A REGULARITY STRUCTURE FOR THE QUASILINEAR GENERALIZED KPZ EQUATION

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Abstract. We prove the local well-posedness of a regularity structure formulation of the quasilinear generalized KPZ equation and give an explicit form for a renormalized equation in the full subcritical regime. This is an abstract of author's work [4].

1. Introduction

We consider the one dimensional quasilinear generalized KPZ equation

$$\partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{1.1}$$

with an initial condition $u_0 \in C^\alpha(\mathbb{T})$ for $\alpha \in (0,1)$, where $\mathbb{R}_+ := (0,\infty)$, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, ξ is the spacetime white noise, and a, f, and g are regular enough functions on $\mathbb{R}$. We assume that a takes values in a compact interval of $\mathbb{R}_+$. This equation is an example of *singular stochastic partial differential equations* (SPDEs) of parabolic type. Recall that the spacetime white noise ξ has a (parabolic) regularity $\alpha_0 - 2$ almost surely, for $0 < \alpha_0 < 1/2$. It is then natural to expect a solution u to the equation (1.1) to have a regularity α_0. However, the nonlinear terms $f(u)\xi$ and $g(u)(\partial_x u)^2$ do not make sense unless $\alpha_0 > 1$.

Hairer [14] introduced a groundbreaking theory called *regularity structures* and opened the door to the study of semilinear singular SPDEs. For quasilinear equations, Otto and Weber [16] introduced a variant of regularity structures to study the equation

$$\partial_t u - a(u)\partial_x^2 u = f(u)\xi \tag{1.2}$$

in the regime $\alpha_0 > 2/3$. Otto, Sauer, Smith, and Weber [15] deepened their framework to study the equation with an additive noise

$$\partial_t u - a(u)\partial_x^2 u = \xi \tag{1.3}$$

in the full-subcritical regime $\alpha_0 \in (0,1)$ and obtained an explicit form of a renormalized equation. Meanwhile, Gerencsér and Hairer [12] provided an infinite dimensional regularity structure for the study of the equation (1.1) and obtained a renormalized equation in the regime $\alpha_0 > 1/2$. By implementing some integration by parts-type formulae, Gerencsér [11] obtained a renormalized equation for the equation (1.3) with the spacetime white noise ξ when the mollification of noise is symmetric with respect to x. In the present work, we introduce another variant of regularity structure formulation of the equation (1.1) and give an explicit form for a renormalized equation in the full subcritical regime. Convergences of stochastic objects are left for future, but we expect that a simple modification of Chandra and Hairer's general proof [9] works well.

We mention another approach to singular SPDEs called *paracontrolled calculus* introduced by Gubinelli, Imkeller, and Perkowski [13]. Furlan and Gubinelli [10] and Bailleul, Debussche, and Hofmanová [2] investigated the equation (1.2) on the two dimensional torus with the space white noise ξ, which has a regularity $\alpha_0 - 2$ for $2/3 < \alpha_0 < 1$. These two works are variants of paracontrolled calculus based on different methods: the paracomposition operator in [10] and the initial form of paracontrolled calculus in [2]. In the present work, we reformulate the latter approach in the framework of regularity structures. Bailleul and Mouzard [5] extended the high order paracontrolled calculus based on [2] to deal with the equation (1.1) in the regime $\alpha_0 > 2/5$.

This paper is organized as follows. In Section 2, we describe the main results of [4] without stating some precise definitions. In Section 3, we briefly review the locall well-posedness result of the regularity structure formulation for (1.1). In Section 4, we outline the sketch of the proof of main results.

Notations. We represent by $z = (t, x) \in \mathbb{R}^2$ a generic spacetime variable, for which we set

$$\|z\|_{\mathfrak{s}} := |t|^{1/2} + |x|.$$

We also set for any multiindex $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$,

$$z^{\mathbf{k}} := t^{k_1} x^{k_2}, \qquad |\mathbf{k}|_{\mathfrak{s}} := 2k_1 + k_2.$$

2. Main results

2.1. **Regularity structure formulation of** (1.1)**.** Following [2, 5], we set $L^{a(v)} := a(v)\partial_x^2$ for an appropriate spacetime function v and rewrite the equation (1.1) under the form

$$(\partial_t - L^{a(v)} + c)u = f(u)\xi + g(u)(\partial_x u)^2 + cu + \big(a(u) - a(v)\big)\partial_x^2 u \tag{2.1}$$

for a large positive constant c.

Remark 2.1. *The choice of v depends on the initial condition u_0. Typically, we choose a spacetime function $v(t, x) = e^{t\partial_x^2} u_0$ or a t-independent function $v(x) = e^{\delta\partial_x^2} u_0$ with sufficiently small $\delta > 0$. See* [4, Section 2.1] *for other possible choices.*

We consider the equation (2.1) as a 'perturbation' of the semilinear equation. We reformulate (2.1) as a system of equations for *modelled distributions* (see Definition 3.2) as follows.

$$\begin{cases} \boldsymbol{u} = \mathsf{P}_{<2}(Q^{a(v)} u_0) + \mathsf{K}_\gamma^{a(v),\mathsf{M}}(\boldsymbol{v} + \boldsymbol{w}), \\ \boldsymbol{v} = \mathsf{Q}_{\le 0}\big[f(\boldsymbol{u})\Xi_1 + \big\{g(\boldsymbol{u})(\boldsymbol{D}\boldsymbol{u})^2 + c\boldsymbol{u}\big\}\Xi_2\big], \\ \boldsymbol{w} = \mathsf{Q}_{\le 0}\big[\big\{a(\boldsymbol{u}) - a(\mathsf{P}_{<2} v)\big\}\big\{\boldsymbol{D}^2 \mathsf{P}_{\le 2}(Q^{a(v)} u_0) + \boldsymbol{D}^2 \mathsf{K}_{\gamma+\alpha_0}^{a(v),\mathsf{M}}(\boldsymbol{v} + \boldsymbol{w})\big\}\Xi_3\big], \end{cases} \tag{2.2}$$

where $Q^{a(v)}$ is the Green function of the parabolic operator $\partial_t - L^{a(v)} + c$. See Section 3 for the definition of all notations. One of the key parts of the work [4] is the well-posedness for the equations (2.2) (see Theorem 3.4) up to a positive time $t_0 = t_0(u_0, \mathsf{M}) > 0$ depending on the initial value u_0 and the *model* M (see Definition 3.1), which consists of all stochastic objects to be renormalized. This analytical statement holds in the full subcritical regime $\alpha_0 \in (0, 1)$. Note that, our regularity structure consists of the *infinite* dimensional model space with Banach norms, in contrast to that only *finite* dimensional model spaces were used in the previous researches [14, 8, 9, 7, 1] of semilinear equations. Additionally, our model space is different from the infinite dimensional spaces considered in [12].

2.2. **Main results.** We consider a family of smooth spacetime functions ξ^ε indexed by $\varepsilon \in (0, 1]$ which approximates the white noise ξ as $\varepsilon \to 0$. We can define the *naive interpretation model* M^ε associated with ξ^ε, but we cannot expect the convergence of M^ε as $\varepsilon \to 0$ in general. By following the general procedure by Bruned, Hairer, and Zambotti [8], we can find some spacetime functions $\ell^\varepsilon[\tau^{\boldsymbol{p}}](z)$ called a *renormalization character* indexed by basis elements of the model space and define the associated *BPHZ renormalized model* $\hat{\mathsf{M}}^\varepsilon$. See Section 4.1 for details.

Assumption 1. *There exists a renormalization character ℓ^ε such that the BPHZ renormalized model $\hat{\mathsf{M}}^\varepsilon$ converges to some model $\hat{\mathsf{M}}$ as $\varepsilon \to 0$.*

While the convergence of $\hat{\mathsf{M}}^\varepsilon$ is stated as an assumption in [4], we expect to be able to prove it by a modification of Chandra and Hairer's proof for semilinear cases [9]. Then, by following [1, 7], we can state the first main result of [4]. Below, $\Upsilon[\tau^{\boldsymbol{p}}]$ is a smooth function on $\mathbb{R}^3$ indexed by basis elements of the model space, which has a role of coefficients of Butcher series. Moreover, $S[\tau^{\boldsymbol{p}}]$ is a positive integer determined by the graph structure of $\tau^{\boldsymbol{p}}$. See Section 4.2 for details.

Theorem 2.2. *Let $u_0 \in C^\alpha(\mathbb{T})$ with $\alpha > 0$ and choose any appropriate function v on $\mathbb{R}_+ \times \mathbb{T}$ as in Remark 2.1. Under Assumption 1, the solution u^ε to the renormalized equation*

$$\partial_t u^\varepsilon - a(u^\varepsilon)\partial_x^2 u^\varepsilon = f(u^\varepsilon)\xi^\varepsilon + g(u^\varepsilon)(\partial_x u^\varepsilon)^2 + \sum_{\tau^{\boldsymbol{p}}} \frac{\ell^\varepsilon[\tau^{\boldsymbol{p}}]}{S[\tau^{\boldsymbol{p}}]} \Upsilon[\tau^{\boldsymbol{p}}](u^\varepsilon, \partial_x u^\varepsilon, v) \tag{2.3}$$

starting from u_0 converges in $C([0,t_0)\times\mathbb{T})$ for a random time $t_0 = t_0(u_0, \hat{\mathsf{M}})$ in probability as $\varepsilon \to 0$. In the last term, $\tau^{\boldsymbol{p}}$ in the sum runs over infinitely many symbols and $\Upsilon[\tau^{\boldsymbol{p}}]$ is at most linear with respect to $\partial_x u^\varepsilon$.

It should be noted that the renormalization character $\ell^\varepsilon[\tau^{\boldsymbol{p}}]$ depends on the choice of v. In general, its dependence is nonlocal in the sense that $\ell^\varepsilon[\tau^{\boldsymbol{p}}](z)$ is not of the form $f\big(v(z)\big)$ with some function f on $\mathbb{R}$. Nevertheless, we assume that $\ell^\varepsilon[\tau^{\boldsymbol{p}}]$ can be traded off with a local function of $a(v)$ up to an ε-uniform remainder and we get the second main result of [4]. See Section 4.3 for the definition of the analytic function $\lambda \mapsto l_\lambda^\varepsilon[\tau^{\boldsymbol{p}}]$.

Assumption 2. *There exist ε-independent constants $C(\tau)$ and $m > 0$ such that*

$$\big|\ell^\varepsilon[\tau^{\boldsymbol{p}}](z) - l^\varepsilon_{a(v(z))}[\tau^{\boldsymbol{p}}]\big| \le C(\tau)\, m^{|\boldsymbol{p}|}$$

holds for any $\boldsymbol{p} \in \mathbb{N}^{E_\tau}$ and $z \in \mathbb{R}_+ \times \mathbb{T}$.

Theorem 2.3. *Under Assumptions 1 and 2, there exist smooth functions $\Upsilon_0[\tau]$ on $\mathbb{R}^2$ indexed by only finitely many symbols τ such that the last term of (2.3) is of the form*

$$\sum_\tau \frac{l^\varepsilon_{a(u^\varepsilon)}[\tau]}{S[\tau]} \Upsilon_0[\tau](u^\varepsilon, \partial_x u^\varepsilon) + O(1), \tag{2.4}$$

for an ε-uniform $O(1)$ term.

Assumption 2 is too strong to believe that it holds in the full subcritical regime, but we can prove it for some particular cases studied by [2, 12, 11].

3. Local well-posedness of the system (2.2)

3.1. **Construction of the regularity structure.** Our model space is generated by the family of symbols $\mathbb{B} = \bigcup_{i=1}^3 \mathbb{B}^i_\bullet \cup \bigcup_{i=1}^3 \mathbb{B}^i_\circ$ recursively defined as follows.

- For each $i \in \{1,2,3\}$, the primitive symbol Ξ_i is contained in $\mathbb{B}^i_\circ$.
- If $\tau_1, \tau_2, \dots, \tau_n \in \mathbb{B}_\circ := \bigcup_{i=1}^3 \mathbb{B}^i_\circ$, then

$$X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_0(\tau_i) \in \mathbb{B}^1_\bullet, \qquad X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_0(\tau_i),\ X^{\mathbf{k}}\mathcal{I}_1(\tau_1)\prod_{i=2}^n \mathcal{I}_0(\tau_i),\ X^{\mathbf{k}}\mathcal{I}_1(\tau_1)\mathcal{I}_1(\tau_2)\prod_{i=3}^n \mathcal{I}_0(\tau_i) \in \mathbb{B}^2_\bullet,$$

$$X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_0(\tau_i),\ X^{\mathbf{k}}\mathcal{I}_2(\tau_1)\prod_{i=2}^n \mathcal{I}_0(\tau_i) \in \mathbb{B}^3_\bullet$$

  for any $\mathbf{k} \in \mathbb{N}^2$, where the multiplications are commutative and $\prod_{i\in\emptyset} := 1$ by the convention.
- For each $i \in \{1,2,3\}$, if $\tau \in \mathbb{B}^i_\bullet$ then $\tau\Xi_i \in \mathbb{B}^i_\circ$.

The noise symbol Ξ_1 represents the noise. The other noise symbols Ξ_2 and Ξ_3 represent the constant function 1, but they are not useful until the definition of $\Upsilon[\tau^{\boldsymbol{p}}]$ (Section 4.2). The symbol $X^{\mathbf{k}}$ represents the basis of Taylor series. The operator $\mathcal{I}_0$ represents the inverse operator $(\partial_t - L^{a(v)} + c)^{-1}$, and $\mathcal{I}_1$ and $\mathcal{I}_2$ represents its first and second derivatives with respect to x, respectively. We can see that each element of $\mathbb{B}_\circ$ above are used to represent the right-hand side of the equation (2.1). As usual, each symbol of $\mathbb{B}$ can be represented as a *rooted tree* with node decorations $\{\Xi, X^{\mathbf{k}}\}$ and edge decorations $\{\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2\}$. We define the homogeneity of each element $\tau \in \mathbb{B}$ by setting $|\Xi_1| := \alpha_0 - 2$ for any fixed $\alpha_0 \in (0,1)$, $|\Xi_2| = |\Xi_3| := 0$, and

$$\left| X^{\mathbf{k}} \Xi_i \prod_{j=1}^n \mathcal{I}_{n_j}(\tau_j) \right| := |\mathbf{k}|_{\mathfrak{s}} + |\Xi_i| + \sum_{j=1}^n \big(|\tau_j| + 2 - n_j \big),$$

for $\mathbf{k} \in \mathbb{N}^2$, $i \in \{1,2,3,4\}$, $\tau_j \in \mathbb{B}_\circ$, and $n_j \in \{0,1,2\}$, where we set $\Xi_4 := 1$. In the previous studies on semilinear equations, it is important that the subset $\{\tau\,;\,|\tau| < \gamma\}$ is finite for any $\gamma \in \mathbb{R}$. While this property does not hold in the present case because the operator $\mathcal{I}_2$ preserves the homogeneity, we have the following.

Proposition 3.1 ([4, Proposition 8])**.** *The set $A := \{|\tau^{\boldsymbol{p}}|\,;\,\tau^{\boldsymbol{p}} \in \mathbb{B}\}$ is locally finite and bounded from below.*

To classify infinitely many trees into finitely many classes, we contract consecutive operators $\Xi_3\mathcal{I}_2$ into one operator with an additional edge decoration. Precisely, we perform the contraction

$$\mathcal{I}_n\big((\Xi_3\mathcal{I}_2)^{\circ p}(\tau)\big) \quad \to \quad \mathcal{I}_n^p(\tau)$$

for any $\tau \in \mathbb{B}_\circ \setminus \Xi_3\mathcal{I}_2(\mathbb{B}_\circ)$ at each branch of the tree. After the contraction, we can represent each element of $\mathbb{B}$ by the unique minimum form

$$\tau^{\boldsymbol{p}},$$

where $\boldsymbol{p} : E_\tau \to \mathbb{N}$ is an edge decoration given for the edge set E_τ of τ. Therefore, there exists a finite set $\mathbb{B}^0$ and we can write $\mathbb{B} = \{\tau^{\boldsymbol{p}}\,;\,\tau \in \mathbb{B}^0,\ \boldsymbol{p} : E_\tau \to \mathbb{N}\}$.

We pick a positive number m and define $T_\beta^{(m)}$ for each $\beta \in A$ as the completion of the linear space spanned by $\tau^{\boldsymbol{p}} \in \mathbb{B}$ with $|\tau^{\boldsymbol{p}}| = \beta$ under the ℓ^2 norm

$$\Bigg\| \sum_{|\tau^{\boldsymbol{p}}|=\beta} c_{\tau^{\boldsymbol{p}}} \tau^{\boldsymbol{p}} \Bigg\|_{\beta,m}^2 := \sum_{|\tau^{\boldsymbol{p}}|=\beta} |c_{\tau^{\boldsymbol{p}}}|^2 m^{2|\boldsymbol{p}|},$$

where $|\boldsymbol{p}| := \sum_{e \in E_\tau} \boldsymbol{p}(e)$. We define the *model space* as an algebraic sum

$$T^{(m)} = \bigoplus_{\beta \in A} T_\beta^{(m)}.$$

The following statement is proved by a general procedure as in [8].

Proposition 3.2 ([4, Section 2.2])**.** *There exists a group $G^{(m)}$ of continuous automorphisms on $T^{(m)}$ such that $(T^{(m)}, G^{(m)})$ is a regularity structure.*

The number m is to be determined after the estimates of stochastic objects are fixed. The existence of such m is implicitly stated in Assumption 1. For simplicity, we omit the letter 'm' and write T instead of $T^{(m)}$ in what follows.

The following notations are used in what follows.

- A *sector* is a closed subspace S of T such that $(S, G|_S)$ is a regularity structure. In particular, sectors $T_\circ$ and U spanned by $\mathbb{B}_\circ$ and $\{X^{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^2} \cup \mathcal{I}_0(\mathbb{B})$ respectively are important.

- A *regularity* of a sector S is a minimum number β such that $S \cap T_\beta \neq \{0\}$.
- Denote by $\mathsf{Q}_\beta : T \to T_\beta$ the canonical projection map and write $\mathsf{Q}_{<\gamma} := \sum_{\beta<\gamma} \mathsf{Q}_\beta$.

3.2. **Well-posedness of the system** (2.2)**.** To consider regularities of spacetime distributions, we introduce a spacetime elliptic operator $\mathcal{L}^{a(v)} := (\partial_t - L^{a(v)})(\partial_t + \partial_x^2)$ and the associated heat semigroup $\mathcal{Q}_\theta^{a(v)} := e^{\theta \mathcal{L}^{a(v)}}$ $(\theta > 0)$. It is important that $\mathcal{Q}_\theta^{a(v)}$ satisfies the anisotropic Gaussian estimate ([4, Proposition 4]). For $\beta < 0$, we define the space $\mathcal{C}_{\mathfrak{s}}^\beta(a(v))$ as the completion of the set of bounded continuous functions on $\mathbb{R}^2$ under the norm

$$\|f\|_{\mathcal{C}_{\mathfrak{s}}^\beta(a(v))} := \sup_{0<\theta\le 1} \theta^{-\beta/4} \|\mathcal{Q}_\theta^{a(v)} f\|_{L^\infty(\mathbb{R}^2)}.$$

Let $K^{a(v)}(\cdot,\cdot)$ be a 'main part' of the Green function of $(\partial_t - L^{a(v)} + c)^{-1}$ (see [4, Section 2.1] for a precise definition) and write

$$\big\{(\partial_x^k K^{a(v)})(\eta)\big\}(z) := \int_{\mathbb{R}^2} \partial_x^k K^{a(v)}(z,z')\eta(z')dz'$$

for any spacetime functions/distributions η. Note that ∂_x acts on the first variable of $K^{a(v)}$.

Definition 3.1 ([4, Definitions 9 and 13])**.** *An* admissible model $\mathsf{M} = (\Pi, \Gamma)$ *consists of continuous operators* $\Pi_z : T \to \mathcal{C}_{\mathfrak{s}}^{-2}(a(v))$ *and* $\Gamma_{z'z} \in G$ *indexed by* $z, z' \in \mathbb{R}^2$ *with the following properties.*

- *(Chen's relations)* $\Pi_{z'}\Gamma_{z'z} = \Pi_{z'}$, $\Gamma_{zz} = \mathrm{Id}$, $\Gamma_{z''z'}\Gamma_{z'z} = \Gamma_{z''z}$ *for all* $z, z', z'' \in \mathbb{R}^2$.
- *(Regularity) For any* $\tau \in T_\beta$, $\gamma \le \beta$, $z, z' \in \mathbb{R}^2$, *and* $\theta \in (0,1]$,

$$\|\mathsf{Q}_\gamma \Gamma_{z'z}\tau\|_{\gamma,m} \lesssim \|z'-z\|_{\mathfrak{s}}^{\beta-\gamma}\|\tau\|_{\beta,m}, \qquad \big|\mathcal{Q}_\theta^{a(v)}(\Pi_z\tau)(z)\big| \lesssim \theta^{\beta/4}\|\tau\|_{\beta,m}.$$

- *(Admissibility) For any* $\tau \in T_\beta^{(m)}$ *with* $\beta < 0$, $z \in \mathbb{R}^2$, *and* $n \in \{0,1,2\}$,

$$\big\{\Pi_z(\mathcal{I}_n\tau)\big\}(\cdot) = \big\{(\partial_x^n K^{a(v)})(\Pi_z\tau)\big\}(\cdot) - \sum_{k<\beta+2-n} \frac{(\cdot - x)^k}{k!}\big\{(\partial_x^{n+k}K^{a(v)})(\Pi_z\tau)\big\}(z).$$

- *(Periodicity)* $\Gamma_{(z'+(0,1))(z+(0,1))} = \Gamma_{z'z}$ *and* $\{\Pi_{z+(0,1)}(\cdot)\}(z'+(0,1)) = \{\Pi_z(\cdot)\}(z')$ *in distributional sense for all* $z, z' \in \mathbb{R}^2$.

Definition 3.2 ([4, Definition 11])**.** *Let* S *be a sector and pick* $\eta \le \gamma$ *and* $t_0 > 0$. *Denote by* $\mathcal{D}^{\gamma,\eta}(0,t_0;S)$ *the set of functions* $\boldsymbol{u} : (0,t_0) \times \mathbb{T} \to S_{<\gamma} := \mathsf{Q}_{<\gamma}(S)$ *equipped with the norm*

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathcal{D}^{\gamma,\eta}(0,t_0;S)} := \max_{\beta<\gamma}\sup_{0<s<t_0} &\Big\{(s\wedge 1)^{\{(\beta-\eta)\vee 0\}/2}\sup_{t\ge s}\|\mathsf{Q}_\beta \boldsymbol{u}(z)\|_{\beta,m}\Big\} \\ &+ \max_{\beta<\gamma}\sup_{0<s<t_0}\Big\{(s\wedge 1)^{(\gamma-\eta)/2}\sup_{t,t'\ge s}\frac{\|\mathsf{Q}_\beta\{\boldsymbol{u}(z') - \Gamma_{z'z}\boldsymbol{u}(z)\}\|_{\beta,m}}{\|z'-z\|_{\mathfrak{s}}^{\gamma-\beta}}\Big\}.\end{aligned}$$

Each element of $\mathcal{D}^{\gamma,\eta}(0,t_0;S)$ is called a *modelled distribution.* We also define the space $\mathcal{D}^{\gamma,\eta} = \mathcal{D}^{\gamma,\eta}(\mathbb{R}\times\mathbb{T};T)$ in the same way and state the famous reconstruction theorem.

Theorem 3.3 ([4, Theorem 12])**.** *Let* $\eta \le \gamma$ *and* $\gamma > 0$. *For any admissible model* M, *there exists a unique continuous linear operator* $\mathsf{R}^{\mathsf{M}} : \mathcal{D}^{\gamma,\eta} \to \mathcal{C}_{\mathfrak{s}}^{\eta\wedge(\alpha_0-2)}(a(v))$ *such that the bound*

$$\big|\mathcal{Q}_\theta^{a(v)}\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{u} - \Pi_z\boldsymbol{u}(z)\big)(z)\big| \lesssim \big(|t| \vee \theta^{1/4}\big)^{\eta\wedge(\alpha_0-2)-\gamma}\,\theta^{\gamma/4}\|\boldsymbol{u}\|_{\mathcal{D}^{\gamma,\eta}}$$

holds uniformly over for any $\boldsymbol{u} \in \mathcal{D}^{\gamma,\eta}$, $z \in \mathbb{R}^2$, *and* $\theta \in (0,1]$. *Moreover,* $\mathsf{R}^{\mathsf{M}}\boldsymbol{u}$ *is a spatially periodic distribution.*

To apply Theorem 3.3 to $\boldsymbol{u} \in \mathcal{D}^{\gamma,\eta}(0,t_0;T)$, we consider a whole time extension $\widetilde{\boldsymbol{u}} \in \mathcal{D}^{\gamma,\eta}(\mathbb{R}\times\mathbb{T};T)$ such that $\widetilde{\boldsymbol{u}}|_{(-\infty,0]\times\mathbb{T}} = 0$ and define the reconstruction $\mathsf{R}^{\mathsf{M}}\widetilde{\boldsymbol{u}}$. Although such an extension is not unique, the value of $\mathsf{R}^{\mathsf{M}}\widetilde{\boldsymbol{u}}$ on the subset $(0,t_0)\times\mathbb{T}$ is unique because of the locality of R^{M}. See [3, Section 4.3] for details. We can define the following operations on modelled distributions and make sense of the right-hand sides of (2.2).

- (Multilevel Schauder [4, Theorem 15]) Recall that $\alpha \in (0,1]$ is a regularity of the initial condition u_0. Pick $\gamma \in (0,\alpha)$, $\eta \in (\alpha-2,\gamma]$, and $\gamma' \le \gamma+2$. Then there exists a continuous linear map
$$\mathsf{K}_{\gamma'}^{a(v),\mathsf{M}} : \mathcal{D}^{\gamma,\eta}(0,t_0;T_\circ) \to \mathcal{D}^{\gamma',(\eta+2)\wedge\alpha_0}(0,t_0;U)$$
such that $u = \mathsf{R}^{\mathsf{M}}\mathsf{K}_{\gamma'}^{a(v),\mathsf{M}}\boldsymbol{v}$ solves $(\partial_t - L^{a(v)} + c)u = \mathsf{R}^{\mathsf{M}}\boldsymbol{v}$ on $(0,t_0)$ with $u|_{t=0}$.
- (Multiplication) Let S_1 and S_2 are sectors of regularities α_1 and α_2 respectively, and such that the product $S_1\times S_2 \to T$ is defined. Then for any $\boldsymbol{u}_i \in \mathcal{D}^{\gamma_i,\eta_i}(S_i)$ ($i\in\{1,2\}$), we have
$$\mathsf{Q}_{<\gamma}(\boldsymbol{u}_1\cdot\boldsymbol{u}_2) \in \mathcal{D}^{\gamma,\eta}$$
with $\gamma = (\gamma_1+\alpha_2)\wedge(\gamma_2+\alpha_1)$ and $\eta = (\eta_1+\alpha_2)\wedge(\eta_2+\alpha_1)\wedge(\eta_1+\eta_2)$. Moreover, the mapping $(\boldsymbol{u}_1,\boldsymbol{u}_2)\mapsto \mathsf{Q}_{<\gamma}(\boldsymbol{u}_1\cdot\boldsymbol{u}_2)$ is locally Lipschitz continuous.
- (Composition) For any $\boldsymbol{u}\in\mathcal{D}^{\gamma,\eta}(U)$ and a function $h\in C^\kappa(\mathbb{R})$ with $\kappa \ge \max\{\gamma/\alpha,1\}$, we define
$$h(\boldsymbol{u}) := \mathsf{Q}_{<\gamma}\left(\sum_{n=0}^{\infty}\frac{h^{(n)}(u_0)}{n!}(\boldsymbol{u}-u_0X^{(0,0)})^n\right),$$
where u_0 denotes the $X^{(0,0)}$-component of $\boldsymbol{u}$. Then $h(\boldsymbol{u})\in\mathcal{D}^{\gamma,\eta}$, and the mapping $\boldsymbol{u}\mapsto h(\boldsymbol{u})$ is locally Lipschitz continuous.
- (Differentiation) Define $\boldsymbol{D}$ as a linear operator on T such that
$$\boldsymbol{D}X^{(k_1,k_2)} := k_2X^{(k_1,k_2-1)}\mathbf{1}_{k_2>0}, \qquad \boldsymbol{D}\mathcal{I}_n(\tau) := \mathcal{I}_{n+1}(\tau)\mathbf{1}_{n<2}.$$
Let $n\in\{1,2\}$. If $\gamma>n$, then the map $\mathcal{D}^{\gamma,\eta}(U)\ni\boldsymbol{u}\mapsto\boldsymbol{D}^n\boldsymbol{u}\in\mathcal{D}^{\gamma-n,\eta-n}$ is continuous and satisfies $\mathsf{R}^{\mathsf{M}}\boldsymbol{D}^n\boldsymbol{u} = \partial_x^n\mathsf{R}^{\mathsf{M}}\boldsymbol{u}$ for any $\boldsymbol{u}\in\mathcal{D}^{\gamma,\eta}(U)$.
- (Lift of regular functions [4, Lemma 16]) Let w be either of v as chosen in Remark 2.1 or $Q^{a(v)}u_0$. Then
$$(\mathsf{P}_{<\gamma}w)(z) := \sum_{|\mathbf{k}|_{\mathfrak{s}}<\gamma}(\partial_z^{\mathbf{k}}w)(z)\frac{X^{\mathbf{k}}}{\mathbf{k}!}$$
belongs to $\mathcal{D}^{\gamma,\eta}$ for any $\gamma\in(1,2+\alpha)\setminus\{2\}$ and $\eta\le\alpha$.

Theorem 3.4 ([4, Theorem 17])**.** *Let $\alpha\in(0,\alpha_0)$. For any $u_0\in C^\alpha(\mathbb{T})$, we choose an appropriate function v on $\mathbb{R}_+\times\mathbb{T}$ as in Remark 2.1. Then for any admissible model M, there exists sufficiently small $t_0 = t_0(u_0,\mathsf{M})$ such that system (2.2) has a unique solution $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ in the class*
$$\mathcal{D}^{\gamma,\alpha}(0,t_0;U)\times\mathcal{D}^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0;T_\circ)\times\mathcal{D}^{\gamma+\alpha_0-2,\alpha-2}(0,t_0;T_\circ)$$
for any $\gamma\in(2-\alpha_0,2-\alpha_0+\alpha)$. The time t_0 can be chosen to be a lower semicontinuous function of (u_0,M) and the solution map from (u_0,M) to $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ is locally Lipschitz continuous.

4. Proof of main results

4.1. **BPHZ renormalized model.** First we define a naive interpretation model $\mathsf{M}^\varepsilon = (\Pi^\varepsilon,\Gamma^\varepsilon)$ associated with the smooth approximation ξ^ε. An important point of [8] is that all models we consider take the form
$$\Pi_z = \mathsf{\Pi}\circ F_z^{-1}, \qquad \Gamma_{z'z} = F_{z'}\circ F_z^{-1}$$

with some continuous operators $\mathsf{\Pi} : T \to \mathcal{C}_{\mathfrak{s}}^{-2}(a(v))$ and $F_z \in G$, and it is sufficient to define $\mathsf{\Pi}$ to construct the model M. See [8, Section 6.2] and [4, Definition 9] for details. For each $\varepsilon \in (0, 1]$, we define the linear map $\mathsf{\Pi}^\varepsilon : T \to C^\infty(\mathbb{R}^2)$ by

$$(\mathsf{\Pi}^\varepsilon X^{\mathbf{k}})(z) = z^{\mathbf{k}}, \qquad \mathsf{\Pi}^\varepsilon \Xi_1 = \xi^\varepsilon, \qquad \mathsf{\Pi}^\varepsilon \Xi_i = 1 \quad (i \in \{2, 3\}),$$
$$\mathsf{\Pi}^\varepsilon \mathcal{I}_n \tau = (\partial_x^n K^{a(v)})(\mathsf{\Pi}^\varepsilon \tau), \qquad \mathsf{\Pi}^\varepsilon(\tau_1 \cdots \tau_n) = (\mathsf{\Pi}^\varepsilon \tau_1) \cdots (\mathsf{\Pi}^\varepsilon \tau_n).$$

We call the associated admissible model $\mathsf{M}^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ a *naive interpretation.*

Since we cannot expect the convergence of naive interpretations as $\varepsilon \to 0$, we try to find a natural convergent transform $\hat{\mathsf{M}}^\varepsilon$ of M^ε. It is known that such a transform is unique in some sense ([8, Theorem 6.18]) and called a *BPHZ renormalized model.* We follow Bruned's recursive formulae [6] to define this transform. Let $\Delta_r^- : T \to T \otimes T$ be a splitting map defined as in [6, Section 4.1] and define the linear map $R_\ell : T \to T$ by

$$R_\ell(z)\tau = \big(\ell[\cdot](z) \otimes \mathrm{Id}\big)\Delta_r^- \tau = \sum_{\tau^{(1)}, \tau^{(2)}} \ell[\tau^{(1)}](z)\tau^{(2)}$$

for some spacetime functions $\ell[\tau](z)$ indexed by $\tau \in \mathbb{B}$, where we use Sweedler's notation $\Delta_r^- \tau = \sum \tau^{(1)} \otimes \tau^{(2)}$ for simplicity. (The letter 'r' means that R_ℓ cancels divergences occurring at the root of the tree.) We call such a map (satisfying some additional conditions – see [4, Section 4.3.1]) a *renormalization character*. For any ℓ, we can define linear maps $\mathsf{\Pi}^{\varepsilon,\ell}$ and $\mathsf{\Pi}_\times^{\varepsilon,\ell}$ as follows.

$$(\mathsf{\Pi}^{\varepsilon,\ell}\tau)(z) = \big\{\mathsf{\Pi}_\times^{\varepsilon,\ell}\big(R_\ell(z)\tau\big)\big\}(z),$$
$$\mathsf{\Pi}_\times^{\varepsilon,\ell}(\tau_1 \cdots \tau_n) = (\mathsf{\Pi}_\times^{\varepsilon,\ell}\tau_1) \cdots (\mathsf{\Pi}_\times^{\varepsilon,\ell}\tau_n), \qquad \mathsf{\Pi}_\times^{\varepsilon,\ell}\mathcal{I}_n\tau = (\partial_x^n K^{a(v)})(\mathsf{\Pi}^{\varepsilon,\ell}\tau).$$

Proposition 4.1 ([6, Proposition 3.16])**.** *The map* $\mathsf{\Pi}^{\varepsilon,\ell}$ *defines a model* $\mathsf{M}^{\varepsilon,\ell}$.

For example, we consider the symbol $\tau = (\mathcal{I}_1(\Xi))^2$. Since $R_\ell(z)\tau = \tau + \ell[\tau](z)X^{(0,0)}$ in this case, by choosing

$$\ell[\tau](z) = -\int_{(\mathbb{R}^2)^2} \partial_x K^{a(v)}(z, z_1)\partial_x K^{a(v)}(z, z_2)\mathbb{E}[\xi^\varepsilon(z_1)\xi^\varepsilon(z_2)]dz_1 dz_2, \tag{4.1}$$

we have

$$\Pi^{\varepsilon,\ell}\tau(z) = \int_{(\mathbb{R}^2)^2} \partial_x K^{a(v)}(z, z_1)\partial_x K^{a(v)}(z, z_2) : \xi^\varepsilon(z_1)\xi^\varepsilon(z_2) : dz_1 dz_2,$$

where $: (\cdot) :$ means Wick product. It is not difficult to show the convergence of the above quantity by a similar way to [14, Section 10]. Assumption 1 says that we can choose an appropriate renormalization character ℓ^ε for $\hat{\mathsf{M}}^\varepsilon := \mathsf{M}^{\varepsilon,\ell}$ to converge as $\varepsilon \to 0$.

4.2. **Proof of Theorem 2.2.** We can see the action of the character ℓ on the equation by following Bailleul and Bruned's simple approach [1]. Note that each tree $\tau \in \mathbb{B}$ can be written as

$$\tau = X^{\mathbf{k}}\Xi_i \prod_{\nu=1}^m \mathcal{I}_{m_\nu}(\sigma_\nu)^{\beta_\nu},$$

where $(m_\mu, \sigma_\mu) \neq (m_\nu, \sigma_\nu)$ for any $\mu \neq \nu$ uniquely up to the order of multiplications. By using this representation, we inductively define $S : \mathbb{B} \to \mathbb{N}$ by

$$S[\tau] := \mathbf{k}! \prod_{\nu=1}^m \big\{S[\sigma_\nu]^{\beta_\nu}\beta_\nu!\big\}.$$

Moreover, we define smooth functions $\Upsilon[\tau](\mathsf{u},\mathsf{v})$ of $\mathsf{u}=(\mathsf{u}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^2}$ and $\mathsf{v}=(\mathsf{v}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^2}$ for each $\tau\in\mathbb{B}$ as follows.

$$\begin{aligned}\Upsilon[\Xi_1](\mathsf{u},\mathsf{v}) &:= f(\mathsf{u}_{(0,0)}),\\ \Upsilon[\Xi_2](\mathsf{u},\mathsf{v}) &:= g(\mathsf{u}_{(0,0)})\mathsf{u}_{(0,1)}^2 + c\mathsf{u}_{(0,0)},\\ \Upsilon[\Xi_3](\mathsf{u},\mathsf{v}) &:= \big(a(\mathsf{u}_{(0,0)})-a(\mathsf{v}_{(0,0)})\big)\mathsf{u}_{(0,2)},\\ \Upsilon\left(X^{\mathbf{k}}\Xi_i\prod_{j=1}^n \mathcal{I}_{n_j}(\tau_j)\right) &:= \big(\partial^{\mathbf{k}}D_{(0,n_1)}\cdots D_{(0,n_i)}\Upsilon[\Xi_i]\big)\prod_{j=1}^n\Upsilon[\tau_j],\end{aligned}$$

where $D_{\mathbf{k}}=D_{\mathsf{u}_{\mathbf{k}}}$ is the differential operator with respect to $\mathsf{u}_{\mathbf{k}}$ and $\partial^{\mathbf{k}}$ is the differential operator defined by

$$\partial^{\mathbf{e}} := \sum_{\mathbf{k}\in\mathbb{N}^2}(\mathsf{u}_{\mathbf{k}+\mathbf{e}}D_{\mathsf{u}_{\mathbf{k}}}+\mathsf{v}_{\mathbf{k}+\mathbf{e}}D_{\mathsf{v}_{\mathbf{k}}}) \qquad (\mathbf{e}\in\{(1,0),(0,1)\})$$

and $\partial^{\mathbf{k}} := (\partial^{(1,0)})^{k_1}(\partial^{(0,1)})^{k_2}$.

Theorem 4.2 ([4, Proposition 24])**.** *Let $(\boldsymbol{u}^{\varepsilon,\ell},\boldsymbol{v}^{\varepsilon,\ell},\boldsymbol{w}^{\varepsilon,\ell})$ be the solution to* (2.2) *with respect to the model $\mathsf{M}^{\varepsilon,\ell}$. Then one can choose $t_0'\in(0,t_0)$ small enough for $u^{\varepsilon,\ell}:=\mathsf{R}^{\mathsf{M}^{\varepsilon,\ell}}\boldsymbol{u}^{\varepsilon,\ell}$ to solve*

$$\partial_t u^{\varepsilon,\ell} - a(u^{\varepsilon,\ell})\partial_x^2 u^{\varepsilon,\ell} = f(u^{\varepsilon,\ell})\xi^\varepsilon + g(u^{\varepsilon,\ell})(\partial_x u^{\varepsilon,\ell})^2 + \sum_{|\tau^{\boldsymbol{p}}|<0}\frac{\ell^\varepsilon[\tau^{\boldsymbol{p}}]}{S[\tau^{\boldsymbol{p}}]}\Upsilon[\tau^{\boldsymbol{p}}]\big(u^{\varepsilon,\ell},\partial_x u^{\varepsilon,\ell},v\big) \tag{4.2}$$

on $(0,t_0')\times\mathbb{T}$, with initial condition u_0. In the last term, $\Upsilon[\tau^{\boldsymbol{p}}]$ depends only on $\mathsf{u}_{(0,0)},\mathsf{u}_{(0,1)},\mathsf{v}_{(0,0)}$ and at most linear with respect to $\mathsf{u}_{(0,1)}$.

Theorem 2.2 is immediately obtained from Theorems 3.4 and 4.2.

4.3. **Proof of Theorem 2.3.** The edge decoration $\boldsymbol{p}$ has the following two important roles.

- [4, Lemma 25] There exist v-independent smooth functions $\Upsilon_0[\tau]$ such that
$$\Upsilon[\tau^{\boldsymbol{p}}](\mathsf{u}_{(0,0)},\mathsf{u}_{(0,1)},\mathsf{v}_{(0,0)}) = \big(a(\mathsf{u}_{(0,0)})-a(\mathsf{v}_{(0,0)})\big)^{|\boldsymbol{p}|}\Upsilon_0[\tau](\mathsf{u}_{(0,0)},\mathsf{u}_{(0,1)}).$$
- [4, Lemma 26] For each $\lambda>0$, let $Z^\lambda(t,x)$ be the Green function of $(\partial_t-\lambda\partial_x^2+c)^{-1}$, and define the renormalization character $l_\lambda^\varepsilon[\tau^{\boldsymbol{p}}]$ by replacing $K^{a(v)}$ in the definition of $\ell^\varepsilon[\tau^{\boldsymbol{p}}]$ (as in (4.1)) with Z^λ. Then $l_\lambda^\varepsilon[\tau^{\boldsymbol{p}}]$ is analytic in λ and
$$\frac{1}{n!}\partial_\lambda^n l_\lambda^\varepsilon[\tau^{\mathbf{0}}] = \sum_{|\boldsymbol{p}|=n} l_\lambda^\varepsilon[\tau^{\boldsymbol{p}}].$$

Proof of Theorem 2.3. Under Assumption 2, we can trade off the character $\ell^\varepsilon[\tau^{\boldsymbol{p}}](\cdot)$ in the last term of (2.3) by $l^\varepsilon_{a(v(\cdot))}[\tau^{\boldsymbol{p}}]$ up to an ε-uniform remainder and have

$$\begin{aligned}&\sum_{|\tau^{\boldsymbol{p}}|<0}\frac{l^\varepsilon_{a(v)}[\tau^{\boldsymbol{p}}]}{S[\tau^{\boldsymbol{p}}]}\Upsilon[\tau^{\boldsymbol{p}}](u^\varepsilon,\partial_x u^\varepsilon,v) = \sum_{|\tau^{\mathbf{0}}|<0}\frac{1}{S[\tau^{\mathbf{0}}]}\sum_{\boldsymbol{p}\in\mathbb{N}^{E_\tau}} l^\varepsilon_{a(v)}[\tau^{\boldsymbol{p}}]\Upsilon_0[\tau^{\boldsymbol{p}}](u^\varepsilon,\partial_x u^\varepsilon,v)\\ &= \sum_{|\tau^{\mathbf{0}}|<0}\frac{1}{S[\tau^{\mathbf{0}}]}\Upsilon_0[\tau](u^\varepsilon,\partial_x u^\varepsilon)\sum_{n=0}^\infty\big(a(u^\varepsilon)-a(v)\big)^n\sum_{|\boldsymbol{p}|=n} l^\varepsilon_{a(v)}[\tau^{\boldsymbol{p}}]\\ &= \sum_{|\tau^{\mathbf{0}}|<0}\frac{1}{S[\tau^{\mathbf{0}}]}\Upsilon_0[\tau](u^\varepsilon,\partial_x u^\varepsilon) l^\varepsilon_{a(u^\varepsilon)}[\tau^{\mathbf{0}}].\end{aligned}$$

In the first equality, we use [4, Lemma 18]. □

Finally, we show some examples satisfying Assumption 2. See [4, Section 4.5] for details.

4.3.1. *Two dimensional generalized PAM.* We consider the equation

$$\partial_t u - a(u)\Delta u = f(u)\xi, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2$$

with the space white noise ξ. Although the spatial dimension is two, similar arguments to above work well. In this case we can choose $2/3 < \alpha_0 < 1$. For example, we consider the renormalization character of the symbol

$$\tau = \Xi_1 \mathcal{I}_0(\Xi_1).$$

Similarly to (4.1), we can choose the character

$$\ell^\varepsilon[\tau](t,x) = \int_{\mathbb{R}^3} K^{a(v)}\big((t,x),(s,y)\big)\mathbb{E}[\xi^\varepsilon(x)\xi^\varepsilon(y)]dsdy.$$

By using the classical Levi's parametrix method, we can trade off $K^{a(v)}$ with $Z_{t-s}^{a(v(t,x))}(x-y)$ (see [4, Proposition 28]) and have that

$$\ell^\varepsilon[\tau](t,x) = \int_{(-\infty,t)\times\mathbb{R}^2} Z_{t-s}^{a(v(t,x))}(x,y)\mathbb{E}[\xi^\varepsilon(x)\xi^\varepsilon(y)]dsdy + O(1) = l^\varepsilon_{a(v(t,x))}[\tau] + O(1).$$

We can perform a similar calculation for the symbol $\Xi_3\mathcal{I}_0(\Xi_1)\mathcal{I}_2(\Xi_1)$ and have the following.

Corollary 4.3 ([2, Theorem 1])**.** *There exists a diverging constant c^ε such that the solution to*

$$\partial_t u^\varepsilon - a(u^\varepsilon)\Delta u^\varepsilon = f(u^\varepsilon)\xi^\varepsilon - c^\varepsilon\left(\frac{f'f}{a} - \frac{a'f^2}{a^2}\right)(u^\varepsilon)$$

converges locally in time as $\varepsilon \to 0$.

4.3.2. *One dimensional generalized KPZ equation with regularized noise.* Let ξ be a stationary Gaussian noise on $\mathbb{R}\times\mathbb{T}$ which is slightly more regular that the white one (e.g. let η be a white noise and consider $\xi = (1-\Delta)^{-\alpha}\eta$ with $\alpha > 0$). In this case we can choose $1/2 < \alpha_0 < 2/3$. We can perform similar calculations to above for the equation

$$\partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}$$

and obtain the following result.

Corollary 4.4 ([12, Equation (1.2)])**.** *There exists a smooth function $C^\varepsilon_{(\cdot)}$ such that the solution to*

$$\partial_t u^\varepsilon - a(u^\varepsilon)\partial_x^2 u^\varepsilon = f(u^\varepsilon)\xi^\varepsilon + g(u^\varepsilon)(\partial_x u^\varepsilon)^2 - C^\varepsilon_{a(u^\varepsilon)}\left(f'f + \frac{gf^2}{a} - \frac{a'f^2}{a}\right)(u^\varepsilon)$$

converges locally in time as $\varepsilon \to 0$.

4.3.3. *One dimensional quasilinear stochastic heat equation.* We consider the equation

$$\partial_t u - a(u)\partial_x^2 u = \xi, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}$$

with the spacetime white noise ξ on $\mathbb{R}\times\mathbb{T}$. In this case we can choose $2/5 < \alpha_0 < 1/2$. The only difference with above cases is that we cannot trade off the function $K^{a(v)}$ in the integral

$$\ell^\varepsilon[\Xi_3\mathcal{I}_0(\Xi_1)\mathcal{I}_2(\Xi_1)](z) = \int_{(\mathbb{R}^2)^2} K^{a(v)}(z,z')\partial_x^2 K^{a(v)}(z,z'')\mathbb{E}[\xi^\varepsilon(z')\xi^\varepsilon(z'')]dz'dz'' \tag{4.3}$$

with $Z_{t-\cdot}^{a(v(t,x))}(x-\cdot)$ up to an integrable remainder. Instead, we choose a time-independent smooth function $v(x)$ and derive that

$$K^{a(v)}(z,z') = Z_{t-t'}^{a(v(x))}(x-x') + a'(v(x))v'(x)Y^x_{t-t'}(x-x') + \cdots$$

by continuing the decomposition based on Levi's parametrix method. Here $Y_t^x(x')$ is an odd function with respect to x' ([4, Proposition 29]). Because of symmetries of Z and Y, the integral

$$\int_{((-\infty,t)\times\mathbb{R})^2} Z_{t-t'}^{a(v(x))}(x-x')\partial_x^2 Y_{t-t''}^x(x-x'')\mathbb{E}[\xi^\varepsilon(t',x')\xi^\varepsilon(t'',x'')]dt'dx'dt''dx''$$

is equal to zero, if we approximate the noise by $\xi^\varepsilon = \rho^\varepsilon * \xi$ with a *spatially even mollifier* ρ^ε. Therefore, we can trade off $K^{a(v)}$ in (4.3) by $Z^{a(v(x))}$ and obtain the following result.

Corollary 4.5 ([11, Theorem 1.1])**.** *There exist smooth functions $C_i^\varepsilon(\cdot)$ for each $i \in \{1,2,3\}$ such that the solution to*

$$\partial_t u^\varepsilon - a(u^\varepsilon)\partial_x^2 u^\varepsilon = \xi^\varepsilon - \left\{C_1^\varepsilon(a(u^\varepsilon))a'(u^\varepsilon) + C_2^\varepsilon(a(u^\varepsilon))(a'a'')(u^\varepsilon) + C_3^\varepsilon(a(u^\varepsilon))(a'(u^\varepsilon))^3\right\}$$

converges locally in time as $\varepsilon \to 0$.

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