# ON THE $\Phi_3^3$-MEASURE

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ABSTRACT. This is a résumé of the construction part of the $\Phi_3^3$-measure of the paper "Stochastic quantization of the $\Phi_3^3$-model" by Oh, Tolomeo, and the author. In this note, we give an outline of the proof of the normalizability of the $\Phi_3^3$-measure in the weakly nonlinear regime.

## 1. INTRODUCTION

This is a résumé of the construction part of the $\Phi_3^3$-measure in [17] by Oh, Tolomeo, and the author. The (non-)construction of focusing Gibbs measures initiated Lebowitz, Rose, and Speer [15]. They studied the focusing $\Phi_1^p$-measure on the one dimensional case:

$$d\rho(u) = Z^{-1} \exp\left(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx\right) d\mu(u),$$

where $\mu$ denotes the periodic Wiener measure on the torus $\mathbb{T}$ and $Z$ is the normalized constant. Since the potential part $\int_{\mathbb{T}} |u|^p dx$ is unbounded from above, this measure is not normalizable for $p > 2$. Lebowitz, Rose, and Speer [15] proposed to consider the following two options to construct the focusing measure:

- an $L^2$-cutoff formulation:

$$d\rho(u) = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}} |u|^2 dx \le K\}} \exp\left(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx\right) d\mu(u), \tag{1.1}$$

  where $\mathbf{1}_S$ denotes the indicator function of a set $S$.
- a taming by the $L^2$-norm:

$$d\rho(u) = Z^{-1} \exp\left(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx - A\Big(\int_{\mathbb{T}} u^2 dx\Big)^q\right) d\mu(u) \tag{1.2}$$

  for some $A > 0$ and $q = q(p)$. The expression (1.2) is referred to as the generalized grand-canonical Gibbs measure.

Lebowitz, Rose, and Speer [15] proved the following in (1.1):

- normalizability for $\Phi_1^p$-measure if either of the following holds:
  - $2 < p < 6$ and $K > 0$;
  - $p = 6$ and $0 < K < \|Q\|_{L^2}^2$, where $Q$ is the optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ such that $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|Q'\|_{L^2(\mathbb{R})}^2$.

- non-normalizability for $\Phi_1^p$-measure if either of the following holds:
  - $p > 6$ and $K > 0$;
  - $p = 6$ and $K > \|Q\|_{L^2}^2$.

In a recent work [19], Oh, Sosoe, and Tolomeo proved that the focusing $L^2$-critical Gibbs measure $\rho$ in (1.1) (with $p = 6$) is indeed constructible at the optimal mass threshold $K = \|Q\|_{L^2(\mathbb{R})}^2$. This completes the program in the one-dimensional case. See Carlen, Fröhlich, and Lebowitz [10] for the construction of the generalized grand-canonical Gibbs measure.

When $d \geq 2$, the support of the Wiener measure is larger than $L^2(\mathbb{T}^d)$. A proper renormalization is needed to define the potential part (even for the defocusing case). It is known that the Wick renormalization is enough in the two-dimensional case. Indeed, in [5], Bourgain reported Jaffe's construction of a $\Phi_2^3$-measure endowed with a Wick-ordered $L^2$-cutoff:

$$d\rho = Z^{-1} \mathbf{1}_{\{\int_{\mathbb{T}^2} :u^2:\, dx \leq K\}} e^{\frac{1}{3}\int_{\mathbb{T}^2} :u^3:\, dx} d\mu(u),$$

where $:u^2:$ and $:u^3:$ denote the Wick powers of $u$, and $\mu$ denotes the massive Gaussian free field on $\mathbb{T}^2$. See also [18]. Moreover, in [5], Bourgain instead constructed the following generalized grand-canonical formulation of the $\Phi_2^3$-measure:

$$d\rho(u) = Z^{-1} e^{\frac{1}{3}\int_{\mathbb{T}^2} :u^3:\, dx - A\left(\int_{\mathbb{T}^2} :u^2:\, dx\right)^2} d\mu(u)$$

for sufficiently large $A > 0$.

Brydges and Slade [8] showed that the focusing Gibbs measure $\rho$ with the quartic interaction is not normalizable as a probability measure. See also [18] for an alternative proof. Furthermore, the same non-normalizability applies for higher order interaction.

Oh, Tolomeo, and the author [17] studied the three-dimensional case. More precisely, they considered the following generalized grand-canonical formulation of the $\Phi_3^3$-measure

$$d\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, dx - A\left|\int_{\mathbb{T}^3} :u^2:\, dx\right|^\gamma\right) d\mu(u) \qquad (1.3)$$

for suitable $A, \gamma > 0$. They proved that the following phase transition occurs.

**Theorem 1.1** (Theorem 1.1 in [17]). *The following phase transition holds for the $\Phi_3^3$-measure in* (1.3).

(i) (weakly nonlinear regime). *Let $0 < |\sigma| \ll 1$ and $\gamma = 3$. Then, by introducing a further renormalization, the $\Phi_3^3$-measure $\rho$ in* (1.3) *exists as a probability measure, provided that $A = A(\sigma) > 0$ is sufficiently large. In this case, the resulting $\Phi_3^3$-measure $\rho$ and the massive Gaussian free field $\mu$ on $\mathbb{T}^3$ are mutually singular.*

(ii) (strongly nonlinear regime). *When* $|\sigma| \gg 1$, *the* $\Phi^3_3$*-measure in* (1.3) *is not normalizable for any* $A > 0$ *and* $\gamma > 0$. *Furthermore, the truncated* $\Phi^3_3$*-measures* $\rho_N$ *do not have a weak limit, as measures on* $\mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$, *even up to a subsequence.*

This result says that the $\Phi^3_3$-model is critical in terms of the measure construction. In the weakly nonlinear regime, the $\Phi^3_3$-measure $\rho$ is constructed only as a weak limit of the truncated $\Phi^3_3$-measures.

As for the non-normalizability result in Theorem 1.1 (ii), the proof is based on a refined version of the machinery introduced in [16] and [18], which was in turn inspired by the work of Tolomeo and Weber [22] on the non-construction of the Gibbs measure for the focusing cubic nonlinear Schrödinger equation (NLS) on the real line, giving an alternative proof of Rider's result [21]. However, there is an additional difficulty in proving Theorem 1.1 (ii) due to the singularity of the $\Phi^3_3$-measure with respect to the base massive Gaussian free field $\mu$.

**Remark 1.2.** When $\gamma = 3$, the first part $\int_{\mathbb{T}^3} :u^3: dx$ and the taming part $\big| \int_{\mathbb{T}^3} :u^2: dx\big|^3$ in (1.3) does not have the same scaling property. However, we need $\|u\|^2_{H^1}$ and $\|u\|^6_{L^2}$ to control $\|u\|^3_{L^3}$, since the Gagliardo-Nirenberg inequality yields that

$$\|u\|^3_{L^3} \lesssim \|u\|^{\frac{3}{2}}_{L^2} \|u\|^{\frac{3}{2}}_{\dot{H}^1} \lesssim \|u\|^6_{L^2} + \|u\|^2_{\dot{H}^1}.$$

This is the reason why we choose $\gamma = 3$ in Theorem 1.1 (i). Note that the $H^1$-norm appears when we apply the variational approach as in [2].

## 2. A Hartree-type interaction

We compare Theorem 1.1 to the Gibbs measure with a Hartree-type interaction. Let $V = \langle \nabla \rangle^{-\beta}$ be the Bessel potential of order $\beta > 0$. In [6], Bourgain first constructed the focusing Gibbs measure with a Hartree-type interaction (for complex-valued $u$), endowed with a Wick-ordered $L^2$-cutoff:

$$d\rho(u) = Z^{-1} \mathbf{1}_{\{\int_{\mathbb{T}^3} :|u|^2: dx \le K\}} e^{\frac{\sigma}{4} \int_{\mathbb{T}^3} (V * :|u|^2:) :|u|^2: dx} d\mu(u),$$

for $\beta > 2$. In [16], Oh, Tolomeo, and the author continued the study of the focusing Hartree $\Phi^4_3$-measure in the generalized grand-canonical formulation (with $\sigma > 0$):

$$d\rho(u) = Z^{-1} \exp\bigg( \frac{\sigma}{4} \int_{\mathbb{T}^3} (V * :u^2:) :u^2: \, dx - A \bigg| \int_{\mathbb{T}^3} :u^2: \, dx \bigg|^\gamma \bigg) d\mu(u) \tag{2.1}$$

and established the following phase transition in two respects:

**Theorem 2.1** (Theorem 1.1 in [16]). *Given* $\beta > 0$, *let* $V$ *be the Bessel potential of order* $\beta$. *Let* $\sigma > 0$. *Then, the following statements hold*:

- *Let $\beta > 2$ and* $\max\big(\frac{\beta+1}{\beta-1}, 2\big) \le \gamma < 3$ *with $\gamma > 2$ when $\beta = 3$. Then, the focusing Hartree Gibbs measure $\rho$ in* (2.1) *exists as a limit of the truncated Gibbs measures, provided that $A > 0$ is sufficiently large.*
- *Let $1 < \beta < 2$. Then, the focusing Hartree Gibbs measure $\rho$ in* (2.1) *is not normalizable (i.e. $Z = \infty$) for any $A, \gamma > 0$.*
- (critical case). *Let $\beta = 2$. Then, by choosing $\gamma = 3$, the focusing Hartree Gibbs measure $\rho$ in* (1.3) *exists in the weakly nonlinear regime ($0 < \sigma \ll 1$), provided that $A = A(\sigma) > 0$ is sufficiently large. On the other hand, in the strongly nonlinear regime (i.e. $\sigma \gg 1$), the focusing Hartree Gibbs measure $\rho$ in* (2.1) *is not normalizable for any $\gamma > 0$ and any $A > 0$.*

*Furthermore, when the focusing Hartree Gibbs measure $\rho$ exists, it is equivalent to the base massive Gaussian free field $\mu$.*

We point out that the Gibbs measure is constructed as a strong limit in the theorem above. Theorem 2.1 provides a complete picture on the construction of the Hartree Gibbs measures on $\mathbb{T}^3$, which is of particular interest in the focusing case due to its critical nature at $\beta = 2$. The most important novelty in Theorem 2.1 is the non-normalizability of the focusing Hartree Gibbs measure for (i) $\beta < 2$ or (ii) $\beta = 2$ and $\sigma \gg 1$. See also [22, 18].

In terms of scaling, the focusing Hartree $\Phi_3^4$-model with $\beta = 2$ corresponds to the $\Phi_3^3$-model and as such, they share some common features. For example, they are both critical with a phase transition, depending on the size of the coupling constant $\sigma$. At the same time, however, there are some differences. While the focusing Hartree $\Phi_3^4$-measure with $\beta = 2$ is absolutely continuous with respect to the base massive Gaussian free field $\mu$, the $\Phi_3^3$-measure studied in [17] is singular with respect to the base massive Gaussian free field $\mu$. As mentioned above, this singularity of the $\Phi_3^3$-measure causes an additional difficulty in proving non-normalizability in the strongly nonlinear regime $|\sigma| \gg 1$.

**Remark 2.2.** In the defocusing case ($\sigma < 0$), the Gibbs measure $\rho$ in (2.1) corresponds to the well-studied $\Phi_3^4$-measure when $\beta = 0$ and $A = 0$. The construction of the $\Phi_3^4$-measure is one of the early achievements in constructive Euclidean quantum field theory; see [12, 13, 11, 20, 9, 1, 2, 14]. For an overview of the constructive program with respect to the $\Phi_3^4$-model, see the introductions in [1, 14].

## 3. Construction of the $\Phi_3^3$-measure

In this section, we describe a renormalization procedure and also a taming by the Wick-ordered $L^2$-norm required to construct the $\Phi_3^3$-measure in (1.3) and make a precise statement (Theorem 3.1). For this purpose, we first fix some notations. Let $\mu$ denote a Gaussian measure with the Cameron-Martin space $H^1(\mathbb{T}^3)$, formally defined by

$$d\mu = Z^{-1}e^{-\frac{1}{2}\|u\|^2_{H^1}}du = Z^{-1}\prod_{n\in\mathbb{Z}^3} e^{-\frac{1}{2}\langle n\rangle^2|\widehat{u}(n)|^2}d\widehat{u}(n),$$

where $\langle\,\cdot\,\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$.

Define the index sets $\Lambda$ and $\Lambda_0$ by

$$\Lambda = \bigcup_{j=0}^{2}\mathbb{Z}^j\times\mathbb{N}\times\{0\}^{2-j} \qquad \text{and} \qquad \Lambda_0 = \Lambda\cup\{(0,0,0)\}$$

such that $\mathbb{Z}^3 = \Lambda\cup(-\Lambda)\cup\{(0,0,0)\}$. Then, let $\{g_n\}_{n\in\Lambda_0}$ be a sequence of mutually independent standard complex-valued[1] Gaussian random variables and set $g_{-n} := \overline{g_n}$ for $n\in\Lambda_0$. We now define random distributions $u = u^\omega$ by the following Gaussian Fourier series:

$$u^\omega = \sum_{n\in\mathbb{Z}^3}\frac{g_n(\omega)}{\langle n\rangle}e_n, \tag{3.1}$$

where $e_n = e^{in\cdot x}$. From the condition $g_{-n} = \overline{g_n}$, $u^\omega$ is a real-valued. Denoting by $\mathrm{Law}(X)$ the law of a random variable $X$ (with respect to the underlying probability measure $\mathbb{P}$), we then have

$$\mathrm{Law}(u) = \mu$$

for $u$ in (3.1). Note that $\mathrm{Law}(u) = \mu$ is supported on $H^s(\mathbb{T}^3)$ for $s < -\frac{1}{2}$ but not for $s \geq -\frac{1}{2}$.

We now consider the $\Phi^3_3$-measure. Since $u$ in the support of the massive Gaussian free field $\mu$ is merely a distribution, the cubic potential energy is not well defined and thus a proper renormalization is required to give a meaning to the potential energy. In order to explain the renormalization process, we first study the regularized model.

Given $N\in\mathbb{N}$, we denote by $\pi_N$ the frequency projector onto the (spatial) frequencies $\{n=(n_1,n_2,n_3)\in\mathbb{Z}^3 : \max_{j=1,2,3}|n_j|\leq N\}$, defined by

$$\pi_N f = \sum_{n\in\mathbb{Z}^3}\chi_N(n)\widehat{f}(n)e_n,$$

associated with a Fourier multiplier $\chi_N$:

$$\chi_N(n) = \mathbf{1}_Q\big(N^{-1}n\big),$$

where $Q$ denotes the unit cube in $\mathbb{R}^3$ centered at the origin:

$$Q = \big\{\xi = (\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3 : \max_{j=1,2,3}|\xi_j|\leq 1\big\}.$$

[1] This means that $g_0\sim\mathcal{N}_{\mathbb{R}}(0,1)$ and $\operatorname{Re} g_n, \operatorname{Im} g_n\sim\mathcal{N}_{\mathbb{R}}(0,\frac{1}{2})$ for $n\neq 0$.

Let $u$ be as in (3.1) and set $u_N = \pi_N u$. For each fixed $x \in \mathbb{T}^3$, $u_N(x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\sigma_N = \mathbb{E}\big[u_N^2(x)\big] = \sum_{n\in\mathbb{Z}^3} \frac{\chi_N^2(n)}{\langle n\rangle^2} \sim N \longrightarrow \infty,$$

as $N \to \infty$. Note that $\sigma_N$ is independent of $x \in \mathbb{T}^3$ due to the stationarity of $\mu$. We define the Wick powers $:\!u_N^2\!:$ and $:\!u_N^3\!:$ by setting

$$:\!u_N^2\!: = u_N^2 - \sigma_N \quad \text{and} \quad :\!u_N^3\!: = u_N^3 - 3\sigma_N u_N.$$

This suggests us to consider the following renormalized potential energy:

$$R_N(u) = -\frac{\sigma}{3}\int_{\mathbb{T}^3} :\!u_N^3\!:\, dx + A\bigg|\int_{\mathbb{T}^3} :\!u_N^2\!:\, dx\bigg|^\gamma. \tag{3.2}$$

As in the case of the $\Phi_3^4$-measure in [2], the renormalized potential energy $R_N(u)$ in (3.2) is divergent (as $N \to \infty$) and thus we need to introduce a further renormalization. This leads to the following renormalized potential energy:

$$R_N^\diamond(u) = R_N(u) + \alpha_N, \tag{3.3}$$

where $\alpha_N$ is a diverging constant (as $N \to \infty$). See (3.14) in [17] for the precise definition.

Finally, we define the truncated (renormalized) $\Phi_3^3$-measure $\rho_N$ by

$$d\rho_N(u) = Z_N^{-1} e^{-R_N^\diamond(u)} d\mu(u), \tag{3.4}$$

where the partition function $Z_N$ is given by

$$Z_N = \int e^{-R_N^\diamond(u)} d\mu(u).$$

Then, we have the following construction of the $\Phi_3^3$-measure.

**Theorem 3.1.** *There exist $\sigma_0 > 0$ such that the following statement holds. Let $0 < |\sigma| < \sigma_0$. Then, by choosing $\gamma = 3$ and $A = A(\sigma) > 0$ sufficiently large, we have the uniform exponential integrability of the density*:

$$\sup_{N\in\mathbb{N}} Z_N = \sup_{N\in\mathbb{N}} \Big\| e^{-R_N^\diamond(u)} \Big\|_{L^1(\mu)} < \infty \tag{3.5}$$

*and the truncated $\Phi_3^3$-measure $\rho_N$ in* (3.4) *converges weakly to a unique limit $\rho$, formally given by*

$$d\rho(u) = Z^{-1} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} :\!u^3\!:\, dx - A\bigg|\int_{\mathbb{T}^3} :\!u^2\!:\, dx\bigg|^3 - \infty\bigg) d\mu(u).$$

*In this case, the resulting $\Phi_3^3$-measure $\rho$ and the base massive Gaussian free field $\mu$ are mutually singular.*

As in case of the $\Phi^4_3$-measure in [2],

$$\sup_{N\in\mathbb{N}} \Big\| e^{-R^\diamond_N(u)} \Big\|_{L^p(\mu)} < \infty$$

holds only for $p = 1$ due to the second renormalization introduced in (3.3). See also [16, 7] for a similar phenomenon in the case of the defocusing Hartree $\Phi^4_3$-measure. We point out that the renormalized potential energy $R^\diamond_N(u)$ in (3.3) does *not* converge to any limit and neither does the density $e^{-R^\diamond_N(u)}$, which is essentially the source of the singularity of the $\Phi^3_3$-measure with respect to the massive Gaussian free field $\mu$.

As in [16], following the variational approach introduced by Barashkov and Gubinelli [2], we use the Boué-Dupuis variational formula in [4] and [23] to prove Theorem 3.1. In fact, we make use of the Boué-Dupuis variational formula in almost every single step of the proof.

In proving Theorem 3.1, we first use the variational formula to establish the uniform exponential integrability (3.5) of the truncated density $e^{-R^\diamond_N(u)}$, from which tightness of the truncated $\Phi^3_3$-measure $\rho_N$ in (3.4) follows. Due to the singularity of the $\Phi^3_3$-measure, we need to apply a change of variables in the variational formulation and thus we need to treat the taming part more carefully than that for the focusing Hartree $\Phi^4_3$-measure studied in [16]. This fact also reflects the critical nature of the $\Phi^3_3$-measure.

We prove uniqueness of the limiting $\Phi^3_3$-measure. Our main strategy is to follow the approach introduced in [16] and compare two (arbitrary) subsequences $\rho_{N_{k_1}}$ and $\rho_{N_{k_2}}$, using the variational formula. We point out, however, that, due to the critical nature of the $\Phi^3_3$-measure, our uniqueness argument becomes more involved than that in [16, Subsection 6.3] for the subcritical defocusing Hartree $\Phi^4_3$-measure. In particular, we need to make use of a certain orthogonality property to eliminate a problematic term.

In proving the singularity of the $\Phi^3_3$-measure, we once again follow the direct approach introduced in [16], making use of the variational formula. We point out that the proof of the singularity of the $\Phi^4_3$-measure by Barashkov and Gubinelli [3] goes through the shifted measure. On the other hand, as in [16], our proof is based on a direct argument without referring to shifted measures.

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