{ij}(t)$ and the standard price $\hat{p}_{ij}$ is preferably small to reduce the complexity in the service as much as possible. Accordingly, from (12), the average of the squared discrepancies, evaluated as

$$V_{\mathrm{p}}(x(t)) = \frac{1}{n^2}\sum_{i,j\in\mathcal{N}} \left(\mathrm{E}\left[\pi\left(\bar{x}_i(t), \bar{x}_j(t)\right)\right]\right)^2, \tag{13}$$

should be as small as possible. For simplicity, we assume that $\pi : \mathbb{Z}_+^2 \to \kappa\mathbb{Z}$ is an affine function as

$$\pi\left(\bar{x}_i, \bar{x}_j\right) = \hat{\pi}_{\mathrm{a}}\bar{x}_i + \hat{\pi}_{\mathrm{b}}\bar{x}_j + \hat{\pi}_{\mathrm{c}},\ \ \hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}} \in \kappa\mathbb{Z}, \tag{14}$$

where $\hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}} \in \kappa\mathbb{Z}$ are design parameters.

Now, we expect to minimize $V_{\mathrm{x}}(x(t))$ and $V_{\mathrm{p}}(x(t))$ by using a uniform, distributed dynamic pricing policy. According to these objectives, we evaluate their terminal values with constants $\mu, \nu > 0$ as

$$\begin{aligned} V\big(\hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}}\big) = \max_{m\in\mathbb{Z}} \lim_{t\to\infty} \big(V_{\mathrm{x}}(\bar{x}(t)) + \mu V_{\mathrm{p}}(x(t))\big) \\ + \nu\Big(\hat{\pi}_{\mathrm{a}}^2 + \hat{\pi}_{\mathrm{b}}^2 + \hat{\pi}_{\mathrm{c}}^2\Big), \end{aligned} \tag{15}$$

including the regularization term of the design parameters. The maximum with respect to $m \in \mathbb{Z}$ is considered in (15) for designing a scalable policy π. Here, a scalable policy indicates that it is implementable regardless of the number m of vehicles.

Now, the following optimization problem is formulated as

$$\min_{\hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}} \in \kappa\mathbb{Z}} V\big(\hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}}\big). \tag{16}$$

Finally, the problem tackled in this paper is summarized.

Problem 1: Solve the optimization problem in (16) for the one-way car-sharing system consisting of the models of (1)–(10) by designing a uniform, distributed dynamic pricing policy of (12) with a function $\pi : \mathbb{Z}_+^2 \to \kappa\mathbb{Z}$ of the form (14).

III. MAIN RESULT

In this section, we derive a necessary condition of the best dynamic pricing policy as a solution to the optimization problem (16). The proofs of the theorems are given in the Appendix.

A. SOLUTION TO PROBLEM 1

To solve (16), the graph topology representing the locations of the stations as Figure 1 plays an important role. Let $\Gamma \in \mathbb{R}^{n\times n}$ be the adjacency matrix of the graph, whose (i,j)-entry γ_{ij} indicates the ease of walking between stations i and k. The graph Laplacian is defined as

$$L = \mathrm{diag}(\Gamma\mathbf{1}) - \Gamma \in \mathbb{R}^{n\times n}. \tag{17}$$

Then, L has non-negative eigenvalues and n orthogonal eigenvectors. There is one zero-eigenvalue with the corresponding eigenvalue $\mathbf{1}/\sqrt{n}$ [19]. The number of zero-eigenvalues of L is denoted by ξ. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues such that $\lambda_1 = \cdots = \lambda_\xi = 0$ and $0 < \lambda_{\xi+1} \le \cdots \le \lambda_n$. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be the corresponding orthonormal eigenvectors with $v_1 = \mathbf{1}/\sqrt{n}$.

To solve (16) in an analytical way, we approximate ϕ with a piece-wise linear function around the possible difference of prices such as

$$\phi(p) = \begin{cases} -\hat{\phi}p & \text{if } \ p < 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

for a coefficient $\hat{\phi} > 0$. This approximation is valid because the price change is expected to be relatively small by restricting $V_p(x(t))$ in (13).

Under this setting, the following is derived.

Theorem 1: Assume that (i) ϕ is given by (18) with a coefficient $\hat{\phi} > 0$, (ii) $\hat{p}_{ij} = \hat{p}_{ji}$ holds, (iii) (5) is satisfied, and (iv) the following two equations hold:

$$v_i^\top\Big(\Delta - \Delta^\top\Big)\mathbf{1} = 0\ \ i \in \{1, \ldots, \xi\} \tag{19}$$

$$\mathrm{E}[\bar{x}(0)] = \frac{\bar{m}}{n}\mathbf{1}. \tag{20}$$

For $x_i(t) \in \mathbb{Z}_+$, $i \in \mathcal{N}$ satisfying (1)–(10), the function $\pi : \mathbb{Z}_+^2 \to \kappa\mathbb{Z}$ in (12) of the form (14) is a solution to the optimization problem (16) only if the parameters $\hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}} \in \kappa\mathbb{Z}$ satisfy

$$\hat{\pi}_{\mathrm{a}} \in \big\{\lfloor\pi_{\mathrm{a}*}\rfloor_\kappa, \lceil\pi_{\mathrm{a}*}\rceil_\kappa, \pi_{\mathrm{a}+}\big\}, \tag{21}$$

$$\hat{\pi}_{\mathrm{a}} \le \pi_{\mathrm{a}+} \tag{22}$$

$$\hat{\pi}_{\mathrm{b}} = -\hat{\pi}_{\mathrm{a}}, \tag{23}$$
$$\hat{\pi}_{\mathrm{c}} = 0, \tag{24}$$

where

$$\pi_{\mathrm{a}*} = \sqrt{\frac{\|h\|}{2\sqrt{2n\nu}}},$$
$$\pi_{\mathrm{a}+} = \left\lfloor \frac{1}{\hat{\phi}\lambda_n \sum_{i,j\in\mathcal{N}} \gamma_{ij}} \right\rfloor_\kappa,$$
$$h = \frac{\sum_{i=\xi+1}^{n} \lambda_i^{-1} v_i v_i^\top (\Delta - \Delta^\top)\mathbf{1}}{\hat{\phi}\sum_{i,j\in\mathcal{N}} \gamma_{ij}}, \tag{25}$$

and $\Delta \in \mathbb{R}^{n\times n}$ is the matrix whose (i, j)-entries are δ_{ij}. The minimum of the objective function in (16) is derived as

$$\min_{\hat{\pi}_{\mathrm{a}},\hat{\pi}_{\mathrm{b}},\hat{\pi}_{\mathrm{c}}\in\kappa\mathbb{Z}} V\big(\hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}}\big) = \min_{i\in\{1,2\}} \left(\frac{\|h\|^2}{4n\pi_{\mathrm{a}i}^2} + 2\nu\pi_{\mathrm{a}i}^2 \right) + c, \tag{26}$$

where $\pi_{\mathrm{a}1} = \lfloor \pi_{\mathrm{a}*} \rfloor_\kappa$, $\pi_{\mathrm{a}2} = \lceil \pi_{\mathrm{a}*} \rceil_\kappa$, and c is a constant.

The assumption (19) always holds if the graph with the adjacency matrix Γ is connected. Otherwise, this equation requires that the number of the vehicles is balanced in each connected component of G, which is achievable by the operator's relocations. The assumption (20) indicates that the vehicles are evenly located at the initial time. Note that even if these assumptions are not satisfied, Theorem 1 provides practically appropriate solutions.

Equations (23) and (24) imply that the dynamic pricing policy π in (14) is effective when it is skew-symmetry, i.e., $\pi(\bar{x}_i, \bar{x}_j) = -\pi(\bar{x}_j, \bar{x}_i)$.

Equation (26) indicates the limit of the performance that can be achieved with dynamic pricing. If we are allowed to relocate stations, there is space to improve the performance. The following theorem shows this by giving the optimized value of the problem (16), i.e., (26), with approximation.

Theorem 2: Under the conditions in Theorem 1, assume that $\pi_{\mathrm{a}+} \geq \lceil \pi_{\mathrm{a}*} \rceil_\kappa$. Then,

$$\min_{\hat{\pi}_{\mathrm{a}},\hat{\pi}_{\mathrm{b}},\hat{\pi}_{\mathrm{c}}\in\kappa\mathbb{Z}} V\big(\hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}}\big) = \frac{\sqrt{2\nu}\left|\sum_{i,j\in\mathcal{N}}\big(\delta_{ij} - \delta_{ij}\big)\right|}{n\hat{\phi}\lambda_{\xi+1}\sum_{i,j\in\mathcal{N}}\gamma_{ij}} + O\big(\kappa, \lambda_{\xi+1}/\lambda_{\xi+2}\big), \tag{27}$$

holds, where $O(\cdot)$ represents the Landau symbol.

B. DESIGN STRATEGY BY USING THE THEOREMS

By using Theorem 1, we can determine the parameters $\hat{\pi}_{\mathrm{a}}$, $\hat{\pi}_{\mathrm{b}}$, and $\hat{\pi}_{\mathrm{c}}$ of the uniform, distributed dynamic pricing policy (12) with (14) to reduce the unevenness with less price change. Note that $\hat{\pi}_{\mathrm{c}} = 0$ from (24) and that $\hat{\pi}_{\mathrm{b}}$ is determined by $\hat{\pi}_{\mathrm{a}}$ from (23). The procedure to choose $\hat{\pi}_{\mathrm{a}}$ from the three candidates in (21) is given as

(a) If $\pi_{\mathrm{a}+} \leq \lfloor \pi_{\mathrm{a}*} \rfloor_\kappa$, then $\hat{\pi}_{\mathrm{a}} = \pi_{\mathrm{a}+}$.
(b) Else if $\pi_{\mathrm{a}+} < \lceil \pi_{\mathrm{a}*} \rceil_\kappa$, then $\hat{\pi}_{\mathrm{a}} = \lfloor \pi_{\mathrm{a}*} \rfloor_\kappa$.
(c) Otherwise, $\hat{\pi}_{\mathrm{a}} = \mathrm{argmin}_{i=1,2}(\|^2/(4n\pi_{\mathrm{a}i}^2) + 2\nu\pi_{\mathrm{a}i}^2)$.

Theorem 2 indicates how stations must be located to enhance the performance of dynamic pricing. Because γ_{ik} represents the ease of walking between stations i and k, it is a function of the positions of stations i and k, $\rho_i, \rho_k \in \mathbb{R}^2$, as (9), say $\gamma_{ik}(\rho_i, \rho_k)$. Then, from (17), the graph Laplacian can be regarded as a function of $\rho = [\rho_1\ \rho_2\ \cdots\ \rho_n]$, say $L(\rho)$, and so is its smallest non-zero eigenvalue, say $\lambda_{\xi+1}(\rho)$. Let us define

$$\Lambda(\rho) = \lambda_{\xi+1}(\rho) \sum_{i,j\in\mathcal{N}} \gamma_{ik}(\rho_i, \rho_k). \tag{28}$$

Then, we can minimize the objective function $V(\hat{\pi}_{\mathrm{a}}, \hat{\pi}_{\mathrm{b}}, \hat{\pi}_{\mathrm{c}})$ in (27) by maximizing $\Lambda(\rho)$. This criterion can be used to consider the stations' locations to enhance the performance of dynamic pricing, as discussed in Section IV.

In operation, prices just have to be adjusted according to the proposed dynamic pricing policy, and then the determined locations are ensured to be the most effective. In this way, the proposed method can systematically integrate the strategic and operational problems in that the station locations are determined based on the demand changed by the customers' shifting due to dynamic pricing. This is possible because the effect of dynamic pricing is evaluated in the closed form in Theorem 2. Although other papers [17], [18] have considered strategic and operational problems at once, the effect of dynamic pricing is first considered in this paper.

IV. STATION RELOCATION PROBLEM

In this section, we consider the relocation of the stations to enhance the control performance while the convenience of the customers is preserved. Theorem 2 shows that the control performance depends on the positions of the stations, $\rho = [\rho_1\ \rho_2\ \cdots\ \rho_n] \in \mathbb{R}^{2\times n}$, and the function $\Lambda(\rho)$ in (28) must increase to enhance the performance. We formulate this issue with an optimization problem and solve it with the Particle Swarm Optimization (PSO) algorithm because this problem is non-convex and cannot be solved in an efficient way.

A. PARTICLE SWARM OPTIMIZATION

PSO is an optimization method modeled after the behavior of animals in groups, such as flocks of birds [24]. In PSO, particles within the search space exchange information with each other and search for a better solution based on their experience of their own position and that of other particles. Given an objective function $f : \mathbb{R}^m \to \mathbb{R}$, we search for the optimal solution of the optimization problem

$$\max_{X\in\mathbb{R}^m} f(X) \tag{29}$$

by creating a particle swarm with P particles. Let $X_p^k \in \mathbb{R}^m$ and $V_p^k \in \mathbb{R}^m$ be the position and velocity of particle $p \in \{1, 2, \ldots, P\}$ in the kth update, respectively. We assign the initial position $X_p^0 \in \mathbb{R}^m$ and initial velocity $V_p^0 \in \mathbb{R}^m$ of particle p randomly in the feasible region. Then, we update particles' positions and velocities according to [25] as

$$V_p^{k+1} = \omega V_p^k + c_1 r_1 \left(X_{\mathrm{Pbest}}^k - X_p^k \right) + c_2 r_2 \left(X_{\mathrm{Gbest}}^k - X_p^k \right) \quad (30)$$

$$X_p^{k+1} = X_p^k + V_p^k, \quad (31)$$

where $\omega, c_1, c_2 > 0$ are the weighting parameters and $r_1, r_2 \in [0, 1]$ are random numbers. Here, the value of the function $f(X_p^k)$ is evaluated at each update k, with each particle's best position X_{Pbest}^k and all particles' best position X_{Gbest}^k. We repeat Eqs. (30) and (31) until the number of iterations N is reached.

B. FORMULATING THE OPTIMIZATION PROBLEM

Let us consider how to use PSO to solve the station relocation problem. According to the discussion after Theorem 2, increasing $\Lambda(\rho)$ in (28) alleviates the uneven distribution of the number of vehicles. On the other hand, the new locations might not be suitable to demand distribution. Hence, in arranging the stations, we want to prevent the discrepancy between the original demand distribution and the stations' new locations, i.e., the distance cost for users. This distance cost is defined as

$$J(\rho) = \int_{\mathcal{Q}} \min_{i \in \mathcal{N}} h(\|q - \rho_i\|) \psi(q) dq,$$

where $\psi : \mathcal{Q} \to \mathbb{R}_+$ is a function representing the demand distribution and $h : \mathbb{R}_+ \to \mathbb{R}$ is a monotonically non-decreasing function representing the distance cost for customers to walk from some position q to a station i. Here, we assume that the stations are originally arranged according to the distribution of users' demand, which can be modeled with the function

$$\psi(q) = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \left(\delta_{ij} + \delta_{ji} \right) e^{-\eta(\|q - \rho_i\|)},$$

where the exponential function indicates that demand decreases according to the distance from each station. The term $\sum_{j \in \mathcal{N}} (\delta_{ij} + \delta_{ji})$ indicates the demand in station i, used as an origin and destination. The distance cost is assumed to be the squared distance

$$h(\|q - \rho_i\|) = \|q - \rho_i\|^2.$$

With this in mind, we set the objective function as

$$f(\rho) = \Lambda(\rho) - \alpha J(\rho), \quad (32)$$

for a weight parameter $\alpha > 0$. Consequently, we just need to increase $f(\rho)$ by updating ρ with PSO.

V. NUMERICAL EXAMPLES

Simulations were conducted with the MATLAB to illustrate the effectiveness of the developed methods.

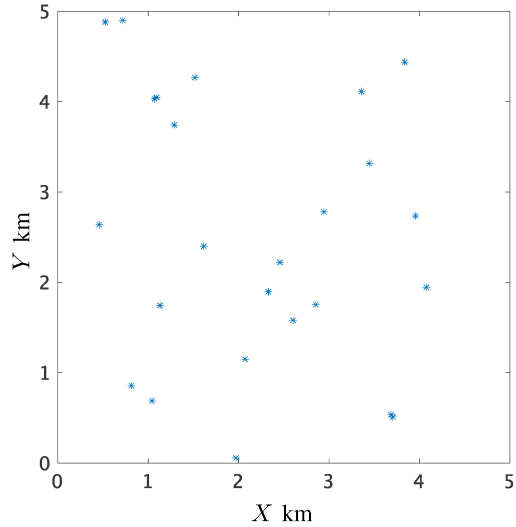

FIGURE 3. Location of $n = 25$ stations.

A. COMPARISON BETWEEN DP AND NON-DP CONDITIONS

Let $n = 25$ be the number of the stations. The stations were located on an area of about 5×5 km as drawn by blue squares in Figure 3. The capacities $x_i^{\max}$ of the parking slots were set from 11 to 20. The total number of sharing vehicles was $m = 248$. The target car-sharing system was composed of the models of (1)–(10). The uniform, distributed dynamic pricing policy is given as (12) with (14). The interval when prices change was set to 15 min. The other parameters were given as follows:

- expectation of demand $\delta_{ij} = 6.30 \times 10^{-4}$ to 1.14,
- price elasticity of demand $\hat{\phi} = 1.00 \times 10^{-4}$,
- ease of shifting stations as (9), where the position $\rho_i \in \mathbb{R}^2$ of station i is determined from Figure 3 and $\eta = 0.750$.

We designed the pricing policy π in (14) for the objective function in (15) with $\mu = \nu = 0.01$ and $\kappa = 1$. According to Theorem 1, the parameters $\hat{\pi}_{\mathrm{a}} = -\hat{\pi}_{\mathrm{b}} = 7$ and $\hat{\pi}_{\mathrm{c}} = 0$ were derived from (21), (23), and (24).

We conducted simulations under two conditions: (a) without dynamic pricing (DP), i.e., with a fixed price, and (b) with DP under the proposed policy. Figure 4 shows the variance of the numbers of reserved parking slots at stations $\mathrm{var}(x_i(t)) = \sum_{i=1}^n (x_i(t) - \sum_{j=1}^n x_j(t)/n)^2/n$, indicating the unevenness, (a) without DP (dashed lines) and (b) with DP (solid lines). Figures 5 and 6 show the values of the unsatisfied demands $\sum_{ij} u_{ij}(t)$ and the maximum price of traveling $\max_{ij} p_{ij}(t)$, respectively, (a) without DP (dashed lines) and (b) with DP (solid lines). Figure 5 indicates that the unsatisfied demand number with DP (solid line) was smaller than that without DP (dashed line).

B. COMPARISON BETWEEN ORIGINAL AND RELOCATED STATIONS

Next, we optimize the location of stations with PSO. We conducted the update equations (30) and (31) with $\omega = 0.2$,

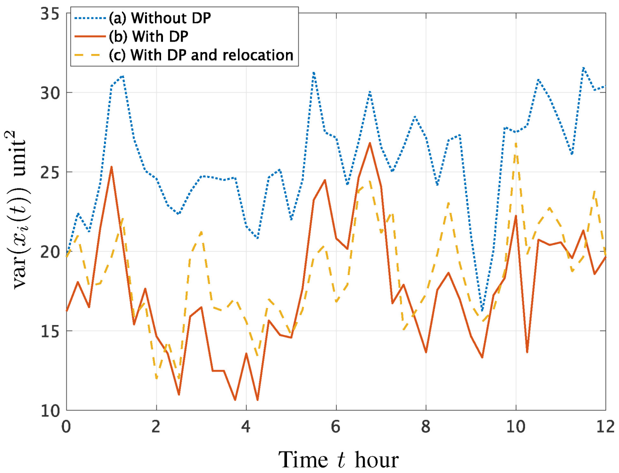

FIGURE 4. Variance of the numbers of parking vehicles $\mathrm{var}(x_i(t))$ (a) without DP (dashed line), (b) with DP (solid line), and (c) with DP and relocation (broken line) for $n = 25$ stations.

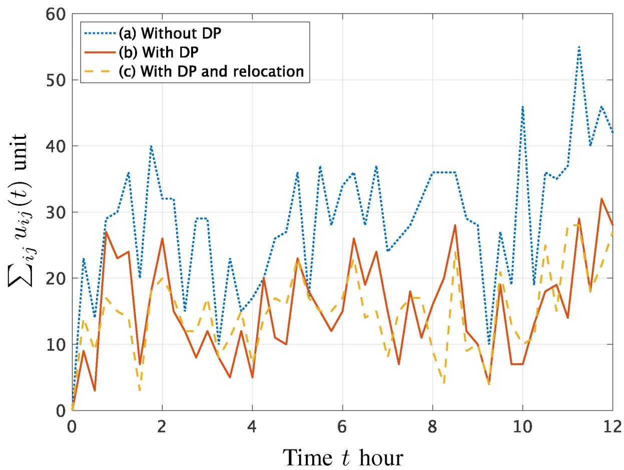

FIGURE 5. Numbers of unsatisfied demands $\sum_{ij} u_{ij}(t)$ (a) without DP (dashed line), (b) with DP (solid line), and (c) with DP and relocation (broken line) for $n = 25$ stations.

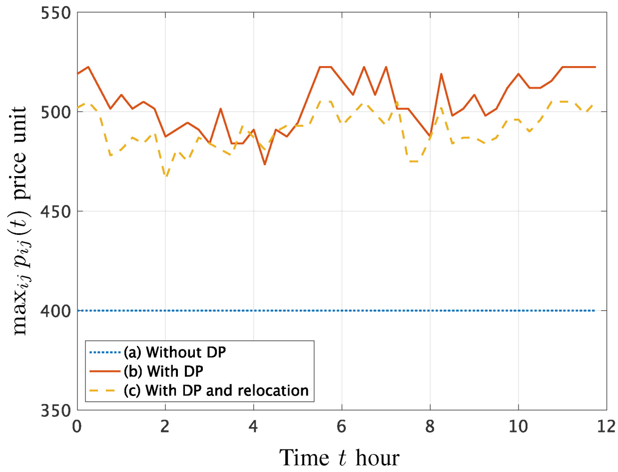

FIGURE 6. Maximum price $\max_{ij} p_{ij}(t)$ (a) without DP (dashed line), (b) with DP (solid line), and (c) with DP and relocation (broken line) for $n = 25$ stations.

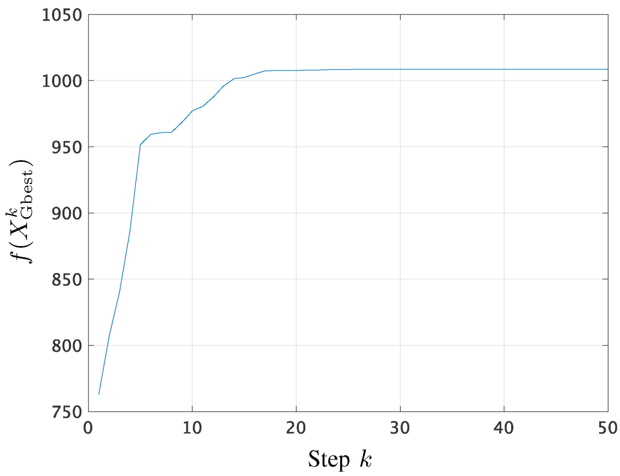

FIGURE 7. Update of the objective function $f(X^k_{\mathrm{Gbest}})$.

$c_1 = 0.1$, and, $c_2 = 0.2$ for the objective function $f(\rho)$ in (32) with $\alpha = 0.01$. The value of $f(X^k_{\mathrm{Gbest}})$ for $k = 1$ to $k = 50$, where X^k_{Gbest} is all particles' best position, is shown in Figure 7. We can see that the best value is obtained about step $k = 20$. The relocated stations are depicted in Figure 8, which is slightly different from the original ones in Figure 3.

We conducted simulations with the relocated stations. In the same setting as the original locations, the parameters $\hat{\pi}_{\mathrm{a}} = -\hat{\pi}_{\mathrm{b}} = 6$ and $\hat{\pi}_{\mathrm{c}} = 0$ were obtained. Figures 5 shows the values of the unsatisfied demands $\sum_{ij} u_{ij}(t)$, which indicates that the unsatisfied demands (b) with DP (solid line) and (c) that with DP and relocation (broken line) are almost the same. Figure 6 shows the maximum price of traveling $\max_{ij} p_{ij}(t)$, which suggests that the prices (c) with DP and relocation (broken line) are lower than (b) that with DP (solid line).

C. RESULTS OF THE SIMULATIONS

The simulation results in Section V-A suggest that introducing DP decreases the unevenness of reserved parking slots and unsatisfied demand by about 31% and 47%, respectively, on average with dynamic pricing. The maximum price of dynamic pricing is about 26% more than the standard price on average. This shows that the benefits of dynamic pricing can be obtained with reasonable price adjustments. Section V-B shows that the station relocation can restrain the change of prices by about 13% on average while the unevenness and unsatisfied demand are kept. This indicates that the effect of dynamic pricing can be enhanced by the station relocation.

The uniform, distributed dynamic pricing policy proposed in this paper is scalable, meaning that it is applicable regardless of the number of stations. For example, for $n = 50$ stations, the same simulation was conducted. Figures 9 and 10 show the variance of the numbers of reserved parking slots at stations $\mathrm{var}(x_i(t))$ and the values of the unsatisfied demands $\sum_{ij} u_{ij}(t)$ in the simulation. These

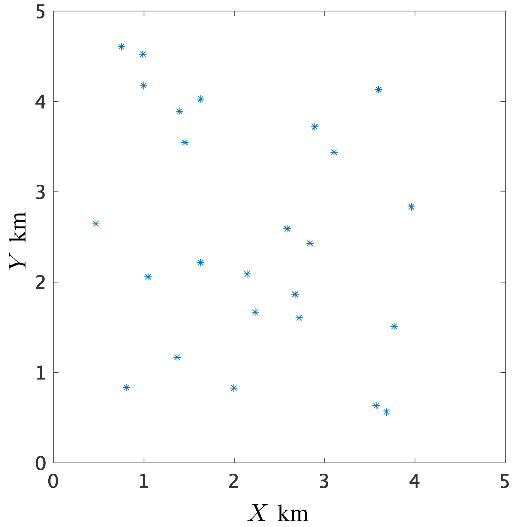

FIGURE 8. Relocated stations.

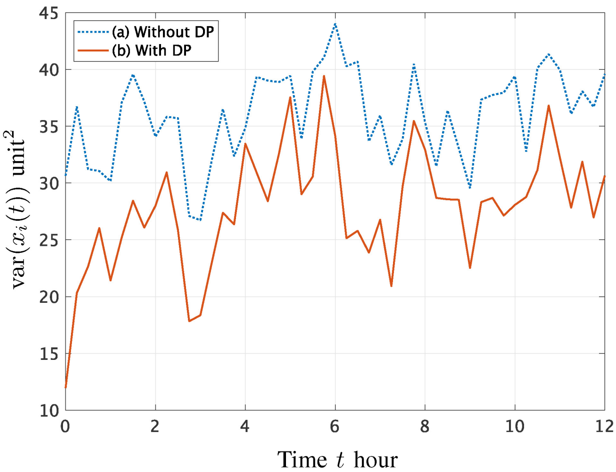

FIGURE 9. Variance of the numbers of parking vehicles $\mathrm{var}(x_i(t))$ (a) without DP (dashed line), (b) with DP (solid line), and (c) with DP and relocation (broken line) for $n = 50$ stations.

figures indicate that similar results were obtained regardless of the number of stations. In contrast, the station relocation method given in Section IV using PSO is not scalable and its computation burden increases rapidly for a large-scale problem. However, this is not a critical issue because the station relocation is a strategic problem and does not need to be conducted realtime.

VI. CONCLUSION

This paper addressed an optimal control problem for one-way car-sharing service under dynamic pricing. First, this system was modeled with differential equations including stochastic processes to represent uncertain human behavior. Then, a uniform, distributed dynamic pricing policy is designed as a solution to the optimization problem regarding the unevenness of the vehicles. This result indicates that the effectiveness of the dynamic pricing depends

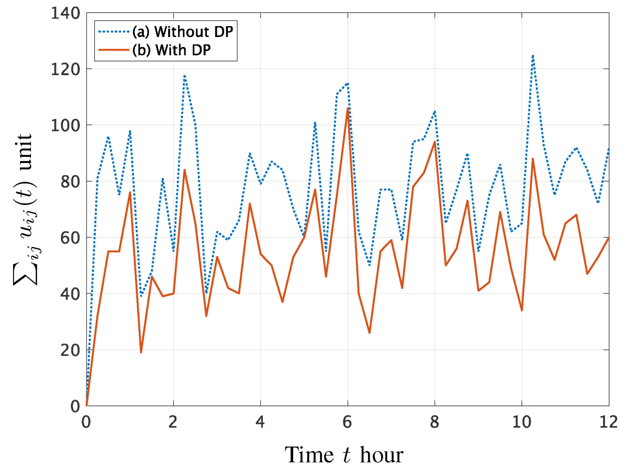

FIGURE 10. Numbers of unsatisfied demands $\sum_{ij} u_{ij}(t)$ (a) without DP (dashed line), (b) with DP (solid line), and (c) with DP and relocation (broken line) for $n = 50$ stations.

on the network topology of stations. Accordingly, we proposed an optimization method for the station relocation with PSO. Finally, the effectiveness of dynamic pricing and station relocation was demonstrated through comparisons in simulations. The future work includes the experiment in actual car-sharing service such as Ha:mo Ride Toyota to confirm the practicality of the developed methods.

In a realistic situation, the number of customers possibly increases or decreases to use other transportation in addition to the shift of links. Furthermore, we should consider travel time to apply the proposed method for the situation where the sampling time is not long enough. The effect of dynamic pricing can be involved in more realistic situations, where a finite number of positions are regarded as candidates and other objectives are additionally considered. The future work includes solving a more practical problem including these issues.

APPENDIX A
NOTATION

The notation used in this paper is given. Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, and $\mathbb{R}_+$ denote the sets of integers, non-negative integers, real numbers, and non-negative real numbers, respectively. The identity matrix is denoted as $I \in \mathbb{R}^{n\times n}$, the unit vector with the ith entry 1 is denoted as $e_i \in \mathbb{R}^n$, and $\mathbf{1} = [1 \ \cdots \ 1]^\top$. For real numbers $a_1, \ldots, a_n \in \mathbb{R}$, or a vector $a = [a_1 \ \cdots \ a_n]^\top \in \mathbb{R}^n$, $\mathrm{diag}(a_1, \ldots, a_n)$, or $\mathrm{diag}(a)$, represents the diagonal matrix of which ith diagonal entry is a_i. The Euclidean norm of the vector space $\mathbb{R}^n$ is denoted by $\|\cdot\|$. For a real number $\kappa \in \mathbb{R}$, $\kappa\mathbb{Z} = \{\ldots, -2\kappa, -\kappa, 0, \kappa, 2\kappa, \ldots\}$ represents the set of multiples of κ. The floor and ceiling functions of $x \in \mathbb{R}$ with respect to κ are defined as

$$\lfloor x \rfloor_\kappa = \max\{y \in \kappa\mathbb{Z} : y \leq x\},$$
$$\lceil x \rceil_\kappa = \min\{y \in \kappa\mathbb{Z} : y \geq x\}.$$

For a stochastic variable $x : \mathbb{Z}_+ \to \mathbb{Z}_+$ of time $t \in \mathbb{Z}_+$ and certain value $x \in \mathbb{Z}_+$, the probability that $x(t) = x$ is expressed as $\Pr(x(t) = x) \in [0, 1]$. The expectation of a function $f : \mathbb{Z}_+ \to \mathbb{R}$ of $x(t)$ is defined as

$$\mathrm{E}\big[f(x(t))\big] = \sum_{x=0}^{\infty} f(x)\Pr(x(t) = x).$$

For random variables $x, y : \mathbb{Z}_+ \to \mathbb{Z}_+$ of time t and values $x, y \in \mathbb{Z}_+$, the joint probability that $x(t) = x, y(t) = y$ is expressed as $\Pr(x(t) = x, y(t) = y) \in [0, 1]$. The conditional probability that $x(t) = x$ under the condition that $y(t) = y$ is defined as

$$\Pr(x(t) = x | y(t) = y) = \frac{\Pr(x(t) = x, y(t) = y)}{\Pr(y(t) = y)}.$$

The conditional expectation of a function $f : \mathbb{Z}_+ \to \mathbb{R}$ of $x(t)$ under the condition that $y(t) = y$ is defined as

$$\mathrm{E}\big[f(x(t))|y(t) = y\big] = \sum_{x=0}^{\infty} f(x)\Pr(x(t) = x|y(t) = y).$$

APPENDIX B
PROOF OF THEOREM 1

Theorem 1 is proved after some preliminaries. First, we show that the system in question is of the same form as a consensus control system with external disturbance, which has been studied in the control engineering field [19].

Lemma 1: Consider the system given in Theorem 1. The expectation of the discrepancy, $\bar{x}_i(t) = x_i(t) - x_i^{\max}/2$, varies as follows:

$$\begin{aligned}&\mathrm{E}[\bar{x}(t+1)]\\ &= \Big(I - (\hat{\pi}_{\mathrm{a}} - \hat{\pi}_{\mathrm{b}})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}L\Big)\mathrm{E}[\bar{x}(t)] + \Big(\Delta - \Delta^\top\Big)\mathbf{1},\end{aligned} \tag{33}$$

where $\bar{x}(t) = [\bar{x}_1(t)\ \cdots\ \bar{x}_n(t)]^\top \in \mathbb{R}^n$.

Proof: From (4), we obtain

$$\bar{x}(t+1) = \bar{x}(t) + \Big(R(t) - R^\top(t)\Big)\mathbf{1}, \tag{34}$$

where $R(t) \in \mathbb{R}^{n\times n}$ is the matrix with (i, j)-entry $r_{ij}(t)$. Taking the conditional expectation of (34), we obtain

$$\begin{aligned}&\mathrm{E}[\bar{x}(t+1)|\bar{x}(t) = \bar{x}]\\ &= \mathrm{E}[\bar{x}(t)|\bar{x}(t) = \bar{x}] + \mathrm{E}\Big[\Big(R(t) - R^\top(t)\Big)\mathbf{1}|\bar{x}(t) = \bar{x}\Big]\\ &= \Big(I - \hat{\phi}\big(\hat{\pi}_{\mathrm{a}} - \hat{\pi}_{\mathrm{b}}\big)\Big(\mathbf{1}^\top\Gamma\mathbf{1}\Big)L\Big)\bar{x} + \Big(\Delta - \Delta^\top\Big)\mathbf{1}\end{aligned} \tag{35}$$

by using the equation

$$\begin{aligned}&\mathrm{E}\Big[\Big(D(t) - D^\top(t)\Big)\mathbf{1}|\bar{x}(t) = \bar{x}\Big]\\ &= \Big(\Delta - \Delta^\top\Big)\mathbf{1} - \hat{\phi}\big(\hat{\pi}_{\mathrm{a}} - \hat{\pi}_{\mathrm{b}}\big)\Big(\mathbf{1}^\top\Gamma\mathbf{1}\Big)L\bar{x},\end{aligned} \tag{36}$$

which is shown later. Taking the expectations of the both sides of (35), we obtain

$$\begin{aligned}&\sum_{\bar{x}} \mathrm{E}[\bar{x}(t+1)|\bar{x}(t) = \bar{x}]\Pr(\bar{x}(t) = \bar{x})\\ &= \sum_{\bar{x}}\Big(\Big(I - (\hat{\pi}_{\mathrm{a}} - \hat{\pi}_{\mathrm{b}})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}L\Big)\bar{x} + \Big(\Delta - \Delta^\top\Big)\mathbf{1}\Big)\\ &\quad\times \Pr(\bar{x}(t) = \bar{x}),\end{aligned}$$

which results in (33).

The remainder of the proof is to derive (36). From (10), (18), and $\gamma_{ik} = \gamma_{ki}$,

$$\begin{aligned}&\mathrm{E}\big[s_{ij,k\ell}(t)|P(t) = P\big] - \mathrm{E}\big[s_{k\ell,ij}(t)|P(t) = P\big]\\ &= \gamma_{ik}\gamma_{j\ell}\phi\big(p_{ij} - p_{k\ell}\big) - \gamma_{ki}\gamma_{\ell j}\phi\big(p_{k\ell} - p_{ij}\big)\\ &= \gamma_{ik}\gamma_{j\ell}\big(\phi\big(p_{ij} - p_{k\ell}\big) - \phi\big(-\big(p_{ij} - p_{k\ell}\big)\big)\big)\\ &= -\gamma_{ik}\gamma_{j\ell}\hat{\phi}\big(p_{ij} - p_{k\ell}\big)\end{aligned}$$

is obtained. From the above equation, (7), and (8),

$$\begin{aligned}&\mathrm{E}\big[r_{ij}(t)|P(t) = P\big]\\ &= \mathrm{E}\Big[\hat{d}_{ij}(t)\Big] + \sum_{k\ell\in\mathcal{N}^2}\big(\mathrm{E}\big[s_{ij,k\ell}(t)|P(t) = P\big]\\ &\qquad - \mathrm{E}\big[s_{k\ell,ij}(t)|P(t) = P\big]\big)\\ &= \delta_{ij} - \sum_{k\ell\in\mathcal{N}^2}\gamma_{ik}\gamma_{j\ell}\hat{\phi}\big(p_{ij} - p_{k\ell}\big)\end{aligned} \tag{37}$$

is obtained because $r_{ij}(t) = d_{ij}(t)$ holds from (5). By arranging (37) into matrix form, we obtain

$$\mathrm{E}[R(t)|P(t) = P] = \Delta - \hat{\phi}(\mathrm{diag}(\Gamma\mathbf{1})P\mathrm{diag}(\Gamma\mathbf{1}) - \Gamma P\Gamma),$$

which yields

$$\begin{aligned}&\mathrm{E}[R(t)\mathbf{1}|P(t) = P]\\ &= \Delta\mathbf{1} - \hat{\phi}(\mathrm{diag}(\Gamma\mathbf{1})P\mathrm{diag}(\Gamma\mathbf{1}) - \Gamma P\Gamma)\mathbf{1}\\ &= \Delta\mathbf{1} - \hat{\phi}(\mathrm{diag}(\Gamma\mathbf{1})P\Gamma\mathbf{1} - \Gamma P\Gamma\mathbf{1})\\ &= \Delta\mathbf{1} - \hat{\phi}(\mathrm{diag}(\Gamma\mathbf{1}) - \Gamma)P\Gamma\mathbf{1}\\ &= \Delta\mathbf{1} - \hat{\phi}LP\Gamma\mathbf{1}.\end{aligned}$$

Similarly,

$$\mathrm{E}\Big[R^\top(t)\mathbf{1}|P(t) = P\Big] = \Delta^\top\mathbf{1} - \hat{\phi}LP^\top\Gamma\mathbf{1}$$

is derived, and thus

$$\begin{aligned}&\mathrm{E}\Big[\Big(R(t) - R^\top(t)\Big)\mathbf{1}|P(t) = P\Big]\\ &= \Big(\Delta - \Delta^\top\Big)\mathbf{1} - \hat{\phi}L\Big(P - P^\top\Big)\Gamma\mathbf{1}\end{aligned} \tag{38}$$

is obtained. From (12), (14), and the assumption $\hat{p}_{ij} = \hat{p}_{ji}$,

$$\begin{aligned}p_{ij}(t) - p_{ji}(t) &= \pi\left(\bar{x}_i(t), \bar{x}_j(t)\right) - \pi\left(\bar{x}_j(t), \bar{x}_i(t)\right)\\ &= \big(\hat{\pi}_{\mathrm{a}} - \hat{\pi}_{\mathrm{b}}\big)\big(\bar{x}_i(t) - \bar{x}_j(t)\big)\end{aligned}$$

holds, which can be represented as

$$P(t) - P^\top(t) = \big(\hat{\pi}_{\mathrm{a}} - \hat{\pi}_{\mathrm{b}}\big)\Big(\bar{x}(t)\mathbf{1}^\top - \mathbf{1}\bar{x}^\top(t)\Big). \tag{39}$$

From (38) and (39),

$$\begin{aligned}&\mathrm{E}\Big[\Big(R(t) - R^\top(t)\Big)\mathbf{1}|\bar{x}(t) = \bar{x}\Big]\\ &= \Big(\Delta - \Delta^\top\Big)\mathbf{1} - \hat{\phi}L\big(\hat{\pi}_{\mathrm{a}} - \hat{\pi}_{\mathrm{b}}\big)\Big(\bar{x}\mathbf{1}^\top - \mathbf{1}\bar{x}^\top\Big)\Gamma\mathbf{1}\\ &= \Big(\Delta - \Delta^\top\Big)\mathbf{1} - \hat{\phi}\big(\hat{\pi}_{\mathrm{a}} - \hat{\pi}_{\mathrm{b}}\big)\Big(\mathbf{1}^\top\Gamma\mathbf{1}\Big)L\bar{x}\end{aligned}$$

is obtained, where we use the property of the graph Laplacian $L\mathbf{1} = 0$. Hence, we have derived (36). ∎

Second, the following lemma is derived from the stability analysis of the result of consensus control systems.

Lemma 2: Consider the system given in Theorem 1. The limit $\lim_{t\to\infty} \mathrm{E}[\bar{x}(t)]$ exists if and only if

$$(\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\lambda_n \sum_{i,j\in\mathcal{N}} \gamma_{ij} < 2 \tag{40}$$

holds. Under this condition,

$$\lim_{t\to\infty} \mathrm{E}[\bar{x}(t)] = \frac{\sum_{i=\xi+1}^{n} \lambda_i^{-1} v_i v_i^\top (\Delta - \Delta^\top)\mathbf{1}}{(\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\sum_{i,j\in\mathcal{N}}\gamma_{ij}} + \frac{\bar{m}\mathbf{1}}{n} + \sum_{i=2}^{\xi} v_i v_i^\top \bar{x}(0) \tag{41}$$

holds, where $\bar{m} = m - m^{\max}/2$.

Proof: Assume that there exists a vector $\hat{x} \in \mathbb{R}^n$ satisfying $\lim_{t\to\infty} \mathrm{E}[\bar{x}(t)] = \hat{x}$. Then, from (33),

$$L\hat{x} = \frac{(\Delta - \Delta^\top)\mathbf{1}}{(\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}} \tag{42}$$

is obtained. Moreover, from (33),

$$\mathrm{E}\Big[v_i^\top \bar{x}(t+1)\Big] = \mathrm{E}\Big[v_i^\top \bar{x}(t)\Big] + v_i^\top\Big(\Delta - \Delta^\top\Big)\mathbf{1}$$

is achieved, which leads to

$$v_i^\top \hat{x} = v_i^\top \bar{x}(0) = v_i^\top \frac{\bar{m}\mathbf{1}}{n} = \begin{cases} \frac{\bar{m}}{\sqrt{n}} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

from (19) and (20) because of the existence of $\hat{x}$.

From (42) and (43),

$$\left(L + \sum_{i=1}^{\xi} v_i v_i^\top\right)\hat{x} = \frac{(\Delta - \Delta^\top)\mathbf{1}}{(\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}} + \frac{\bar{m}\mathbf{1}}{n} \tag{44}$$

holds, and the inverse matrix on the left-hand side in (44) is expressed as follows:

$$\begin{aligned}\left(L + \sum_{i=1}^{\xi} v_i v_i^\top\right)^{-1} &= \Big(V\mathrm{diag}\big(1,\ldots,1,\lambda_{\xi+1},\ldots,\lambda_n\big)V^\top\Big)^{-1} \\ &= V\mathrm{diag}\Big(1,\ldots,1,\lambda_{\xi+1}^{-1},\ldots,\lambda_n^{-1}\Big)V^\top,\end{aligned} \tag{45}$$

where $V = [v_1\ v_2\ \cdots\ v_n]$ is the orthogonal matrix. From (44) and (45), $\hat{x}$ is derived as follows:

$$\begin{aligned}\hat{x} &= \left(L + \sum_{i=1}^{\xi} v_i v_i^\top\right)^{-1}\left(\frac{(\Delta - \Delta^\top)\mathbf{1}}{(\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}} + \frac{\bar{m}}{n}\mathbf{1}\right) \\ &= \Big(V\mathrm{diag}\Big(1,\ldots,1,\lambda_{\xi+1}^{-1},\ldots,\lambda_n^{-1}\Big)V^\top\Big) \\ &\quad\times\left(\frac{(\Delta - \Delta^\top)\mathbf{1}}{(\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}} + \frac{\bar{m}}{n}\mathbf{1}\right) \\ &= \frac{V\mathrm{diag}\Big(0,\ldots,0,\lambda_{\xi+1}^{-1},\ldots,\lambda_n^{-1}\Big)V^\top(\Delta - \Delta^\top)\mathbf{1}}{(\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}} + \frac{\bar{m}\mathbf{1}}{n}.\end{aligned} \tag{46}$$

From (2) and (3), (46) is reduced to (41).

We show that $\lim_{t\to\infty} \mathrm{E}[\bar{x}(t)]$ exists if and only if (40) holds. Let $\tilde{x}(t) = \bar{x}(t) - \hat{x}$, which is governed by the equation

$$\begin{aligned}\mathrm{E}\big[\tilde{x}(t+1)\big] &= -\hat{x} + \Big(I - (\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}L\Big)\big(\mathrm{E}\big[\tilde{x}(t)\big] + \hat{x}\big) \\ &\quad + \Big(\Delta - \Delta^\top\Big)\mathbf{1} \\ &= \Big(I - (\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}L\Big)\mathrm{E}\big[\tilde{x}(t)\big]\end{aligned} \tag{47}$$

from (33) and (42). In descending order, the eigenvalues of the matrix $I - (\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}L$ are given as 1 and

$$1 - (\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}\lambda_2, \ldots, 1 - (\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b})\hat{\phi}\mathbf{1}^\top\Gamma\mathbf{1}\lambda_n. \tag{48}$$

Hence, $\mathrm{E}[\tilde{x}(t)]$ in (47) converges if and only if the last term in (48) is more than -1 from the convergence condition for discrete-time systems. This condition corresponds to (40). ∎

Finally, we prove Theorem 1 from Lemmas 1 and 2.

Proof of Theorem 1: Assume that the solution to the optimization problem of (16) is obtained as $\pi : \mathbb{Z}_+^2 \to \kappa\mathbb{Z}$ in (12) of the form (14) with certain $\hat{\pi}_\mathrm{a}, \hat{\pi}_\mathrm{b}, \hat{\pi}_\mathrm{c} \in \kappa\mathbb{Z}$. Therefore, $\lim_{t\to\infty} \mathrm{E}[\bar{x}(t)]$ exists, and thus (40) holds from Lemma 2. This limit is derived as (41), which is reduced to

$$\lim_{t\to\infty} \mathrm{E}[\bar{x}(t)] = \frac{h}{\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b}} + \frac{\bar{m}\mathbf{1}}{n} \tag{49}$$

with h in (25). From (13), (14), and (49),

$$\begin{aligned}&\lim_{t\to\infty} V_\mathrm{p}(x(t)) \\ &= \frac{1}{n^2}\lim_{t\to\infty}\sum_{i,j\in\mathcal{N}}\big(\mathrm{E}\big[\hat{\pi}_\mathrm{a}\bar{x}_i(t) + \hat{\pi}_\mathrm{b}\bar{x}_j(t) + \hat{\pi}_\mathrm{c}\big]\big)^2 \\ &= \frac{1}{n^2}\sum_{i,j\in\mathcal{N}}\Big((\hat{\pi}_\mathrm{a}e_i + \hat{\pi}_\mathrm{b}e_j)^\top \lim_{t\to\infty}\mathrm{E}[\bar{x}(t)] + \hat{\pi}_\mathrm{c}\Big)^2 \\ &= \frac{1}{n^2}\sum_{i,j\in\mathcal{N}}\left(\frac{(\hat{\pi}_\mathrm{a}e_i + \hat{\pi}_\mathrm{b}e_j)^\top h}{\hat{\pi}_\mathrm{a} - \hat{\pi}_\mathrm{b}} + \frac{(\hat{\pi}_\mathrm{a} + \hat{\pi}_\mathrm{b})\bar{m}}{n} + \hat{\pi}_\mathrm{c}\right)^2\end{aligned} \tag{50}$$

is derived. Under the assumption, $\max_{m\in\mathbb{Z}_+}\lim_{t\to\infty} V_\mathrm{p}(x(t))$ is bounded; thus, $\hat{\pi}_\mathrm{a} + \hat{\pi}_\mathrm{b} = 0$ holds from (50) and $\bar{m} = m - m^{\max}/2$. Hence, (23) is derived and (50) is reduced to

$$\lim_{t\to\infty} V_\mathrm{p}(x(t)) = \frac{1}{n^2}\sum_{i,j\in\mathcal{N}}\left(\frac{(e_i - e_j)^\top h}{2} + \hat{\pi}_\mathrm{c}\right)^2. \tag{51}$$

From (4) and (6), $x_i^+(t) = x_i(t+1)$ holds because $r_{ij}(t) = d_{ij}(t)$ holds from the assumption. Then, from (11) and (49),

$$\begin{aligned}\lim_{t\to\infty} V_\mathrm{x}(x(t)) &= \lim_{t\to\infty}\frac{1}{n}\|\mathrm{E}\big[C\bar{x}^+(t)\big]\|^2 \\ &= \lim_{t\to\infty}\frac{1}{n}\|\mathrm{E}[C\bar{x}(t+1)]\|^2\end{aligned}$$

$$= \lim_{t\to\infty} \frac{1}{n} \mathrm{E}[\bar{x}(t+1)]^\top C \mathrm{E}[\bar{x}(t+1)]$$
$$= \frac{1}{n} \frac{\|h\|^2}{(2\hat{\pi}_\mathrm{a})^2} \tag{52}$$

is obtained, where $C = I - \mathbf{1}\mathbf{1}^\top/n$. Here, we use the equations $C^2 = C$, $Ch = h$, and $C\mathbf{1} = 0$. From (15), (51), and (52),

$$V(\hat{\pi}_\mathrm{a}, \hat{\pi}_\mathrm{b}, \hat{\pi}_\mathrm{c}) = \frac{\|h\|^2}{4n\hat{\pi}_\mathrm{a}^2} + \frac{\mu}{n^2} \sum_{i,j\in\mathcal{N}} \left(\frac{(e_i - e_j)^\top h}{2} + \hat{\pi}_\mathrm{c} \right)^2 + \nu\left(2\hat{\pi}_\mathrm{a}^2 + \hat{\pi}_\mathrm{c}^2\right) \tag{53}$$

is derived. Each term in (53) can be minimized individually for $\hat{\pi}_\mathrm{a}$ and $\hat{\pi}_\mathrm{c}$. By partially differentiating (53) with $\hat{\pi}_\mathrm{c}$, we obtain

$$\frac{\partial V(\hat{\pi}_\mathrm{a}, \hat{\pi}_\mathrm{b}, \hat{\pi}_\mathrm{c})}{\partial \hat{\pi}_\mathrm{c}} = \frac{2\mu}{n^2} \sum_{i,j\in\mathcal{N}} \left(\frac{(e_i - e_j)^\top h}{2} + \hat{\pi}_\mathrm{c} \right) + 2\nu\hat{\pi}_\mathrm{c}$$
$$= 2(\mu + \nu)\hat{\pi}_\mathrm{c},$$

which is zero only for $\hat{\pi}_\mathrm{c} = 0$. Hence, $V(\hat{\pi}_\mathrm{a}, \hat{\pi}_\mathrm{b}, \hat{\pi}_\mathrm{c})$ is minimized by $\hat{\pi}_\mathrm{c} = 0$, which yields (24). Now, (53) is reduced to (26).

For $\hat{\pi}_\mathrm{a}$, from (23) and (40), (22) must be satisfied. If $\hat{\pi}_\mathrm{a}$ could take a real value, $\hat{\pi}_\mathrm{a} = \hat{\pi}_{\mathrm{a}*}$ would be the solution to (53). From the convexity of (53) with respect to $\hat{\pi}_\mathrm{a}$, the optimizer $\hat{\pi}_\mathrm{a} \in \kappa\mathbb{Z}$ is derived by rounding $\pi_{\mathrm{a}*}$ into $\kappa\mathbb{Z}$. Then, the first two terms in (21) are obtained. If (22) is not satisfied, the solution is given by the boundary of the inequality. Then, the last term in (21) is derived. ■

APPENDIX C
PROOF OF THEOREM 2

From the assumption and (24), either $\hat{\pi}_\mathrm{a} = \lceil \pi_{\mathrm{a}*} \rceil_\kappa$ or $\hat{\pi}_\mathrm{a} = \lfloor \pi_{\mathrm{a}*} \rfloor_\kappa$ holds. Hence, $|\hat{\pi}_\mathrm{a} - \pi_{\mathrm{a}*}| \le \kappa$ holds, which leads to $\hat{\pi}_\mathrm{a} = \pi_{\mathrm{a}*} + O(\kappa)$. From (25), (26) is reduced to

$$\frac{1}{n}\frac{\|h\|^2}{4\hat{\pi}_\mathrm{a}^2} + 2\nu\hat{\pi}_\mathrm{a}^2 = \frac{1}{n}\frac{\|h\|^2}{4(\pi_{\mathrm{a}*} + O(\kappa))^2} + 2\nu(\pi_{\mathrm{a}*} + O(\kappa))^2$$
$$= \frac{1}{n}\frac{\|h\|^2}{4\pi_{\mathrm{a}*}^2} + 2\nu\pi_{\mathrm{a}*}^2 + O(\kappa) = \sqrt{\frac{2\nu}{n}}\|h\| + O(\kappa)$$
$$= \sqrt{\frac{2\nu}{n}} \frac{\left\| \lambda_{\xi+1}^{-1} \sum_{i=\xi+1}^{n} \frac{\lambda_{\xi+1}}{\lambda_i} v_i v_i^\top (\Delta - \Delta^\top)\mathbf{1} \right\|}{\hat{\phi}\sum_{i,j\in\mathcal{N}} \gamma_{ij}} + O(\kappa)$$
$$= \sqrt{\frac{2\nu}{n}} \frac{\left\| \lambda_{\xi+1}^{-1} v_{\xi+1} v_{\xi+1}^\top (\Delta - \Delta^\top)\mathbf{1} + O(\lambda_{\xi+1}/\lambda_{\xi+2}) \right\|}{\hat{\phi}\sum_{i,j\in\mathcal{N}} \gamma_{ij}} + O(\kappa)$$
$$= \sqrt{\frac{2\nu}{n^3}} \frac{\left\| \lambda_{\xi+1}^{-1} \mathbf{1}\mathbf{1}^\top (\Delta - \Delta^\top)\mathbf{1} \right\|}{\hat{\phi}\sum_{i,j\in\mathcal{N}} \gamma_{ij}} + O(\kappa, \lambda_{\xi+1}/\lambda_{\xi+2})$$
$$= \frac{\sqrt{2\nu}\left| \sum_{i,j\in\mathcal{N}} (\delta_{ij} - \delta_{ji}) \right|}{n\hat{\phi}\lambda_{\xi+1}\sum_{i,j\in\mathcal{N}} \gamma_{ij}} + O(\kappa, \lambda_{\xi+1}/\lambda_{\xi+2}),$$

which yields (27). ■

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REFERENCES

[1] S. Mizokami, T. Maruyama, J. Hashimoto, T. Mori, and D. Sunaga, "Research and development on the possibility of introducing a one-way car sharing system and new ways to utilize road space (in Japanese)," in *Proc. New Road Technol. Conf.*, 2019, pp. 1–48.

[2] "Toyota to start verification tests in Japan using 'Ha:mo Ride' driving data." Toyota. Feb. 25, 2022. [Online]. Available: https://global.toyota/en/newsroom/corporate/28866957.html

[3] S. Illgen and M. Hock, "Literature review of the vehicle relocation problem in one-way car sharing networks," *Transp. Res. B, Methodol.*, vol. 120, pp. 193–204, Feb. 2019.

[4] G. C. Calafiore, C. Bongiorno, and A. Rizzo, "A control-oriented model for mobility on demand systems," in *Proc. IEEE Conf. Decis. Control (CDC)*, 2018, pp. 999–1004.

[5] S. L. Smith, M. Pavone, M. Schwager, E. Frazzoli, and D. Rus, "Rebalancing the rebalancers: Optimally routing vehicles and drivers in mobility-on-demand systems," in *Proc. Amer. Control Conf. (ACC)*, 2013, pp. 2362–2367.

[6] M. Barth and M. Todd, "Simulation model performance analysis of a multiple station shared vehicle system," *Transp. Res. C, Emerg. Technol.*, vol. 7, no. 4, pp. 237–258, 1999.

[7] D. Jorge, G. Molnar, and G. H. A. Correia, "Trip pricing of one-way station based carsharing networks with zone and time of day price variations" *Transp. Res. B, Methodol.*, vol. 8, no. 2, pp. 461–482, 2015.

[8] M. Xu, Q. Meng, and Z. Liu, "Electric vehicle fleet size and trip pricing for one-way carsharing services considering vehicle relocation and personnel assignment," *Transp. Res. B, Methodol.*, vol. 111, pp. 60–82, May 2018.

[9] R. Xie, W. Wei, Q. Wu, T. Ding, and S. Mei, "Optimal service pricing and charging scheduling of an electric vehicle sharing system," *IEEE Trans. Veh. Technol.*, vol. 69, no. 1, pp. 78–89, Jan. 2020.

[10] L. Wang and W. Ma, "Pricing approach to balance demands for one-way car-sharing systems," in *Proc. 22nd IEEE Intell. Transp. Syst. Conf. (ITSC)*, 2019, pp. 1697–1702.

[11] T. Kamatani, Y. Nakata, and S. Arai, "Dynamic pricing method to maximize utilization of one-way car sharing service," in *Proc. IEEE Int. Conf. Agents (ICA)*, 2019, pp. 65–68.

[12] Z. Haidera, A. Nikolaevb, J. E. Kangb, and C. Kwona, "Inventory rebalancing through pricing in public bike sharing systems," *Eur. J. Oper. Res.*, Vol. 270, pp. 103–117, Oct. 2018.

[13] K. Sumida, K. Sakurama, and T. Aoki, "Demand shift model of a one-way car-sharing system with real-time pricing," in *Proc. 24th IEEE Int. Intell. Transp. Syst. Conf. (ITSC)*, 2021, pp. 3271–3277.

[14] G. H. de A. Correia and A. P. Antunes, "Optimization approach to depot location and trip selection in one-way carsharing systems" *Transp. Res. E, Logist. Transp. Rev.*, vol. 48, no. 1, pp. 233–247, 2012.

[15] Y. Deng and M.-A. Cardin, "Integrating operational decisions into the planning of one-way vehicle-sharing systems under uncertainty," *Transp. Res. C, Emerg. Technol.*, vol. 86, pp. 407–424, Jan. 2018.

[16] K. Zhang, Y. Takano, Y. Wang, and A. Yoshise, "Optimizing the strategic decisions for one-way station-based carsharing systems: A mean-CVaR approach" *IEEE Access*, vol. 9, pp. 79816–79828, 2021.

[17] J. Wang and G. Hua, "Optimization approach to station location of car sharing system," in *Proc. 15th Int. Conf. Service Syst. Service Manage. (ICSSSM)*, 2018, pp. 1–5.

[18] K. Huanga, G. H. de Almeida Correiab, and K. An, "Solving the station-based one-way carsharing network planning problem with relocations and non-linear demand," *Transp. Res. C, Emerg. Technol.*, vol. 90, pp. 1–17, May 2018.

[19] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[20] Q. Han, Y.-S. Zhou, Y.-X. Tang, X.-G. Tuo, and P. He, "Event-triggered finite-time sliding mode control for leader-following second-order nonlinear multi-agent systems," *IEEE Open J. Intell. Transp. Syst.*, vol. 3, pp. 570–579, 2022.

[21] K. Sakurama and T. Aoki, "Distributed dynamic pricing for car-sharing systems with stochastic demand shift," in *Proc. 25th IEEE Int. Intell. Transp. Syst. Conf. (ITSC)*, 2022, pp. 1970–1975.
[22] (i-Transport Lab., Tokyo, Japan). *SOUND Ver.5, A Traffic Simulator for Wide Area Road Networks*. (Feb. 25, 2022). [Online]. Available: https://www.i-transportlab.jp/en/index/products/sound/
[23] T. Ikeda, K. Sakurama, and K. Kashima, "Multiple sparsity constrained control node scheduling with application to rebalancing of mobility networks," *IEEE Trans. Autom. Control*, vol. 67, no. 8, pp. 4314–4321, Aug. 2022.
[24] J. Kennedy and R. Eberhart, "Particle swarm optimization," in *Proc. IEEE Int. Conf. Neural Netw.*, 1995, pp. 1942–1948.
[25] Y. Shi and R. Eberhart, "A modified particle swarm optimizer," in *Proc. IEEE Int. Conf. Evol. Comput.*, 1998, pp. 69–73.

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