Asymptotic behavior toward radially symmetric stationary solutions of the compressible Navier-Stokes equation

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1. ABSTRACT

This research is a joint work with Professor Shinya Nishibata, Souhei Sugizaki of Tokyo institute of technology and Akitaka Matsumura of Osaka university. The present talk is concerned with the existence and asymptotic behaviors of radially symmetric stationary solutions for the compressible Navier-Stokes equation, describing the motion of viscous barotropic gas without external forces, where boundary and far field data are prescribed on the exterior domain in $\mathbb{R}^n, n \geq 3$. We clear that for both inflow and outflow problems, there exist non trivial stationary solution, and for outflow problem we show that the stationary wave are asymptotic stable in a suitably small neighborhood of the initial data. Furthermore, detailed decay rate of the stationary solutions are derived.

2. INTRODUCTION

In this talk, we consider the compressible Navier-Stokes equation which describes a barotropic motion of viscous gas in the exterior domain Ω to a ball in $\mathbb{R}^n$ $(n \geq 2)$:

$$\begin{cases} \rho_t + \mathrm{div}(\rho U) = 0, \\ (\rho U)_t + \mathrm{div}(\rho U \otimes U) + \nabla p = \nu \triangle U + (\nu + \lambda)\nabla(\mathrm{div} U), \quad t > 0,\ x \in \Omega, \end{cases} \tag{1}$$

where $\Omega = \{x \in \mathbb{R}^n\,(n \geq 2); |x| > 1\}$, $\rho = \rho(t,x) > 0$ is the mass density, $U = (u_1(t,x), \cdots, u_n(t,x))$ is the fluid velocity, and $p = p(\rho)$ is the pressure given by a smooth function of ρ satisfying $p'(\rho) > 0$ $(\rho > 0)$. Furthermore, ν and λ are the viscosity coefficients. In this talk, we focus our attention on the radially symmetric solutions, which have the form

$$\rho(t,x) = \rho(t,r), \quad U(t,x) = \frac{x}{r}u(t,r), \quad r = |x|, \tag{2}$$

where $u(t,r)$ is a scalar function. By plugging (2) to (1), we can rewrite (1) as in the form

$$\begin{cases} (r^{n-1}\rho)_t + (r^{n-1}\rho u)_r = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_r + (n-1)\dfrac{\rho u^2}{r} = \mu\Big(\dfrac{(r^{n-1}u)_r}{r^{n-1}}\Big)_r, \quad t > 0,\ r > 1, \end{cases} \tag{3}$$

where $\mu = 2\nu + \lambda > 0$. Now, we consider the initial boundary value problems to (3) under the initial condition

$$(\rho, u)(0,r) = (\rho_0, u_0)(r), \quad r > 1, \tag{4}$$

the far field condition

$$\lim_{r\to\infty}(\rho, u)(t,r) = (\rho_+, u_+), \quad t > 0, \tag{5}$$

and also the following two types of boundary conditions depending on the sign of the velocity on the boundary

$$\begin{cases} (\rho, u)(t, r_0) = (\rho_b, u_b), \quad t > 0, \quad (u_b > 0), \\ u(t, r_0) = u_b, \quad t > 0, \quad\quad\quad (u_b \leq 0), \end{cases} \tag{6}$$

where $\rho_b, \rho_+ > 0$, u_b and u_+ are given constants.

The case $u_b > 0$ is known as "inflow problem", the case $u_b = 0$ as "impermeable wall problem", and the case $u_b < 0$ as "outflow problem".

When the problems are multi-dimensional ($n \geq 2$), Jiang [2] and Nakamura-Nishibata-Yanagi [8] studied the case $u_\pm = 0$. They study more general compressible Navier-Stokes equation, describing the motion of viscous polytropic ideal gas. Jiang [2] first showed the global asymptotic stability of the constant states, and later Nakamura-Nishibata-Yanagi [8] extended the results to the case with external potential forces. Recently, Hashimoto-Matsumura [4] treated the multi-dimensional problems in more general cases $u_b \neq 0$ and they showed the existence of the radially symmetric stationary solution in a suitably small neighborhood of the far field state.

3. MAIN RESULT

In this talk, we consider the asymptotic stability of the radially symmetric stationary solution obtained in Matsumura-H [5]. The stationary solution $(\tilde{\rho}, \tilde{u})$ is a solution to (3)-(6) which is independent of the time variable. Thus, $(\tilde{\rho}, \tilde{u})$ satisfy the equations written as

$$\begin{cases} (r^{n-1}\tilde{\rho}\tilde{u})_r = 0, \\ \tilde{\rho}\tilde{u}\tilde{u}_r + p(\tilde{\rho})_r = \mu\left(\dfrac{(r^{n-1}\tilde{u})_r}{r^{n-1}}\right)_r, \qquad r \geq 1, \\ \lim_{r\to\infty} (\tilde{\rho}, \tilde{u})(r) = (\rho_+, u_+), \\ (\tilde{\rho}, \tilde{u})(r_0) = (\rho_b, u_b) \ \ (u_b > 0), \quad \tilde{u}(r_0) = u_b \ \ (u_b \leq 0). \end{cases} \tag{7}$$

From the first equation in (7), we easily see it holds

$$r^{n-1}\rho(r)u(r) = m, \qquad r \geq 1, \tag{8}$$

for some constant m, and it also holds from the boundary conditions that

$$m = \rho_b u_b \quad (u_b > 0), \qquad m = \rho(1)u_b \quad (u_b \leq 0),$$

where note that in the case $u_b \leq 0$, m includes the unknown $\rho(1)$ which should be determined later. The formula (8) implies that if $n \geq 2$,

$$u_+ = \lim_{r\to\infty} u(r) = \lim_{r\to\infty} \frac{m}{r^{n-1}\rho_+} = 0.$$

Hence, we need to assume $u_+ = 0$ for the existence of multi-dimensional stationary solutions. Now we state the statement of existence Theorem.

Theorem 3.1. *Let $n \geq 2$ and $u_+ = 0$. Then, for any $\rho_+ > 0$, there exist positive constants ϵ_0 and C satisfying the following:*

(I) *Let* $u_- > 0$. *If* $|u_-| + |\rho_- - \rho_+| \le \epsilon_0$, *there exists a unique smooth solution* (ρ, u) *of the problem* (7) *satisfying*

$$|\rho(r) - \rho_+| \le Cr^{-2(n-1)}(|u_-|^2 + |\rho_- - \rho_+|),$$

$$C^{-1}r^{-(n-1)}|u_-| \le |u(r)| \le Cr^{-(n-1)}|u_-|, \qquad r \ge r_0.$$

Furthermore, for any positive constant h, *there exists a positive constant* C_h *such that it holds*

$$\sup_{r \ge r_0 + h} |\rho(r) - \rho_+| \le C_h |u_-|^2. \tag{9}$$

(II) *Let* $u_- \le 0$. *If* $|u_-| \le \epsilon_0$, *there exists a unique smooth solution* (ρ, u) *of the problem* (7) *satisfying*

$$|\rho(r) - \rho_+| \le Cr^{-2(n-1)}|u_-|^2,$$

$$C^{-1}r^{-(n-1)}|u_-| \le |u(r)| \le Cr^{-(n-1)}|u_-|, \qquad r \ge r_0.$$

Now we are ready to state the result for the asymptotic stability of stationary solution obtained for outflow problem in the following Theorem.

Theorem 3.2. *Let* $u_b < 0$, $n \ge 2$ *and* $u_+ = 0$. *We assume that* σ *be an arbitrary constant satisfying* $0 < \sigma < 1$. *Assume that the initial data (4) belongs to the function space*

$$\rho_0 \in \mathcal{C}^{1+\sigma}[1, \infty), \quad u_0 \in \mathcal{C}^{2+\sigma}[1, \infty).$$

Then there exists a constant ϵ_0 *such that if* $|u_-|$, $\|r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_{H^1} \le \epsilon_0$, *then the initial boundary value problem (3)-(6) has a unique solution* $(\rho, u)(t, r)$ *satisfying*

$$(\rho, u) \in \mathcal{C}^{1+\sigma/2, 1+\sigma} \times \mathcal{C}^{1+\sigma/2, 2+\sigma}([0, T] \times [1, \infty)),$$

for an arbitrary $T > 0$ *and* $(\rho - \rho_+, u - u_+) \in C([0, \infty); H^1(\mathbb{R}))$. *Moreover, the solution* $(\rho, u)(t, r)$ *converges to the stationary wave* $(\tilde{\rho}, \tilde{u})(r)$ *as time tends to infinity. Precisely, it holds that*

$$\lim_{t \to +\infty} \sup_{r \in \mathbb{R}_+} |(\rho, u)(t, r) - (\tilde{\rho}, \tilde{u})(r)| = 0.$$

In the proof on the Theorems, we use spacial weighted energy method proposed by Nakamura-Nishibata-Yanagi [8] and decay rate of stationary solution. $\mathcal{C}^{k+\sigma}$ denotes the Hölder space of continuous functions which have the kth order derivatives of Höölder continuity with exponent σ.

Remark 1. *When* $u_- = 0$, *by* (3)-(5), *we easily see the solution is the trivial constant state* $(\rho, u) \equiv (\rho_+, 0)$. *Therefore, when* $u_- > 0$ *and* $\rho_- \ne \rho_+$, *the estimate* (11) *shows a boundary layer for mass density does appear as* $u_- \to +0$.

4. Preliminary and property of the stationary solution

In this section we translate the problem (7) and we use the symbol $u(r)$ and $\rho(r)$ as the stationary solution of $\tilde{u}(r)$ and $\tilde{\rho}(r)$ for less confusion. We introduce the specific volume v by $v = 1/\rho$ (accordingly, denote $v_\pm$ by $1/\rho_\pm$). Then, by (8)), the velocity u is given in terms of v as

$$u(r) = \frac{m}{r^{n-1}} v(r), \quad r \geq 1, \tag{10}$$

where $m = u_b/v_b$ ($u_b > 0$), and $m = u_b/v(1)$ ($u_b \leq 0$). Substituting (10) into the second equation of (7), we have

$$\frac{m^2}{r^{n-1}} \left(\frac{v}{r^{n-1}} \right)_r + \tilde{p}(v)_r = m\mu \left(\frac{v_r}{r^{n-1}} \right)_r, \tag{11}$$

where $\tilde{p}(v) := p(1/v)$, and it holds $\tilde{p}'(v) < 0\,(v > 0)$ by the assumption on $p(v)$. Now, we further introduce a new unknown function η, as the deviation of v from the far field state v_+, by

$$\eta(r) = v(r) - v_+, \qquad r \geq 1. \tag{12}$$

Plugging (12) into (11), we have

$$m\mu \left(\frac{\eta_r}{r^{n-1}} \right)_r = \tilde{p}(v_+ + \eta)_r + \frac{m v_+}{2} \left(\frac{1}{r^{2(n-1)}} \right)_r + \frac{m^2}{r^{n-1}} \left(\frac{\eta}{r^{n-1}} \right)_r, \tag{13}$$

where $m = u_b/v_-$ ($u_b > 0$), and $m = u_b/(v_+ + \eta(1))$ ($u_b < 0$). Under the far field condition $\eta(\infty) = 0$, the equation (13) is also equivalent to

$$\left(m\mu \frac{\eta_r}{r^{n-1}} - \tilde{p}(v_+ + \eta) - \frac{m^2 v_+}{2r^{2(n-1)}} - \frac{m^2 \eta}{r^{2(n-1)}} + m^2(n-1) \int_r^\infty \frac{\eta(s)}{s^{2n-1}} ds \right)_r = 0, \tag{14}$$

which implies that the function in the parenthesis in the left hand side of (14) is identically equals to a constant c_0 for $r > 1$. Then, it follows from the far field condition that

$$\lim_{r \to \infty} \frac{\eta_r(r)}{r^{n-1}} = \frac{1}{m\mu} (c_0 + \tilde{p}(v_+)),$$

which concludes $c_0 = -\tilde{p}(v_+)$, otherwise it contradicts the far field condition again. Thus, we finally have the following reformulated problem in terms of η:

$$\begin{cases} \eta_r = \dfrac{r^{n-1}}{m\mu} \big(\tilde{p}(v_+ + \eta) - \tilde{p}(v_+) \big) \\[2mm] \qquad + \dfrac{m v_+}{2\mu} \dfrac{1}{r^{n-1}} + \dfrac{m\eta}{\mu r^{n-1}} - \dfrac{m(n-1) r^{n-1}}{\mu} \displaystyle\int_r^\infty \frac{\eta(s)}{s^{2n-1}} ds, \quad r > 1, \\[2mm] \displaystyle\lim_{r \to \infty} \eta(r) = 0, \\[2mm] \eta(1) = \eta_b := v_b - v_+ \ (u_b > 0), \quad \textit{no boundary condition} \ (u_b < 0), \end{cases} \tag{15}$$

where $m = u_b/v_-$ ($u_b > 0$), and $m = u_b/(v_+ + \eta(1))$ ($u_b < 0$). Once the desired solution η of (15) is obtained, the velocity u is immediately obtained by (10) as

$$u(r) = \frac{u_b(v_+ + \eta(r))}{v_- r^{n-1}} \quad (u_b > 0), \quad u(r) = \frac{u_b(v_+ + \eta(r))}{(v_+ + \eta(1)) r^{n-1}} \quad (u_b < 0).$$

The existence theorem for the reformulated problem (15) is the following.

Theorem 4.1 (Hashimoto-Matsumura [5]). *Let $n \geq 2$. Then, for any $v_+ > 0$, there exist positive constants ϵ_0 and C satisfying the following:*

(I) *Let $u_b > 0$. If $|u_b| + |\eta_b| \leq \epsilon_0$, there exists a unique smooth solution η of the problem* (15) *satisfying*

$$|\eta(r)| \leq Cr^{-2(n-1)}(|u_b|^2 + |\eta_b|), \qquad r \geq 1.$$

(II) *Let $u_b < 0$. If $|u_b| \leq \epsilon_0$, there exists a unique smooth solution η of the problem* (15) *satisfying*

$$|\eta(r)| \leq C_0 r^{-2(n-1)}|u_b|^2, \qquad r \geq 1.$$

Theorem 4.1 ensures that if $|u_b|$ is sufficiently small, then it holds that

$$\frac{v_+}{2} \leq v_+ + \eta(1) \leq \frac{3}{2}v_+.$$

In addition to the Theorem 4.1, we obtain the following lemma for the property of stationary solution.

Lemma 4.2. *Let $u_- \leq 0$ and $n \geq 2$. Then, for any $v_+ > 0$, if $|u_b| \ll 1$ we obtain the following properties.*

$$\eta(r) > 0 \quad , \eta_r(r) < 0, \qquad \eta_{rr}(r) > 0 \quad (1 \leqq r < \infty),$$
$$|\eta_r(r)| \leqq C_1 r^{-2n+1}|u_b|^2, \qquad |\eta_{rr}| \leqq C_2 r^{-2n}|u_b|^2.$$

$$u(r) < 0 \quad , u_r > 0 \quad , u_{rr} < 0 \quad (1 \leqq r < \infty)$$
$$C_3^{-1} r^{-n+1}|u_b| \leqq |u(r)| \leqq C_3 r^{-n+1}|u_b|,$$
$$|u_r| \leqq C_4 r^{-n}|u_b|, \quad |u_{rr}| \leqq C_5 r^{-n-1}|u_b|,$$
$$|\rho(r) - \rho_+| \leqq C_6 r^{-2n+2}|u_b|^2,$$
$$\left|\frac{\partial}{\partial r}(\rho(r) - \rho_+)\right| \leqq C_7 r^{-2n+1}|u_b|^2, \quad \left|\frac{\partial^2}{\partial r^2}(\rho(r) - \rho_+)\right| \leqq C_8 r^{-2n}|u_b|^2,$$

where,$C_1, \cdots, C_8$ are positive constants independent of r and $|u_b|$.

5. The energy estimates

In this section, we derive the a priori estimate. $\tilde{u}(r)$ and $\tilde{\rho}(r)$ stand for the stationary solution for (7) here. In order to prove the stability result in Theorem 3.2, it is convenient to regard the solution (ρ, u) as a perturbation from the stationary solution $(\tilde{\rho}, \tilde{u})$. Thus, we define new unknown functions as

$$\phi(r,r) := \rho(t,r) - \tilde{\rho}(r), \qquad \psi(t,r) := u(t,r) - \tilde{u}(r).$$

Subtracting (7) from (3) yields that

$$\begin{cases} \phi_t + u\phi_r + \rho\psi_r = F, \\ \rho(\psi_t + u\psi_r) + P'(\rho)\phi_r - \mu\psi_{rr} = G, \end{cases} \tag{16}$$

where

$$\begin{cases} F := -\tilde{\rho}_r\psi - \tilde{u}_r\phi - \frac{n-1}{r}(\phi u + \tilde{\rho}\psi) \\ G := -(\phi\psi + \tilde{u}\phi + \tilde{r}o\psi)\tilde{u}_r - (P'(\rho) - P'(\tilde{\rho}))\tilde{\rho}_r + \mu(n-1)(\frac{\psi}{r})_r \end{cases} \tag{17}$$

The initial and boundary conditions to the system (16) are derived from (4) and (6),

$$\begin{aligned}&\phi(0,r)=\phi_0(r):=\rho(r)-\tilde{\rho}(r), \qquad \psi(0,r)=\psi_0(r):=u_0(r)-\tilde{u}(r)\\&\psi(t,0)=0.\end{aligned} \tag{18}$$

The local existence of the solution (ϕ,ψ) to the initial boundary value problem (16) and (18) is stated as follows.(Tani, [9])

Lemma 5.1. *Assume that the same conditions in Theorem 3.2 hold. Then there exists a positive constant T_0, depending only on $|\phi_0|_{1+\sigma}$ and $|\psi_0|_{2+\sigma}$, such that the initial boundary value problem (3) - (6) has a unique solution (ϕ,ψ) in the space:*

$$\begin{aligned}&(\phi,\psi)\in\mathcal{C}^{1+\sigma/2,1+\sigma}\times\mathcal{C}^{1+\sigma/2,2+\sigma}([0,T_0]\times[1,\infty)),\\&r^{\frac{n-1}{2}}\phi,\ r^{\frac{n-1}{2}}\psi,\ r^{\frac{n-1}{2}}\phi_r,\ r^{\frac{n-1}{2}}\psi_r\in C([0,T_0];L^2(\mathbb{R}_+)),\\&r^{\frac{n-1}{2}}\psi_r\in L^2(0,T_0;L^2(\mathbb{R}_+)).\end{aligned}$$

Next, we proceed to the a priori estimate in the Sobolev space, which is stated in Proposition 5.2. To show this estimate, it is convenient to use notation

$$N(t):=\sup_{0\le\tau\le t}\|(\phi,\psi)(\tau)\|_1^2.$$

Proposition 5.2. *Let (ϕ,ψ) be a solution to (16), (17) and (18) in a time interval $[0,T]$, which has the same regularity as in Lemma 5.1. Then there exist positive constants ε_0 and C, such that if $N(T)\le\varepsilon_0$, then the following estimate holds for an arbitrary $t\in[0,T]$:*

$$\begin{aligned}&\|(\phi,\psi)(t)\|_1^2+\int_0^t\|\phi_r(\tau)\|^2+\|\psi_r(\tau)\|_1^2+|(\phi,\phi_r)(\tau,1)|^2d\tau\\&\le C\int_1^\infty r^{n-1}(\phi_0^2+\psi_0^2dr+\phi_{0,r}^2+\psi_{0,r}^2)dr.\end{aligned} \tag{19}$$

Proof. The proof is divided into three steps, which are stated in Lemmas 5.3, 5.4 and 5.5. Combining the uniform estimates proved in these three Lemmas gives the desired estimate (19). □

The smallness assumption on $N(T)$ in Proposition 5.2 ensures that if ε_0 is sufficiently small, then there exist certain positive constants c_ρ and C_ρ such that

$$0\le c_\rho\le\rho(t,r)\le C_\rho \quad for \quad t\in[0,T].$$

Next, we introduce an energy form by

$$\mathcal{E}:=\frac{1}{2}\psi^2+\int_{\tilde{\rho}}^{\rho}\frac{P(\eta)-P(\tilde{\rho})}{\eta^2}.$$

By using (16), (17) and (18), we see that $\rho\mathcal{E}$ satisfies the equation by the straightforward computation.

$$
\begin{aligned}
&(\rho\mathcal{E})_t + \{\rho u\mathcal{E} + (P(\rho) - P(\tilde{\rho}))\psi - \mu\psi\psi_r - \mu(n-1)\frac{\psi^2}{2r}\}_r \\
&+ \mu\psi_r^2 + \frac{n-1}{2}(\frac{\rho u}{r}\psi^2 + \mu\frac{\psi^2}{r^2}) \\
&+ (n-1)\left\{\frac{\rho u}{r}\int_{\tilde{\rho}}^{\rho}\frac{P(\eta)-P(\tilde{\rho})}{\eta^2}d\eta + \frac{P(\rho)-P(\tilde{\rho})}{r\rho}(\phi u + \tilde{\rho}\psi)\right\} \\
&+ (n-1)\frac{\tilde{\rho}\tilde{u}}{r}\left(\frac{P(\eta)-P(\tilde{\rho})}{\rho} - P'(\tilde{\rho})\frac{\rho-\tilde{\rho}}{\tilde{\rho}}\right) \\
&= -\left(\rho\psi^2 + P(\rho) - P(\tilde{\rho}) - P'(\tilde{\rho})(\rho-\tilde{\rho})\right)\tilde{u}_r - \frac{\mu}{\tilde{\rho}}\phi\psi\left(\frac{(r^{n-1}\tilde{u})_r}{r^{n-1}}\right)_r
\end{aligned}
$$

We are in a position to state the basic energy estimate.

Lemma 5.3. *There exist positive constants* ε_1 *and* C *such that if* $N(T) + |u_b| \leq \varepsilon_1$, *then it holds that*

$$
\begin{aligned}
&\int_1^\infty r^{n-1}\phi^2 + r^{n-1}\psi^2 dr + |u_b|\int_0^t \phi(1,\tau)^2 d\tau \\
&\quad + \int_0^t\int_1^\infty r^{n-3}\psi^2 + r^{n-1}\psi_r^2 + |u_b|r^{-1}\psi^2 dr d\tau \\
&\leq C\int_1^\infty r^{n-1}\phi_0^2 + r^{n-1}\psi_0^2 dr + C|u_b|^3\int_0^t \|\phi_r(\tau)\|^2 d\tau,
\end{aligned}
$$

for an arbitrary $t \in [0,T]$.

Next, we state the L^2-estimates for the first derivatives, ϕ_r and ψ_r. To derive this, we use a difference quotient of a function φ with respect to r defined as

$$
\varphi_h = \varphi_h(r) := \frac{\varphi(r+h) - \varphi(r)}{h}, \qquad for \qquad h > 0.
$$

In addition, the following identity holds

$$
(\varphi\omega)_h = \varphi_h\omega^h + \varphi\omega_h,
$$

where we have defined $\varphi^h := \varphi(r+h)$. The statement of the L^2-estimates for the first derivatives, ϕ_r is the following.

Lemma 5.4. *There exist positive constants ε_2 and C such that if $N(T)+|u_b|\le\varepsilon_2$, then the following estimate holds for any $t\in[0,T]$:*

$$\begin{aligned}&\int_1^\infty r^{n-1}\phi_r^2dr+|u_b|\int_0^t\phi_r(\tau,1)^2d\tau\\&\quad+\int_0^t\int_1^\infty|u_b|\frac{\phi_r^2}{r}+r^{n-1}\phi_r^2+r^{n-1}\tilde{u}_r\phi_r^2drd\tau\\&\le C\int_1^\infty r^{n-1}\phi_0^2+r^{n-1}\psi_0^2dr+\int_1^\infty r^{n-1}\phi_{0,r}^2dr+C|u_b|^3\int_0^t\|\phi_r(\tau)\|^2d\tau\\&\quad+N(t)\int_0^t|u_b|^2\phi(\tau,1)^2d\tau+CN(t)\int_0^t\|\psi_r\|_1^2d\tau\\&\qquad+C(|u_b|+N(t))\int_0^t\int_1^\infty r^{n-1}\phi_r^2+r^{n-3}\psi^2+r^{n-1}\psi_r^2drd\tau.\end{aligned}$$

The statement of the L^2-estimates for the first derivatives, ψ_r is the following.

Lemma 5.5. *There exist positive constants ε_3 and C such that if $N(T)+|u_b|\le\varepsilon_3$, then it holds that for $t\in[0,T]$:*

$$\begin{aligned}&\int_1^\infty r^{n-1}\psi_r^2dr+\int_0^t\int_1^\infty r^{n-1}\psi_{rr}^2drd\tau\\&\le C\int_1^\infty r^{n-1}\phi_0^2+r^{n-1}\psi_0^2dr+\int_1^\infty r^{n-1}(\phi_{0,r}^2+\psi_{0,r}^2)dr+C|u_b|^3\int_0^t\|\phi_r(\tau)\|^2d\tau\\&+C(|u_b|+N(t))\int_0^t\int_1^\infty r^{n-1}\phi_r^2+r^{n-3}\psi^2+r^{n-1}\psi_r^2drd\tau\\&+CN(t)\int_0^t r^{n-1}\psi_{rr}^2d\tau.\end{aligned}$$

Once the Sobolev estimates are obtained, we can proceed to derive the Hölder estimates. A priori estimate in the Sobolev space (19) in Lemma 5.2 ensure the first step of the Höölder estimate:

$$|\rho|_{1/4,1/2}^T\ ,\ |u|_{1/4,1/2}^T<C(E_0), \tag{20}$$

where E_0 is defined by

$$E_0:=\int_1^\infty r^{n-1}(\phi_0^2+\psi_0^2+\phi_{0,r}^2+\psi_{0,r}^2)dr.$$

Once we obtain the basic estimate (20), the higher order estimate in Höölder space is derived by the same strategy as in Kawashima-Nishibata-Zhu [3], Nakamura-Nishibata-Yanagi [8]. The higher order estimate in Höölder space is described as the following proposition.

Proposition 5.6. *Let $0<\sigma<1$. Under the same assumptions in Lemma 5.1, it holds*

$$|\rho|_{1+\frac{\sigma}{2},1+\sigma}^T,\quad |u|_{1+\frac{\sigma}{2},2+\sigma}^T<C(T),$$

where $C(T)$ is a constant depending only on T, $|\rho_0|_{1+\sigma}$, $|u_0|_{2+\sigma}$ and $\|(\phi,\psi)\|_1$.

We can derive the above estimate by using shauder estimate in Lagrangean coordinate. (See [3] and [8] for details.) Finally, combining Lemma 5.1, Proposition 5.2 and Proposition 5.6, we obtain the Theorem 3.2.

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