General remarks on the propagation of chaos in wave turbulence and application to the incompressible Euler dynamics

Anne-Sophie de Suzzoni

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1 Introduction

Propagation of chaos in the context of wave turbulence is the fact that when considering the solution to a Cauchy problem whose initial datum is random and presents independent Fourier coefficients, the Fourier coefficients of the solution at later times remain asymptotically independent.

Consider the Cauchy problem for the incompressible Euler equation on the torus of size $L > 0$,

$$\begin{cases} \partial_t u_L + u_L \cdot \nabla u_L = -\nabla p_L, & x \in L\mathbb{T}^d, \\ \nabla \cdot u_L = 0, & \\ u_L(t=0) = a_L := \sum_{\mathbb{Z}^d_*} \frac{e^{ikx/L}}{(2\pi L)^{d/2}} g_k a(k/L), & \end{cases} \tag{1}$$

where u has values in $\mathbb{R}^d$. The initial datum a_L depends on a function a that is even, bounded, compactly supported and with values in $\mathbb{R}^d$. To ensure that $\nabla \cdot a_L = 0$, we also ask that $\xi \cdot a(\xi) = 0$ for all $\xi \in \mathbb{R}^d$. The randomness of the initial datum comes from the $(g_k)_k$ which are centred normalised jointly Gaussian variables such that $\mathbb{E}(g_k g_l) = \delta_{k+l}$, which means $g_k = \overline{g_{-k}}$ but otherwise they are independent. The fact that $g_k = \overline{g_{-k}}$ ensures that a_L has values in $\mathbb{R}^d$.

Notice that initially Fourier coefficients are independent Gaussian variables. Indeed,

$$\hat{a}_L(\xi) = g_{L\xi} a(\xi).$$

The issue at stake is to understand in which sense the Fourier coefficients at later times, namely

$$\hat{u}_L(t, \xi)$$

remain independent Fourier coefficients at fixed time t, but when L goes to infinity. If we do not consider the asymptotic regime $L \to \infty$, because the equation is nonlinear, the Fourier coefficients are not independent a priori.

The reason why one thinks that propagation of chaos is verified is that as L goes to infinity, each Fourier coefficient of the initial datum has less and less weight, and thus the probability to take two that are independent goes to 0. This is what we will explain in the sequel.

1.1 Motivations

One motivation is the derivation of kinetic equations in the Physics literature. Kinetic equations are the equations that characterise the evolution of the correlations of Fourier coefficients of solutions to Cauchy problems with random initial data. When one derives formally the moments of order 2, one involves moments of higher order depending on the order of the nonlinearity of the equation. To get a closed system, one uses propagation of chaos. The reasons justifying propagation of chaos are combinatory. The more independent Gaussian variables you have, the less probable it becomes to pick two that are not independent. This is what we follow here. We mention the following works about wave turbulence, [22, 4, 18, 19, 25, 24, 2, 21].

On the mathematical treatment of these type of issues, we mention the works by Deng and Hani, [10, 11, 12, 13] on quantum equations : they derive kinetic equations for Schrödinger equations and then deduce propagation of chaos. The tools that are used are oscillating integrals and dispersion.

We also mention the following works [23, 8, 7, 5, 20, 17, 14, 16, 15].

1.2 Wick formula

Our goal is thus to quantify how Fourier coefficients at ulterior time differ from independent centred Gaussian variables. We quantify this by estimating how well Fourier coefficients of the solution at later times satisfy the Wick formula.

Centred jointly Gaussian variables satisfy the Wick formula :

$$\mathbb{E}\Big(\prod_{l=1}^{R} g_l\Big) = \sum_{\underline{\mathfrak{S}_R}} \prod_{l\in S_\sigma^+} \mathbb{E}(g_l g_{\sigma(l)})$$

where $\underline{\mathfrak{S}_R}$ is the set of involutions of $[|1,R|] := [1,R]\cap\mathbb{N}$ without fixed points and

$$S_\sigma^+ = \{l \in [|1,R|] \mid l < \sigma(l)\}.$$

An involution without fixed point is simply a way to pair each of the elements of $[|1,R|]$. In particular, if R is odd, then $\underline{\mathfrak{S}_R} = \emptyset$ and

$$\mathbb{E}\Big(\prod_{l=1}^{R} g_l\Big) = 0.$$

The fact that we restrict the product to S_σ^+ is not to repeat the expectation of the pairs twice.

Remark 1.1. *If R even, and $g_1 = \ldots = g_R = g$ are normalized and centred real Gaussian variables, we find*

$$\mathbb{E}(g^R) = \#\underline{\mathfrak{S}_R} = \frac{R!}{2^{R/2}(R/2)!}$$

which is indeed the R moment of a normalized and centred real Gaussian variable that one can compute by integration by parts. To count the cardinal of $\mathfrak{S}_R$, we take $R/2$ elements of $[|1,R|]$, that is

$$\frac{R!}{(R/2!)^2}.$$

Then, we map each of these elements to elements of the complementary in $[|1,R|]$ that makes

$$\frac{R!}{(R/2)!}.$$

Because the pairs are not ordered, we divide by $2^{R/2}$ and get the result.

In the same way, if $g_1 = \ldots = g_{R/2} = g$, $g_{R/2+1} = \ldots = g_R = \bar{g}$ where g is a complex centred normalized Gaussian variable, we find

$$\mathbb{E}(|g|^R) = (R/2)!.$$

Indeed, $\mathbb{E}(g^2) = \mathbb{E}((\bar{g})^2) = 0$ *and* $\mathbb{E}(|g|^2) = 1$*. Therefore,*

$$\prod_{l\in S_\sigma^+} \mathbb{E}(g_l g_{\sigma(l)}) = \begin{cases} 0 & \text{if } \exists l \in [|1, R/2|],\ \sigma(l) \in [|1, R/2|] \\ 1 & \text{if } \forall l \in [|1, R/2|],\ \sigma(l) \in [|R/2+1, R|] \end{cases}.$$

There are indeed $R/2!$ *involutions contributing, and we get the result, which corresponds indeed to the* R *moment of a normalized and centred complex Gaussian variable.*

Remark 1.2. *This formula characterises the law of a family of jointly Gaussian variables (real or complex). In particular, we see that the law of jointly Gaussian variables is characterised by the correlations*

$$\mathbb{E}(g_l g_m).$$

1.3 Framework

We recall that we consider the incompressible Euler equation (1).

Let P be the Leray projector, that is the 0 order differential operator

$$Pu = u - \nabla \triangle^{-1} (\nabla \cdot u).$$

Note that $P(\nabla p_L) = 0$ and that $\nabla \cdot P = 0$, $P^2 = P$. In other words, P projects on the divergence free functions.

Commuting with P, the equation becomes

$$\partial_t u_L + P(u_L \cdot \nabla u_L) = 0.$$

This equation is invariant under the action of spatial translations.

We consider the Cauchy problem :

$$\begin{cases} \partial_t u_L + P(u_L \cdot \nabla u_L) = 0 \\ u_L(t=0) = a_L := \sum_{\mathbb{Z}_*^d} \frac{e^{ikx/L}}{(2\pi L)^{d/2}} g_k a(k/L) \end{cases}.$$

Remark 1.3 (Remark on the initial datum)**.** *The reason why we sum on* $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{0\}$ *instead of* $\mathbb{Z}^d$ *is the following. We remark that* $\int_{L\mathbb{T}^d} u$ *is a priori conserved by the flow for periodic solutions and thus we choose it null.*

Remark 1.4 (Remark on invariance)**.** *The law of* a_L *is invariant under the action of spatial translations, so is the law of* $u_L(t)$*. This is due to the fact that the law of a Gaussian variable is invariant under the action of* $U(1)$ *(it can be multiplied by a phase).*

This implies that for all $\xi_1, \dots, \xi_R$*,* $x_0 \in \mathbb{R}^d$*, writing* τ_{x_0} *the translation* $\tau_{x_0} u(x) = u(x - x_0)$*,*

$$\mathbb{E}\Big(\prod_{l=1}^R \hat{u}_L(t, \xi_l)\Big) = \mathbb{E}\Big(\prod_{l=1}^R \widehat{\tau_{x_0} u_L}(t, \xi_l)\Big)$$

and thus

$$\mathbb{E}\Big(\prod_{l=1}^R \hat{u}_L(t, \xi_l)\Big) = e^{ix_0(\sum_l \xi_l)} \mathbb{E}\Big(\prod_{l=1}^R \hat{u}_L(t, \xi_l)\Big).$$

We deduce

$$\mathbb{E}\Big(\prod_{l=1}^R \hat{u}_L(t, \xi_l)\Big) \neq 0 \Rightarrow \sum_l \xi_l = 0.$$

We specify the normalisation of the Fourier coefficients : we set

$$\hat{u}_L(t,\xi) = \frac{1}{(2\pi L)^{d/2}} \int_{[-\pi L,\pi L]^d} u_L(x) e^{-i\xi x} dx$$

such that

$$\hat{u}_L(0,\xi) = g_{L\xi} a(\xi).$$

Our goal is to estimate, for a given R (at least locally in time),

$$\sup_{(\xi_l)_l,(j_l)_l,t} \Big| \mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_L^{(j_l)}(t,\xi_l) \Big) - \sum_{\sigma \in \underline{\mathfrak{S}_R}} \prod_{l \in S_\sigma^+} \mathbb{E}\Big(\hat{u}_L^{(j_l)}(t,\xi_l) \hat{u}_L^{(j_{\sigma(l)})}(t,\xi_{\sigma(l)}) \Big) \Big|$$

and prove that it goes to 0 as $L \to \infty$. We precise that we consider $\hat{u}_L^{(j_l)}(t,\xi_l)$ the j_l coefficient of $\hat{u}_L(t,\xi_l)$ because the solution has values in $\mathbb{R}^d$. Namely, if $X \in \mathbb{R}^d$ we write

$$X = (X^{(1)}, \ldots, X^{(d)}).$$

One problem is that a_L is a Gaussian variable so any norm of a_L might be big on a set with positive measure and thus the time of existence of the flow may be very small on non-negligible sets. Therefore, for a given t, the solution $u_L(t)$ is not necessarily well-defined. The first result we state is not on the full solution but on what is sometimes referred to in the literature as *quasi-solutions*.

2 Results

2.1 Quasi-solutions

We define the sequence $(u_{L,n})_n$ in the following way

$$u_{L,0} = a_L, \quad u_{L,n+1} = -\sum_{n_1+n_2=n} \int_0^t P(u_{L,n_1}(\tau) \cdot \nabla u_{L,n_2}(\tau)) d\tau.$$

If the series $\sum_n u_{L,n}$ converges, then the sum is equal to the full solution, that is

$$\sum u_{L,n} = u_L.$$

Remark 2.1. *One may see this last equality as an expansion in the size of the initial datum.*

Notation 2.1 (Quasi-solution)**.** We set

$$u_{L,\le N} = \sum_{n=0}^{N} u_{L,n}.$$

This last function is said to be a quasi-solution.

On quasi-solutions, we prove the following theorem.

Theorem 2.2 (dS, [9])**.** *For all $R \in \mathbb{N}^*$, all $t \in \mathbb{R}$ and all $N \in \mathbb{N}$, there exists $C = C(a,R,N,t)$ such that for all L, all $(j_l)_{1\le l\le R}$, all $(\xi_l)_{1\le l\le R}$, we have*

$$\Big| \mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_{L,\le N}^{(j_l)}(t,\xi_l) \Big) - \sum_{\sigma \in \underline{\mathfrak{S}_R}} \prod_{l \in S_\sigma^+} \mathbb{E}\Big(\hat{u}_{L,\le N}^{(j_l)}(t,\xi_l) \hat{u}_{L,\le N}^{(j_{\sigma(l)})}(t,\xi_{\sigma(l)}) \Big) \Big| \le \frac{C}{L^{d/2}}.$$

Remark 2.2. *One can prove a finer estimate which depends on the parity of R and algebraic relations satisfies by the $(\xi_l)_l$, see Proposition 3.1.*

Remark 2.3. *The constant C is explicit, but it depends badly on N, as in* $(NR)!$.

2.2 Result on the full solution

Because the above constant behaves badly in N, to get a result on the full solution one must get the convergence of the sequence $(u_{L,\leq N})_N$ before considering expectations, which means that we have to resort to classical and deterministic arguments.

The problem is that the typical size of any usual norm of the initial datum a_L behaves badly with L because it is not localised, indeed :

$$\begin{aligned}\|a_L\|_{L^2(L\mathbb{T}^d)} &\sim L^{d/2},\\ \|a_L\|_{L^p(L\mathbb{T}^d)} &\sim L^{d/p}\\ \|a_L\|_{L^\infty} &\sim \sqrt{\ln L}.\end{aligned}$$

By this, we mean that there exists $\alpha_p, \beta_p, \gamma_p > 0$ such that for all L,

$$\begin{aligned}\mathbb{P}(\|a_L\|_{L^p} \in [\alpha_p L^{d/p}, \beta_p L^{d/p}]) &\geq \gamma_p,\\ \mathbb{P}(\|a_L\|_{L^\infty} \in [\alpha_\infty \sqrt{\ln L}, \beta_\infty \sqrt{\ln L}]) &\geq \gamma_\infty.\end{aligned}$$

This might not be true for all functions a defining the initial datum, but there exists examples of a where this is the typical behaviour of the norms.

The idea for the last estimate is that there are circa L^d independent Gaussian variables forming a_L. In some cases, for instance, if a is explicit or smooth enough, this implies that there exist circa L^d space coordinates $x \in L\mathbb{T}^d$ are such that the $a_L(x)$ are independent Gaussian variables.

We change the framework and set

$$\underline{a_L} = \varepsilon(L) a_L$$

with $\varepsilon(L) = O(\frac{1}{\sqrt{\ln L}})$.

Theorem 2.3 (dS, [9])**.** *Let $\theta > 0$. There exists $c(\theta) > 0$ such that for all L, there exists a set $\mathcal{E}_{L,\theta}$ such that*

$$\mathbb{P}(\mathcal{E}_{L,\theta}) \geq 1 - e^{-c\varepsilon(L)^{-2}} \to 1$$

and such that the flow of the Euler equation is well-defined on $[-\theta, \theta]$ when the i.d. is taken in $\mathcal{E}_{L,\theta}$.

What is more, for all $R \in \mathbb{N}^*$, there exists $\theta_0 = \theta_0(R, a)$ and $C = C(a, R)$ such that for all $t \in [-\theta_0, \theta_0]$, for all L, all $(j_l)_{1\leq l\leq R}$, all $(\xi_l)_{1\leq l\leq R}$, we have

$$\Big|\mathbb{E}\Big(\mathbf{1}_{\mathcal{E}_{L,\theta_0}} \prod_{l=1}^R \underline{\hat{u}_L}^{(j_l)}(t,\xi_l)\Big) - \sum_{\sigma\in\underline{\mathfrak{S}_R}} \prod_{l\in S_\sigma^+} \mathbb{E}\Big(\mathbf{1}_{\mathcal{E}_{L,\theta_0}} \underline{\hat{u}_L}^{(j_l)}(t,\xi_l) \underline{\hat{u}_L}^{(j_{\sigma(l)})}(t,\xi_{\sigma(l)})\Big)\Big| \leq \frac{C\varepsilon(L)^R}{L^{d/2}}.$$

Remark 2.4. *If $\varepsilon(L) = o(\frac{1}{\sqrt{\ln L}})$ then the result is global in the sense that the second part of the theorem remains true for any θ_0. In other words, it can be replaced by "What is more, for all $R \in \mathbb{N}^*$, there exists $C = C(a, R, \theta)$ such that for all $t \in [-\theta, \theta]$, for all L, all $(j_l)_{1\leq l\leq R}$, all $(\xi_l)_{1\leq l\leq R}$, we have*

$$\Big|\mathbb{E}\Big(\mathbf{1}_{\mathcal{E}_{L,\theta}} \prod_{l=1}^R \underline{\hat{u}_L}^{(j_l)}(t,\xi_l)\Big) - \sum_{\sigma\in\underline{\mathfrak{S}_R}} \prod_{l\in S_\sigma^+} \mathbb{E}\Big(\mathbf{1}_{\mathcal{E}_{L,\theta}} \underline{\hat{u}_L}^{(j_l)}(t,\xi_l) \underline{\hat{u}_L}^{(j_{\sigma(l)})}(t,\xi_{\sigma(l)})\Big)\Big| \leq \frac{C\varepsilon(L)^R}{L^{d/2}}."$$

3 Strategy of proof

The proof of the first theorem is purely combinatory.

We give an example. The number of Fourier coefficients we consider is $R = 6$. We have

$$u_{L,1}(t) = -\int_0^t P(a_L \cdot \nabla a_L) d\tau = -tP(a_L \cdot \nabla a_L),$$

and

$$\hat{u}^{(1)}_{L,1}(t,\xi) = \frac{t}{L^{d/2}} \sum_{\xi_1+\xi_2=\xi} \psi(\xi_1,\xi_2) g_{L\xi_1} g_{L\xi_2}.$$

with

$$\psi(\xi_1,\xi_2) = \sum_{k=1}^{d} a^{(k)}(\xi_1)\xi_2^{(k)} a^{(1)}(\xi_2) - \sum_{k,j=1}^{d} \frac{(\xi_1+\xi_2)^{(1)}(\xi_1+\xi_2)^{(j)}}{|\xi_1+\xi_2|^2} a^{(k)}(\xi_1)\xi_2^{(k)} a^{(j)}(\xi_2).$$

Note that the expression of ψ does not depend on L.

We deduce

$$\mathbb{E}\Big(\prod_{l=1}^{6} \hat{u}^{(1)}_L(t,\xi_l)\Big) = \frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1},\xi_{l,2}) \mathbb{E}\Big(\prod_l g_{L\xi_{l,1}} g_{L\xi_{l,2}}\Big).$$

A priori, we sum on 6 parameters, which makes the sum of order L^{3d} but the $(g_k)_k$ satisfy the Wick formula, and thus we can pair them which means that we sum on at most 3 parameters and thus the sum is a priori of order 1.

3.1 Orbits

We have that, using the Wick formula

$$\mathbb{E}\Big(\prod_l g_{L\xi_{l,1}} g_{L\xi_{l,2}}\Big) = \sum_{\sigma\in\underline{\mathfrak{S}(6,2)}} \prod_{(l,j)\in S^+_\sigma} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma(l,j)}}) \leq 6!$$

where $\underline{\mathfrak{S}(6,2)}$ is the set of involutions without fixed points of $[|1,6|] \times \{1,2\}$.

If this quantity is not null, then there exists an involution σ without fixed points of $[|1,6|] \times \{1,2\}$ such that for all (l,j), we have $\xi_{l,j} = -\xi_{\sigma(l,j)}$.

Note that since $\xi_l \neq 0$, we have for all $l \in [|1,6|]$, $\sigma(l,1) \neq (l,2)$.

We set for $\mathcal{I} \subseteq [|1,6|]$,

$$\tilde{\sigma}(\mathcal{I}) = \{l \in [|1,6|] \mid \exists l' \in \mathcal{I}, j, j',\ \sigma(l',j') = (l,j)\}$$

and for $l \in [|1,6|]$,

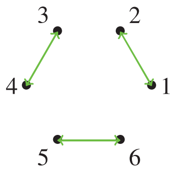
$$o(l) = \bigcup_n \tilde{\sigma}^n(\{l\}).$$

We call $o(l)$ the orbit of l. Of course, l is not an element of $[|1,6|] \times \{1,2\}$ hence this is is an abuse of vocabulary. The set $o(l)$ is the set of all elements $l' \in [|1,6|]$ that one can reach by applying σ several times.

The orbits form a partition of $[|1,6|]$. Since $\sigma(l,1) \neq (l,2)$, each orbit has at least 2 elements. Therefore, there are at most $R/2 = 3$ orbits.

3.2 Examples of orbits and involutions

Involution with 3 orbits



A typical involution of $[|1,6|] \times \{1,2\}$ whose orbits are $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$ as in the above picture is σ_1 given by

$$\begin{array}{lll} \sigma_1(1,1) = (2,2), & \sigma_1(3,1) = (4,2), & \sigma_1(5,1) = (6,2), \\ \sigma_1(2,1) = (1,2), & \sigma_1(4,1) = (3,2), & \sigma_1(6,1) = (5,2). \end{array}$$

Therefore, if

$$\prod_{(l,j)\in S^+_{\sigma_1}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_1(l,j)}}) \neq 0 \quad (= 1)$$

then, we have

$$\begin{array}{lll} \boxed{\xi_{1,1}} + \xi_{2,2} = 0, & \boxed{\xi_{3,1}} + \xi_{4,2} = 0, & \boxed{\xi_{5,1}} + \xi_{6,2} = 0, \\ \xi_{1,2} + \xi_{2,1} = 0, & \xi_{3,2} + \xi_{4,1} = 0, & \xi_{5,2} + \xi_{6,1} = 0. \end{array}$$

We deduce by summing the equalities, since $\xi_l = \xi_{l,1} + \xi_{l,2}$, that

$$\xi_1 + \xi_2 = 0, \quad \xi_3 + \xi_4 = 0, \quad \xi_5 + \xi_6 = 0.$$

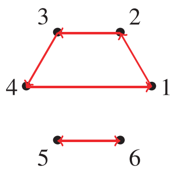
What is more, only three parameters $\xi_{1,1}$, $\xi_{3,1}$ and $\xi_{5,1}$ determine all the $\xi_{l,j}$. We deduce

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1}, \xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma_1}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_1(l,j)}})$$

is a sum on three parameters in $\frac{1}{L}\mathbb{Z}^d_*$, is null if $\xi_1 + \xi_2 \neq 0$ or $\xi_3 + \xi_4 \neq 0$ or $\xi_5 + \xi_6 \neq 0$ and otherwise

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1}, \xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma_1}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_1(l,j)}}) \lesssim 1.$$

Involution with 2 orbits



A typical involution of $[|1,6|] \times \{1,2\}$ whose orbits are $\{1,2,3,4\}$ and $\{5,6\}$ as in the above picture is σ_2 given by

$$\begin{array}{ll} \sigma_2(1,1) = (2,2), & \\ \sigma_2(2,1) = (3,2), & \sigma_2(5,1) = (6,2), \\ \sigma_2(3,1) = (4,2), & \sigma_2(6,1) = (5,2). \\ \sigma_2(4,1) = (1,2), & \end{array}$$

Therefore, if

$$\prod_{(l,j)\in S^+_{\sigma_2}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_2(l,j)}}) \neq 0 \quad (=1)$$

then, we have

$$\begin{aligned} \boxed{\xi_{1,1}} + \xi_{2,2} &= 0, \\ \xi_{2,1} + \xi_{3,2} &= 0, \qquad \boxed{\xi_{5,1}} + \xi_{6,2} = 0, \\ \xi_{3,1} + \xi_{4,2} &= 0, \qquad \xi_{5,2} + \xi_{6,1} = 0. \\ \xi_{4,1} + \xi_{1,2} &= 0, \end{aligned}$$

We deduce by summing the equalities, since $\xi_l = \xi_{l,1} + \xi_{l,2}$, that

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, \quad \xi_5 + \xi_6 = 0.$$

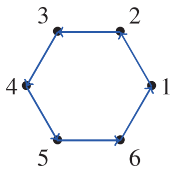
What is more, only two parameters $\xi_{1,1}$ and $\xi_{5,1}$ determine all the $\xi_{l,j}$. We deduce that

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1}, \xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma_2}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_2(l,j)}})$$

is a sum on two parameters in $\frac{1}{L}\mathbb{Z}^d_*$, is null if $\xi_1 + \xi_2 + \xi_3 + \xi_4 \neq 0$ or $\xi_5 + \xi_6 \neq 0$ and otherwise

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1}, \xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma_2}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_2(l,j)}}) \lesssim L^{-d}.$$

Involution with 1 orbit



A typical involution of $[|1,6|] \times \{1,2\}$ whose only orbit is $[|1,6|]$ as in the above picture is σ_3 given by

$$\begin{aligned} \sigma_3(1,1) &= (2,2), \\ \sigma_3(2,1) &= (3,2), \\ \sigma_3(3,1) &= (4,2), \\ \sigma_3(4,1) &= (5,2), \\ \sigma_3(5,1) &= (6,2), \\ \sigma_3(6,1) &= (1,2). \end{aligned}$$

Therefore, if

$$\prod_{(l,j)\in S^+_{\sigma_3}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_3(l,j)}}) \neq 0 \quad (=1)$$

then, we have

$$\begin{aligned} \boxed{\xi_{1,1}} + \xi_{2,2} &= 0, \\ \xi_{2,1} + \xi_{3,2} &= 0, \\ \xi_{3,1} + \xi_{4,2} &= 0, \\ \xi_{4,1} + \xi_{5,2} &= 0, \\ \xi_{5,1} + \xi_{6,2} &= 0, \\ \xi_{6,1} + \xi_{1,2} &= 0. \end{aligned}$$

We deduce by summing the equalities, since $\xi_l = \xi_{l,1} + \xi_{l,2}$, that

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0.$$

What is more, only one parameter $\xi_{1,1}$ determine all the $\xi_{l,j}$. We deduce that

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1}, \xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma_3}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_3(l,j)}})$$

is a sum on one parameter in $\frac{1}{L}\mathbb{Z}^d_*$, is null if $\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 \neq 0$ and otherwise

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1}, \xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma_3}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_3(l,j)}}) \lesssim L^{-2d}.$$

In general, the order in L of

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1}, \xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma(l,j)}})$$

depends only on the number of orbits of σ.

However, the involutions fix conditions on the $(\xi_l)_l$. Namely, if $\xi_1 + \xi_2 = 0$, $\xi_3 + \xi_4 = 0$, $\xi_5 + \xi_6 = 0$ then σ_1, σ_2 and σ_3 will contribute to the expectation. However, if we have only $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ and $\xi_5 + \xi_6 = 0$, then only σ_2 and σ_3 will contribute.

This is summed up in the following proposition.

Proposition 3.1. *Given $(\xi_1, \ldots, \xi_R)$, consider all the partitions of $[|1, R|]$:*

$$[|1, R|] = \bigsqcup_{i\in\mathcal{I}} o_i$$

such that for all $i \in \mathcal{I}$,

$$\sum_{l\in o_i} \xi_l = 0.$$

One has maximal cardinal $\#\mathcal{I}$. Then

$$\mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}^{(j_l)}_{L,\leq N}(t, \xi_l)\Big) = O(L^{d(\#\mathcal{I}-\frac{R}{2})}).$$

As we have already seen $\#\mathcal{I} \leq \frac{R}{2}$ and in case of equality the leading order is

$$\sum_{\sigma\in\underline{\mathfrak{S}_R}} \prod_{l\in S^+_\sigma} \mathbb{E}\Big(\hat{u}^{(j_l)}_{L,\leq N}(t, \xi_l) \hat{u}^{(j_{\sigma(l)})}_{L,\leq N}(t, \xi_{\sigma(l)})\Big)$$

3.3 Idea of proof for the result of the full solution

A traditional way to solve the Cauchy problem for the incompressible Euler equation (or similar equations) is to perform a contraction argument for analytic data or for a regularized problem and extend the result to Sobolev spaces by exploiting the conservation laws of the equation. This is based on compactness and bootstrap arguments. We mention [1].

Here, we solve the problem for analytic data and exploit the Cauchy-Kowalevskaia theorem.

For problems on the torus, see [3], one may consider the following norms

$$\|u\|_\rho = \sum_{k\in\mathbb{Z}^d_*} |\hat{u}(k)| e^{\rho|k|}, \quad \text{or} \quad \|u\|_\rho = \sqrt{\sum_{k\in\mathbb{Z}^d_*} |\hat{u}(k)|^2 e^{\rho|k|}}.$$

However, they behave badly as $L \to \infty$, hence this framework has to be slightly modified.

We define

$$\|u\|_\rho = \sum_{n\in\mathbb{Z}^d} e^{\rho|n|} \|u_n\|_{L^\infty}$$

where u_n is u but localised in frequencies around $n \in \mathbb{Z}^d$ and such that $\sum u_n = u$. The norm $\|\cdot\|_\rho$ is the norm for the initial datum. Indeed, taking $\rho_0 > 0$, we have the following probabilistic property

$$\mathbb{P}(\|\underline{a_L}\|_{\rho_0} \geq A) \leq e^{-c(\rho_0)A^2\varepsilon(L)^{-2}}.$$

As in [6], we define for $\beta \in (0,1)$, $\theta > 0$, $\theta(\rho) = \theta(\rho_0 - \rho)$,

$$\|u\|_{\rho_0,\beta,\theta} = \sup_{\rho\in(0,\rho_0)} \sup_{t\in[0,\theta(\rho))} \Big(\|u(t)\|_\rho + \|\nabla u(t)\|_\rho (\theta(\rho) - t)^\beta \Big).$$

This is the norm in which we perform the contraction argument.

We have bilinear estimates

$$\Big\| \int_0^t P(u(\tau)\cdot\nabla v(\tau)) d\tau \Big\|_{\rho_0,\beta,\theta} \lesssim C(\theta) \|u\|_{\rho_0,\beta,\theta} \|v\|_{\rho_0,\beta,\theta}$$

which allows to perform the contraction argument.

We deduce that there exists $A(\theta)$ such that on

$$\mathcal{E}_{L,\theta} = \{\|\underline{a_L}\|_{\rho_0} \leq A(\theta)\},$$

we have

$$\|\underline{u_{L,n}}\|_{\rho_0,\beta,\theta} \leq 2^{-n} A(\theta).$$

Since we know the dependence of the constant in N in the first theorem, it remains to optimize.

References

[1] Claude Bardos and Said Benachour. Domaine d'analycité des solutions de l'équation d'Euler dans un ouvert de r^n. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 4e série, 4(4):647–687, 1977.

[2] D. J. Benney, Philip Geoffrey Saffman, and George Keith Batchelor. Nonlinear interactions of random waves in a dispersive medium. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 289(1418):301–320, 1966.

[3] Sylvie Benzoni-Gavage, Jean-François Coulombel, and Nikolay Tzvetkov. Ill-posedness of nonlocal Burgers equations. *Adv. Math.*, 227(6):2220–2240, 2011.

[4] R. Brout and I. Prigogine. Statistical mechanics of irreversible processes part viii: general theory of weakly coupled systems. *Physica*, 22(6–12):621 – 636, 1956.

[5] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah. Onset of the wave turbulence description of the longtime behavior of the nonlinear Schrödinger equation. *Invent. Math.*, 225(3):787–855, 2021.

[6] Russel E. Caflisch. A simplified version of the abstract Cauchy-Kowalewski theorem with weak singularities. *Bull. Amer. Math. Soc. (N.S.)*, 23(2):495–500, 1990.

[7] Charles Collot and Pierre Germain. On the derivation of the homogeneous kinetic wave equation. *arXiv preprint arXiv:1912.10368*, 2019.

[8] Charles Collot and Pierre Germain. Derivation of the homogeneous kinetic wave equation: longer time scales. *arXiv preprint arXiv:2007.03508*, 2020.

[9] Anne-Sophie de Suzzoni. General remarks on the propagation of chaos in wave turbulence and application to the incompressible euler dynamics. *arXiv eprints 2206.14744*, 2022.

[10] Yu Deng and Zaher Hani. Full derivation of the wave kinetic equation. *arXiv eprints 2104.11204*, 2021.

[11] Yu Deng and Zaher Hani. On the derivation of the wave kinetic equation for NLS. *Forum Math. Pi*, 9:Paper No. e6, 37, 2021.

[12] Yu Deng and Zaher Hani. Propagation of chaos and the higher order statistics in the wave kinetic theory. *arXiv eprints 2110.04565*, 2021.

[13] Yu Deng and Zaher Hani. Derivation of the wave kinetic equation: full range of scaling laws, 2023.

[14] Andrey Dymov and Sergei Kuksin. On the Zakharov-L'vov stochastic model for wave turbulence. *Dokl. Math*, 101:102–109, 2020.

[15] Andrey Dymov and Sergei Kuksin. Formal expansions in stochastic model for wave turbulence 1: Kinetic limit. *Comm. Math. Phys.*, 382(2):951–1014, 2021.

[16] Andrey Dymov and Sergei Kuksin. Formal expansions in stochastic model for wave turbulence 2: Method of diagram decomposition. *J. Stat. Phys.*, 190(1):Paper No. 3, 42, 2023.

[17] Erwan Faou. Linearized wave turbulence convergence results for three-wave systems. *Communications in Mathematical Physics*, 378, 09 2020.

[18] K. Hasselmann. On the non-linear energy transfer in a gravity-wave spectrum part 1. general theory. *Journal of Fluid Mechanics*, 12(4):481–500, 1962.

[19] K. Hasselmann. On the non-linear energy transfer in a gravity wave spectrum part 2. conservation theorems; wave-particle analogy; irrevesibility. *Journal of Fluid Mechanics*, 15(2):273–281, 1963.

[20] Jani Lukkarinen and Herbert Spohn. Weakly nonlinear Schrödinger equation with random initial data. *Invent. Math.*, 183(1):79–188, 2011.

[21] Sergey Nazarenko. *Wave turbulence*, volume 825 of *Lecture Notes in Physics*. Springer, Heidelberg, 2011.

[22] R. Peierls. Zur kinetischen theorie der wärmeleitung in kristallen. *Annalen der Physik*, 395(8):1055–1101, 1929.

[23] Gigliola Staffilani and Minh-Binh Tran. On the wave turbulence theory for stochastic and random multidimensional KdV type equations. *arXiv eprints 2106.09819*, 2021.

[24] V. E. Zakharov. Weak turbulence in media with a decay spectrum. *Journal of Applied Mechanics and Technical Physics*, 6(4):22–24, July 1965.

[25] V. E. Zakharov and N.N. Filonenko. Energy spectrum for stochastic oscillations of the surface of a liquid. *Dokl. Akad. Nauk SSSR*, 170:1292–1295, 1966.