EVOLUTION OF GAUSSIAN MEASURES AND APPLICATION TO THE ONE DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. In this note, we give an overview of some results obtained in [3], written in collaboration with Nicolas Burq. This latter work is devoted to the study of the one-dimensional nonlinear Schrödinger equation with random initial conditions. Namely, we describe the nonlinear evolution of Gaussian measures and we deduce global well-posedness and scattering results for the corresponding nonlinear Schrödinger equation.

1. Introduction

The motivation of this work is to study the long time behaviour of the nonlinear Schrödinger equation with random initial conditions

$$\begin{cases} i\partial_s U + \partial_y^2 U = |U|^{p-1}U, & (s,y) \in \mathbb{R}\times\mathbb{R}, \\ U(0) = U_0 \in L^2(\mathbb{R}). \end{cases}$$

We will be able to prove:

- almost sure global existence results ($p > 1$)
- almost sure scattering results ($p > 3$).

To prove these results, we will first construct measures on the space of initial data for which we can describe precisely the non trivial evolution by the linear Schrödinger flow. Then we prove that the nonlinear evolution of these measures is absolutely continuous with respect to their linear evolutions, with quantitative estimates on the Radon-Nikodym derivative. To the best of our knowledge, these results are the first ones giving insight, in a non compact setting on the time evolution of the statistical distribution of solutions of a nonlinear PDE. They also are the first ones providing scattering for NLS for large initial data without assuming decay at infinity: our solutions are in a slighly larger space than $L^2(\mathbb{R})$.

1.1. **On invariant measures.**

To begin with, let us recall the definition of a measure left invariant by a one parameter group.

Definition 1.1. *Consider a space X and a one parameter group $(\Phi(t,.))_{t\in\mathbb{R}}$ with*

$$\Phi(t,.) : X \longrightarrow X.$$

A measure μ defined in the space X is called invariant with respect to $(\Phi(t,.))_{t\in\mathbb{R}}$ if for any $\mu-$measurable set A one has

$$\mu\big(\Phi(t,A)\big) = \mu(A), \quad t \in \mathbb{R}.$$

In the case where μ is a probability measure, the Poincaré theorem applies:

Theorem 1.2 (Poincaré theorem)**.** *Let $(X,\mathcal{B},\mu)$ be a probability space and let $\Phi(t,.) : X \longrightarrow X$ be a one parameter group which preserves the probability measure μ.*

(*i*) *Let* $A \in \mathcal{B}$ *be such that* $\mu(A) > 0$*, then there exists* $k \geq 1$ *such that*

$$\mu\big(A \cap \Phi(k, A)\big) > 0.$$

(*ii*) *Let* $B \in \mathcal{B}$ *be such that* $\mu(B) > 0$*, then for* μ*-almost all* $x \in B$*, the orbit* $\big(\Phi(n,x)\big)_{n\in\mathbb{N}}$ *enters infinitely many times in* B*.*

In the case of ordinary differential equations, the Liouville theorem provides a condition so that the Lebesgue measure (possibly with a density) is invariant by the flow of the system. Namely, let $\Omega \subset \mathbb{R}^d$ be an open set and $F : \Omega \longrightarrow \mathbb{R}^d$ a C^∞ function. Consider the ordinary differential equation

$$\begin{cases} \dot{x}(t) = \dfrac{dx}{dt}(t) = F(x(t)), \\ x(0) = x_0. \end{cases}$$

We assume that for all $x_0 \in \mathbb{R}$ the system has a unique solution $\Phi(t, x_0)$, such that $\Phi(0, x_0) = x_0$ and which is defined for all $t \in \mathbb{R}$. The family $(\Phi(t, .)_{t\in\mathbb{R}}$ is a one parameter group of diffeomorphisms such that $\Phi(0, .) = id$, $\Phi(t, \Phi(s, .)) = \Phi(t + s, .)$ for all $s, t \in \mathbb{R}$.

Theorem 1.3 (Liouville theorem)**.** *Denote by* dx *the Lebesgue measure on* Ω *and let* $g : \Omega \longrightarrow [0, +\infty)$ *a* $\mathcal{C}^\infty$ *function. The flowmap* $\Phi(t, .)$ *preserves the measure* gdx *if and only if*

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\big(gF_k\big) = 0.$$

An important class of examples is given by finite dimensional Hamiltonian systems of equations.

1.2. **Invariant measures for the Schrödinger equation on compact manifolds.** Let M a compact manifold. Then there exists a Hilbert basis $(h_n)_{n\geq 0}$ of $L^2(M)$, composed of eigenfunctions of Δ_M and we write

$$-\Delta_M h_n = \lambda_n^2 h_n \quad \text{for all} \quad n \geq 0.$$

Next, consider a probability space $(\Omega, \mathcal{F}, \mathbf{p})$ and let $(g_n)_{n\geq 0}$ be a sequence of independent complex standard Gaussian variables $\mathcal{N}_{\mathbb{C}}(0, 1)$

$$g_n = \frac{1}{\sqrt{2}}(g_{1,n} + i g_{2,n}), \qquad g_{1,n}, g_{2,n} \in \mathcal{N}_{\mathbb{R}}(0, 1).$$

Finally, let $(\alpha_n)_{n\geq 0}$ and define the probability measure μ via the map

$$\omega \quad \longmapsto \quad \gamma(\omega) = \sum_{n=0}^{+\infty} \alpha_n g_n(\omega) h_n, \qquad \mu = \mathbf{p} \circ \gamma^{-1} = \gamma_{\#}\mathbf{p},$$

in other words: for all measurable set A, the measure μ is defined by

$$\mu(A) = \mathbf{p}\big(\omega \in \Omega : \gamma(\omega) \in A\big). \tag{1.1}$$

It is then easy to observe that for any choice of $(\alpha_n)_{n\geq 0}$ the measure μ is invariant by the flow of the linear Schrödinger equation:

Proposition 1.4. *The measure* μ *defined in* (1.1) *is invariant under the flow of the equation*

$$i\partial_s U + \Delta_M U = 0, \quad (s, y) \in \mathbb{R} \times M.$$

Considering the nonlinear equation, it is then natural to look for invariant measures (invariant Gibbs measures, see [2] for example) or quasi-invariant measures (measures for which the nonlinear evolution is absolutely continuous with respect to μ. In this direction, we refer to the works [17, 15, 13, 14]).

Let us sketch the proof of Proposition 1.4, since it is elementary.

Proof. For all $t \in \mathbb{R}$, the random variable

$$e^{it\Delta_M}\gamma(\omega) = \sum_{n=0}^{+\infty} \alpha_n e^{-it\lambda_n^2 t} g_n(\omega) h_n$$

has the same distribution as γ since for all $t \in \mathbb{R}$

$$e^{-it\lambda_n^2 t} g_n(\omega) \sim \mathcal{N}_{\mathbb{C}}(0,1),$$

hence the measure μ defined in (1.1) is invariant by the linear flow. □

1.3. **The non compact case.** Let us now turn to the case of the Schrödinger equation posed on $\mathbb{R}$. Here the situation is dramatically different, since one has:

Proposition 1.5 ([3], Proposition 3.1)**.** *Let $\sigma \in \mathbb{R}$ and consider a probability measure μ on $H^\sigma(\mathbb{R})$. Assume that μ is invariant under the flow Σ_{lin} of equation*

$$\begin{cases} i\partial_s U + \partial_y^2 U = 0, & (s,y) \in \mathbb{R}\times\mathbb{R}, \\ U(0,\cdot) = U_0. \end{cases}$$

Then $\mu = \delta_0$.

By [3, Proposition 3.2 and Proposition 3.3], similar results hold true for the nonlinear equation

$$i\partial_s U + \partial_y^2 U = |U|^{p-1}U.$$

Proof. Let us give the main lines of the argument. Let $\sigma \in \mathbb{R}$ and consider μ an invariant probability measure on $H^\sigma(\mathbb{R})$. Let $\chi \in C_0^\infty(\mathbb{R})$. By invariance of the measure,

$$\int_{H^\sigma(\mathbb{R})} \frac{\|\chi u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} d\mu(u) = \int_{H^\sigma(\mathbb{R})} \frac{\|\chi \Sigma_{lin}(t)u\|_{H^\sigma}}{1+\|\Sigma_{lin}(t)u\|_{H^\sigma}} d\mu(u),$$

and by unitarity of the linear flow in $H^\sigma(\mathbb{R})$, we get

$$\int_{H^\sigma(\mathbb{R})} \frac{\|\chi u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} d\mu(u) = \int_{H^\sigma(\mathbb{R})} \frac{\|\chi \Sigma_{lin}(t)u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} d\mu(u). \tag{1.2}$$

Assume that the r.h.s. of (1.2) tends to 0 when $t \to +\infty$. This implies that $\|\chi u\|_{H^\sigma} = 0$, $\mu-$a.s., and thus $\mu = \delta_0$ since χ is arbitrary.

By continuity of the product by χ in $H^\sigma(\mathbb{R})$ and unitarity of the linear flow in $H^\sigma(\mathbb{R})$, we have

$$\frac{\|\chi\Sigma_{lin}(t)u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} \leq C\frac{\|\Sigma_{lin}(t)u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} = C\frac{\|u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} \leq C.$$

If $v \in \mathcal{C}_0^\infty(\mathbb{R})$ is smooth, by the Leibniz rule and dispersion

$$\|\chi\Sigma_{lin}(t)v\|_{H^\sigma} \leq \|\chi\|_{W^{\sigma,4}} \|\Sigma_{lin}(t)v\|_{W^{\sigma,4}} \leq Ct^{-1/4}\|v\|_{W^{\sigma,4/3}} \longrightarrow 0,$$

when $t \longrightarrow +\infty$. We can conclude with an approximation argument. □

1.4. **Some functional analysis.** Define the harmonic oscillator

$$H = -\partial_x^2 + x^2 .$$

There exists a Hilbert basis $(e_n)_{n\geq 0}$ of $L^2(\mathbb{R})$, composed of eigenfunctions of H and we write

$$He_n = \lambda_n^2 e_n = (2n+1)e_n \quad \text{for all} \quad n \geq 0.$$

We define the harmonic Sobolev space $\mathcal{W}^{\sigma,p}$ by the norm ($\sigma > 0$)

$$\|u\|_{\mathcal{W}^{\sigma,p}} = \|H^{\sigma/2}u\|_{L^p} \equiv \|(-\Delta)^{\sigma/2}u\|_{L^p} + \|\langle x\rangle^\sigma u\|_{L^p}.$$

1.5. **Definition of the Gaussian measure μ_0.** Let $\epsilon > 0$, we define the probability Gaussian measure μ_0 on $\mathcal{H}^{-\epsilon}(\mathbb{R})$ as the distribution of the random variable γ

$$\begin{array}{rcl} \Omega & \longrightarrow & \mathcal{H}^{-\epsilon}(\mathbb{R}) \\ \omega & \longmapsto & \gamma(\omega) = \displaystyle\sum_{n=0}^{+\infty} \frac{1}{\lambda_n} g_n(\omega) e_n, \end{array} \qquad \mu_0 = \mathbf{p}\circ\gamma^{-1} = \gamma_\#\mathbf{p}.$$

Notice that we can interpret μ_0 as the Gibbs measure of the equation $i\partial_t u - Hu = 0$.

We denote by

$$X^0(\mathbb{R}) = \bigcap_{\epsilon>0} \mathcal{H}^{-\epsilon}(\mathbb{R}).$$

Thus $L^2(\mathbb{R}) \subset X^0(\mathbb{R}) \subset \mathcal{H}^{-\epsilon}(\mathbb{R})$. One can show that the measure μ_0 satisfies $\mu_0(L^2(\mathbb{R})) = 0$ and $\mu_0\big(X^0(\mathbb{R})\big) = 1$, since $\mu_0(\mathcal{H}^{-\epsilon}(\mathbb{R})) = 1$ for all $\epsilon > 0$. This shows that the support of μ_0 is essentially composed of L^2 functions. However, one can prove that the support of μ_0 is actually smoother in other L^p scales. For instance, we have the bound:

$$\mu_0\big(\big\{u_0 \in X^0(\mathbb{R}) : \|e^{-itH}u_0\|_{L^\infty((-\pi,\pi);\mathcal{W}^{1/6-,\infty})} \geq R\big\}\big) \leq Ce^{-cR^2}.$$

1.6. **Equivalence of Gaussian measures.** Let μ and ν be two measures. We say that $\mu \ll \nu$ (μ is absolutely continuous with respect to ν) if $\nu(A) = 0 \implies \mu(A) = 0$. Now let us consider the particular case of Gaussian measures. Let $\alpha_n, \beta_n > 0$ and define the measures $\mu = \mathbf{p}\circ\gamma^{-1}$ and $\nu = \mathbf{p}\circ\psi^{-1}$ with

$$\gamma(\omega) = \sum_{n=0}^{+\infty} \frac{1}{\alpha_n} g_n(\omega)e_n, \qquad \psi(\omega) = \sum_{n=0}^{+\infty} \frac{1}{\beta_n} g_n(\omega)e_n.$$

Then the measures μ and ν are absolutely continuous with respect to each other (this means that they have the same zero measure sets) if and only if

$$\sum_{n=0}^{+\infty} (\frac{\alpha_n}{\beta_n} - 1)^2 < +\infty. \tag{1.3}$$

This criterion shows that it is very restrictive for two infinite-dimensional Gaussian measures to be absolutely continuous with respect to each other: the condition (1.3) says in some sense that the map which γ sends on ψ has to be close to the identity. We refer to [3, Appendix B.3] for more details.

More generally, in [3, Section 2.2] we construct a four-parameter family of Gaussian measures based on the symmetries of the Schrödinger equation.

2. Main results

Consider the problem

$$\begin{cases} i\partial_s U + \partial_y^2 U = |U|^{p-1}U, & (s,y) \in \mathbb{R}\times\mathbb{R}, \\ U(0) = U_0 \in X^0(\mathbb{R}). \end{cases} \tag{2.1}$$

2.1. Global existence result.

We are now able to state our global existence result:

Theorem 2.1 ([3], Theorem 2.4)**.** *Let $p > 1$.*

(*i*) *For μ_0-almost every initial data $U_0 \in X^0(\mathbb{R})$, there exists a unique, global in time, solution*

$$U = \Psi(s,0)U_0$$

to (2.1)*.*

(*ii*) *The measures $\Psi(s,0)_\#\mu_0$ and $\Psi_{lin}(s,0)_\#\mu_0$ are equivalent:*

$$\Psi_{lin}(s,0)_\#\mu_0 \ll \Psi(s,0)_\#\mu_0 \ll \Psi_{lin}(s,0)_\#\mu_0. \tag{2.2}$$

(*iii*) *For all $s' \neq s$, the measures $\Psi(s,0)_\#\mu_0$ and $\Psi(s',0)_\#\mu_0$ are mutually singular.*

It is interesting to compare this result with the case where NLS is posed on a compact manifold. In this latter case, the linear flow satisfies $\widetilde{\Psi}_{lin}(s,0)_\#\mu_0 = \mu_0$. Hence, in this context it is natural to try to prove a quasi-invariance result of the form $\widetilde{\Psi}(s,0)_\#\mu_0 \ll \mu_0$. But for NLS on the real line, we are able to prove, using criterion (1.3) that $\Psi_{lin}(s,0)_\#\mu_0$ and μ_0 are mutually singular, hence $\Psi(s,0)_\#\mu_0$ should not be compared to μ_0 but to $\Psi(s,0)_\#\mu_0$ as in (2.2).

2.2. The scattering result.

Theorem 2.2 ([3], Theorem 2.4)**.**

(*i*) *Assume that $p > 1$. Then for μ_0-almost every initial data $U_0 \in X^0(\mathbb{R})$, there exists a constant $C > 0$ such that for all $s \in \mathbb{R}$*

$$\|\Psi(s,0)U_0\|_{L^{p+1}(\mathbb{R})} \leq \begin{cases} C\frac{(1+\log\langle s\rangle)^{1/(p+1)}}{\langle s\rangle^{\frac12-\frac{1}{p+1}}} & \text{if } 1<p<5 \\ \frac{C}{\langle s\rangle^{\frac12-\frac{1}{p+1}}} & \text{if } p\geq 5. \end{cases}$$

(*ii*) *Assume now that $p > 3$. Then there exist $\eta > 0$ and $W_\pm \in L^2(\mathbb{R})$ such that for all $s \in \mathbb{R}$*

$$\|\Psi(s,0)U_0 - e^{is\partial_y^2}(U_0 + W_\pm)\|_{L^2(\mathbb{R})} \leq C\langle s\rangle^{-\eta}.$$

For all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ we have the dispersion bound

$$\|e^{is\partial_y^2}\varphi\|_{L^{p+1}(\mathbb{R})} \leq \frac{C}{|s|^{\frac12-\frac{1}{p+1}}}\|\varphi\|_{L^{(p+1)'}(\mathbb{R})}, \qquad s\neq 0,$$

therefore, the power decay in s is optimal in the case $p \geq 5$ (we do not know if the log is necessary in the case $1 < p < 5$). We also stress that the condition $p > 3$ is optimal in our scattering result, by [1].

We conclude this section by giving a few references on scattering results for NLS.

- Deterministic scattering results for NLS:

Barab [1]	$\longrightarrow$ never scattering when $p \leq 3$
Tsutumi–Yajima [18]	$\longrightarrow$ scattering in L^2 with H^1 data
Nakanishi [11]	$\longrightarrow$ scattering in H^σ
Dodson [8]	$\longrightarrow$ scattering in L^2 when $p = 5$

- Probabilistic scattering results for NLS:

Burq–Thomann–Tzvetkov [4]	$\longrightarrow$ case $d = 1$ and $p \geq 5$
Poiret–Robert–Thomann [16]	$\longrightarrow$ case $d \geq 2$ and $p \geq 3$
Dodson–Lührmann–Mendelson [9]	$\longrightarrow$ case $d = 4$ and $p = 3$
Killip–Murphy–Visan [10]	$\longrightarrow$ case $d = 4$ in the radial setting
Latocca [12]	$\longrightarrow$ case $d \geq 2$ in the radial setting

For results on the Gross-Pitaevskii equation with random perturbations we refer to the works of de Bouard, Debussche and Fukuizumi [5, 6, 7] and references therein.

3. Some key ingredients of the proof

3.1. **The lens transform: compactification in time and space.** There is an explicit transform, called the lens transform, which maps the solutions of NLS to solutions of NLS with harmonic potential. Namely, if $U(s, y)$ is a solution of the problem (2.1), then the function $u(t, x)$ defined for $|t| < \frac{\pi}{4}$ and $x \in \mathbb{R}$ by

$$u(t,x) = \mathscr{L}(U)(t,x) := \frac{1}{\cos^{\frac{1}{2}}(2t)} U\big(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\big) e^{-\frac{ix^2 \tan(2t)}{2}}$$

solves the problem

$$\begin{cases} i\partial_t u - Hu = \cos^{\frac{p-5}{2}}(2t)|u|^{p-1}u, \quad |t| < \dfrac{\pi}{4}, \, x \in \mathbb{R}, \\ u(0, \cdot) = U_0. \end{cases} \tag{3.1}$$

We define the corresponding energy

$$\mathcal{E}(t, u(t)) = \frac{1}{2}\|\sqrt{H}\, u(t)\|^2_{L^2(\mathbb{R})} + \frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}, \tag{3.2}$$

which is not conserved. For $-\frac{\pi}{4} < t < \frac{\pi}{4}$, we define the measure

$$d\nu_t = e^{-\mathcal{E}(t,u)} du d\overline{u} = e^{-\frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})}} d\mu_0$$

which is therefore not invariant by the flow of (3.1).

3.2. **Monotonicity of the measure ν_t.** We are able to bound the nonlinear evolution of ν_0 by ν_t. More precisely, we have:

Proposition 3.1. *For all* $0 \leq |t| < \frac{\pi}{4}$

$$\nu_0(\Phi(t,0)^{-1}A) \leq \begin{cases} \big[\nu_t(A)\big]^{\left(\cos(2t)\right)^{\frac{5-p}{2}}} & \textit{if } 1 \leq p \leq 5 \\ \nu_t(A) & \textit{if } p \geq 5. \end{cases} \tag{3.3}$$

These quantitative estimates will be in the core of our argument. In particular, they allow to extend the globalisation argument of Bourgain relying on invariant measures (see Section 3.4). They are also crucial in the proof of the scattering result. The monotonicity estimate (3.3) has already been used in [4] in the case $p \geq 5$ in order to prove global well posedness results for NLS. Notice that in the case $1 \leq p \leq 5$, the estimates (3.3) are quite accurate for small times, but they deteriorate when $|t|$ is close to $\pi/4$: in this regime we recover the trivial bound $\nu_0(\Phi(t,0)^{-1}A) \leq 1$.

We give here an outline of the proof of Proposition 3.1. Recall the definition (3.2), then a direct computation shows that

$$\frac{d}{dt}\big(\mathcal{E}(t,u(t))\big)=\frac{(5-p)\sin(2t)\cos^{\frac{p-7}{2}}(2t)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}.$$

Next, set $F(t)=\nu_t\big(\Phi(t,0)A\big)$. Then we have to prove that for all $0\leq |t|<\frac{\pi}{4}$

$$F(0)\leq\begin{cases}\big[F(t)\big]^{\big(\cos(2t)\big)^{\frac{5-p}{2}}} & \text{if } 1\leq p\leq 5\\ F(t) & \text{if } p\geq 5.\end{cases}$$

We compute

$$\frac{d}{dt}F(t)=(p-5)\tan(2t)\int_A \alpha\big(t,u(t)\big)e^{-\mathcal{E}(t,u(t))}du_0,$$

where $\alpha(t,u)=\frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})}$. Using the Hölder inequality, we can check that for all $k\geq 1$

$$\frac{d}{dt}F(t)\leq (p-5)\tan(2t)\frac{k}{e}\big(F(t)\big)^{1-\frac{1}{k}}.$$

Next, optimizing with $k=-\log\big(F(t)\big)$ yields

$$\frac{d}{dt}F(t)\leq -(p-5)\tan(2t)\log\big(F(t)\big)F(t).$$

Finally the result follows from the integration of the previous differential inequality.

3.3. **On Radon-Nikodym derivatives.** The bounds obtained in Proposition 3.1 say much more than just an absolute continuity result between two measures. In fact, they provide integrability results on the Radon-Nikodym density, since one has the following general result:

Proposition 3.2 ([3], Proposition 3.5)**.** *Let μ,ν be two finite measures on a measurable space $(X,\mathcal{T})$. Assume that*

$$\mu\ll\nu,$$

and more precisely

$$\exists\, 0<\alpha\leq 1,\quad \exists\, C>0,\quad \forall A\in\mathcal{T},\quad \mu(A)\leq C\nu(A)^{\alpha}. \tag{3.4}$$

By the Radon-Nikodym theorem, there exists a $f\in L^1(d\nu)$ with $f\geq 0$, such that $d\mu=fd\nu$, and we write $f=\dfrac{d\mu}{d\nu}$.

(*i*) *The assertion* (3.4) *is satisfied with $0<\alpha<1$ iff $f\in L^p_w(d\nu)\cap L^1(d\nu)$ with $p=\frac{1}{1-\alpha}$. In other words, $f\in L^1(d\nu)$ and*

$$\nu\big(\{x:\ |f(x)|\geq\lambda\}\big)\leq C'\langle\lambda\rangle^{-p},\qquad \forall\,\lambda>0.$$

(*ii*) *The assertion* (3.4) *is satisfied with $\alpha=1$ iff $f\in L^\infty(d\nu)\cap L^1(d\nu)$.*

3.4. **The Bourgain argument revisited.** Let us now show how local in time solutions can be extended, using the bound (3.3). In order to simplify the presentation of the following argument, we assume that $p \geq 5$. Moreover, we do not give details on the norm $\| \cdot \|$ below, since it does not really play a role in the method. Thus, let us assume the three following facts:

▶ There exists a flow Φ such that the time of existence τ on the ball

$$B_R = \big\{ u \in X^0(\mathbb{R}) \;:\; \|u\| \leq R^{1/2} \big\},$$

is uniform and such that $\tau \sim R^{-\kappa}$ for some $\kappa > 0$.

▶ For all $|t| \leq \tau$

$$\Phi(t,0)\big(B_R\big) \subset \big\{ u \in X^0(\mathbb{R}) \;:\; \|u\| \leq (R+1)^{1/2} \big\}.$$

▶ We have the large deviation estimate $\mu_0(X^0(\mathbb{R}) \backslash B_R) \leq Ce^{-cR}$.

Then for $T \leq e^{cR/2}$ fixed, we define the set of the good data

$$\Sigma_R = \bigcap_{k=-[T/\tau]}^{[T/\tau]} \Phi(k\tau, 0)^{-1}\big(B_R\big).$$

By Proposition 3.1 we have

$$\begin{aligned} \nu_0(X^0(\mathbb{R})\backslash \Sigma_R) &\leq \sum_{k=-[T/\tau]}^{[T/\tau]} \nu_0\Big(\Phi(k\tau,0)^{-1}\big(X^0(\mathbb{R})\backslash B_R\big)\Big) \\ &\leq \sum_{k=-[T/\tau]}^{[T/\tau]} \nu_{k\tau}\big(X^0(\mathbb{R})\backslash B_R\big). \end{aligned}$$

In his original argument, Bourgain [2] considered invariant measures, so that the previous estimate was an indeed equality in his case. We observe here that the monotonicity property (3.3) is sufficient. Next, by definition of ν_t, we have $\nu_t(A) \leq \mu_0(A)$, so that

$$\begin{aligned} \nu_0(X^0(\mathbb{R})\backslash \Sigma_R) &\leq \big(2[T/\tau]+1\big)\mu_0\big(X^0(\mathbb{R})\backslash B_R\big) \\ &\leq ce^{-cR/2} \end{aligned}$$

which shows that Σ_R is a big set of $X^0(\mathbb{R})$ when $R \longrightarrow +\infty$.

We deduce that for all $|t| \leq T$ and $u \in \Sigma_R$

$$\|\Phi(t,0)(u)\| \leq (R+1)^{1/2}.$$

In particular, for $|t| = T \sim e^{cR/2}$

$$\|\Phi(t,0)(u)\| \leq C(\ln|t|+1)^{1/2}.$$

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