Recent results on threshold solutions for the double power nonlinear Schrödinger equations

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Abstract

In this note, we report our recent results in [23], which are concerned with the focusing cubic-quintic nonlinear Schrödinger equation in three space dimensions. Especially, we study the global dynamics of solutions whose energy and mass equal to those of the ground state in a spirit of Duycaerts and Merle [13]. When we try to obtain the corresponding results of [13], we meet several difficulties due to the cubic-quintic nonlinearity. We overcome them by using the one-pass theorem (no return theorem) developed by Nakanishi and Schlag [35].

1 Introduction and main result

In this note, we consider the following nonlinear Schrödinger equation:

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi + |\psi|^4\psi = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^3, \tag{1.1}$$

where Δ is the Laplace operator on $\mathbb{R}^3$. Several studies have been made on the asymptotic behavior of solutions to double power nonlinear Schrödinger equations (see e.g. [1, 6, 7, 19, 20, 21, 22, 25, 26, 27, 29, 31, 32, 36, 37, 38] and references therein). Here, we are concerned with global dynamics of solutions whose mass and energy equal to those of the ground state.

For any $\psi_0 \in H^1(\mathbb{R}^3)$, there exists a unique solution ψ in $C(I_{\max}; H^1(\mathbb{R}^3))$ with $\psi|_{t=0} = \psi_0$ for some interval $I_{\max} = (-T^-_{\max}, T^+_{\max}) \subset \mathbb{R}$, a maximal existence interval including 0. We say that ψ blows up in finite time if $T^+_{\max} < \infty$ or $T^-_{\max} < \infty$. The solution ψ satisfies the following conservation laws of the mass and the energy in this order:

$$\mathcal{M}(\psi(t)) = \mathcal{M}(\psi_0), \qquad \mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0), \tag{1.2}$$

where

$$\mathcal{M}(u) := \frac{1}{2}\|u\|_{L^2}^2, \qquad \mathcal{E}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{4}\|u\|_{L^4}^4 - \frac{1}{6}\|u\|_{L^6}^6 \qquad \text{for } u \in H^1(\mathbb{R}^3)$$

If, in addition, $\psi_0 \in L^2(\mathbb{R}^3, |x|^2dx)$, then the corresponding solution ψ also belongs to $C(I_{\max}; L^2(\mathbb{R}^3, |x|^2dx))$ and satisfies the so-called virial identity:

$$\begin{aligned}\int_{\mathbb{R}^3} |x|^2|\psi(t,x)|^2dx = \int_{\mathbb{R}^3} |x|^2|\psi_0(x)|^2dx + 2t \ \mathrm{Im} \int_{\mathbb{R}^3} x\cdot\nabla\psi_0(x)\overline{\psi_0(x)}dx \\ + 16\int_0^t\int_0^{t'} \mathcal{K}(\psi(t''))dt''dt' \qquad \text{for any } t\in I_{\max},\end{aligned} \tag{1.3}$$

where

$$\mathcal{K}(u) := \|\nabla u\|_{L^2}^2 - \frac{3}{4}\|u\|_{L^4}^4 - \|u\|_{L^6}^6 \qquad \text{for } u\in H^1(\mathbb{R}^3).$$

See e.g. Cazenave [8, Section 6.5] for details.

By a *standing wave*, we mean a solution to (1.1) of the form

$$\psi(t,x) = e^{i\omega t}Q_\omega(x)$$

for some $\omega > 0$ and $Q_\omega \in H^1(\mathbb{R}^3)\setminus\{0\}$. Then, we see that Q_ω should solve the following semilinear elliptic equation:

$$-\Delta Q + \omega Q - |Q|^2Q - |Q|^4Q = 0 \qquad \text{in } \mathbb{R}^3. \tag{1.4}$$

If we define the action functional $\mathcal{S}_\omega$ by

$$\mathcal{S}_\omega(u) := \mathcal{E}(u) + \omega\mathcal{M}(u) \qquad \text{for } u\in H^1(\mathbb{R}^3), \tag{1.5}$$

then $\mathcal{S}'_\omega(Q_\omega) = 0$ in $H^{-1}(\mathbb{R}^3)$ if and only if $Q_\omega \in H^1(\mathbb{R}^3)$ is a weak solution to (1.4). To seek a solution to (1.4), we consider the following minimization problem:

$$m_\omega := \inf\left\{\mathcal{S}_\omega(u) \colon u\in H^1(\mathbb{R}^3)\setminus\{0\},\ \mathcal{K}(u)=0\right\}. \tag{1.6}$$

It is known that if there is a minimizer for m_ω, it satisfies (1.4). Here we call Q_ω a *ground state* to (1.6) provided Q_ω is a minimizer for m_ω. Concerning the existence of a ground state, the following result holds:

Theorem 1.1 ([2, 39])**.** *There exists $\omega_c > 0$ such that m_ω has a ground state for $0 < \omega < \omega_c$ and no ground state for $\omega > \omega_c$.*

Remark 1.1. *We do not know whether the ground state exists or not at $\omega = \omega_c$.*

There are several results on the global dynamics of solutions to nonlinear Schrödinger equations. See e.g. [1, 3, 4, 5, 6, 7, 10, 11, 12, 13, 15, 17, 18, 24, 26, 27, 28, 30, 35, 40, 41] and references therein. Let us recall some of them which are concerned with the following nonlinear Schrödinger equations:

$$i\partial_t\psi + \Delta\psi + |\psi|^{p-1}\psi = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^d. \tag{1.7}$$

where $d\in\mathbb{N}, 1 < p < 2^*-1$ and $2^* = \frac{2d}{d-2}$. The equation (1.7) is scale invariant. More precisely, putting

$$\psi_\lambda(t,x) := \lambda^{\frac{2}{p-1}}\psi(\lambda^2t, \lambda x) \qquad (\lambda > 0), \tag{1.8}$$

we see that if $\psi(t,x)$ satisfies (1.7), so does ψ_λ. The scaling (1.8) preserves the mass $\mathcal{M}$ and the corresponding energy when $p = 1 + 4/d$ and $p = (d+2)/(d-2)$, respectively. Thus, the exponent $p = 1 + 4/d$ is referred to as "mass critical" and $p = (d+2)/(d-2)$ as "energy critical". [1]

It is known that (1.7) has a stationary solution, which neither scatters [2] nor blows up. More precisely, when the energy critical case, (1.7) has the following explicit static solution,

$$W(x) := \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}.$$

The solution W is called by *Aubin-Talenti function.* Similarly, when $1 < p < (d+2)/(d-2)$, the equation (1.7) also has a standing wave $\psi(t,x) = e^{i\omega t}R_\omega$ $(\omega > 0)$. Then, R_ω satisfies the following semilinear equation:

$$-\Delta R + \omega R - |R|^{p-1}R = 0 \qquad \text{in } \mathbb{R}^d. \tag{1.9}$$

For the energy critical case $p = (d+2)/(d-2)$, Kenig and Merle [24] employed the concentration-compactness and showed that the radial solution to (1.7) whose energy is less than that of the Aubin-Talenti function W blows up in finite time or scatters as $t \to \pm\infty$ for $d = 3, 4, 5$. Killip and Visan [28] extended the result of [24] for $d \geq 5$, removing the radial condition. Dodson [10] obtained the corresponding result of [28] for $d = 4$.

For the mass supercritical and the energy subcritical case $1+4/d < p < (d+2)/(d-2)$, Holmer and Roudenko [18] considered the three dimensional cubic nonlinear Schrödinger equation ($d = p = 3$) and proved that the radial solution below the ground state scatters or blows up in finite time. Then, Duyckaerts, Holmer and Roudenko [11] extended the result of [18] to non-radial H^1 initial data. Then, Akahori and Nawa [3] and Fang, Xie and Cazenave [16] extended the result to general dimension and power nonlinearity.

Duyckaerts and Merle [13] studied the threshold solution to the energy critical nonlinear Schrödinger equations, that is, the solution whose energy equals to the Aubin-Talenti function for $d = 3, 4, 5$. They constructed special solutions $W^{\pm}$, which converge to the Aubin-Talenti function W in the positive time direction while W^+ blows up and W^- scatters in the negative time direction, respectively. They also classified the threshold solutions under the radial assumption. Li and Zhang [30] extended the result of [13] to the higher dimensions $d \geq 6$. Duyckaerts and Roudenko [14] studied the threshold solution for the three dimensional cubic nonlinear Schrödinger equations. They also constructed special solutions and classify all solutions (not necessarily radially symmetric) at the threshold level. Recently, Campos, Farah and Roudenko [9] generalized the result

[1] Note that the quintic power nonlinearity $|\psi|^4\psi$ in three space dimensions which is involved in (1.1) corresponds to the energy critical one

[2] Here, we say that a solution scatters if the solution converges to the one of the linear Schrödinger equation

of Duyckaerts and Roudenko [14] to any dimension and any power of the nonlinearity. They also considered the energy critical case and gave an alternative proof of the result of Li and Zhang [30]. See also [4, 5, 6, 12, 17, 32] for the threshold solutions to other nonlinear Schrödinger equations.

In this paper, we address the threshold solution to (1.1). To state our results, we put

$$\begin{aligned} \mathcal{BA}_\omega &:= \left\{u \in H^1_{\mathrm{rad}}(\mathbb{R}^3) \colon \mathcal{S}_\omega(u) = m_\omega,\ \mathcal{M}(u) = \mathcal{M}(Q_\omega)\right\}, \\ \mathcal{BA}_{\omega,+} &:= \{u \in \mathcal{BA}_\omega \colon \mathcal{K}(u) > 0\}, \\ \mathcal{BA}_{\omega,-} &:= \{u \in \mathcal{BA}_\omega \colon \mathcal{K}(u) < 0\}, \\ \mathcal{BA}_{\omega,0} &:= \{u \in \mathcal{BA}_\omega \colon \mathcal{K}(u) = 0\}. \end{aligned}$$

Clearly, we have $\mathcal{BA}_\omega = \mathcal{BA}_{\omega,-} \cup \mathcal{BA}_{\omega,0} \cup \mathcal{BA}_{\omega,+}$. We can easily find that

$$\mathcal{BA}_{\omega,0} = \left\{e^{i\theta} Q_\omega \colon \theta \in \mathbb{R}\right\}. \tag{1.10}$$

In addition, we see that the sets $\mathcal{BA}_{\omega,\pm}$ and $\mathcal{BA}_{\omega,0}$ are invariant under the flow of (1.1). Then, by a similar argument to [13], we can construct the following special solutions to (1.1):

Theorem 1.2. *There exists a sufficiently small $\omega_* > 0$ such that for $\omega \in (0, \omega_*)$, (1.1) has two radial solutions $Q_\omega^+ \in \mathcal{BA}_{\omega,+}$ and $Q_\omega^- \in \mathcal{BA}_{\omega,-}$ satisfying the following:*

(i) $Q_\omega^\pm$ exists on $[0,\infty)$, and there exist constants $e_\omega, C_\omega > 0$ such that

$$dist_{H^1}(Q_\omega^\pm(t), \mathcal{O}(Q_\omega)) \le C_\omega e^{-e_\omega t} \qquad \text{for all } t \ge 0,$$

where

$$\mathrm{dist}_{H^1}\big(u, \mathcal{O}(Q_\omega)\big) := \inf_{\theta \in \mathbb{R}} \|u - e^{i\theta} Q_\omega\|_{H^1}.$$

(ii) $\mathcal{K}(Q_\omega^-) < 0$ and the negative time of existence of Q_ω^- is finite.

(iii) $\mathcal{K}(Q_\omega^+) > 0$, Q_ω^+ exists on $(-\infty,\infty)$ and scatters for negative time, that is, there exists $\phi_- \in H^1(\mathbb{R}^3)$ such that

$$\lim_{t\to-\infty} \|Q_\omega^+(t) - e^{it\Delta}\phi_-\|_{H^1} = 0.$$

We say that $\psi = \phi$ up to the symmetries if there exists $t_0 \in \mathbb{R}$ and θ_0 such that

$$\psi(t,x) = e^{i\theta_0}\phi(t + t_0, x) \qquad \text{or} \qquad \psi(t,x) = e^{i\theta_0}\overline{\phi}(-t + t_0, x).$$

Our main result in [23] is as follows:

Theorem 1.3. *Let $\omega_* > 0$ be a constant given in Theorem 1.2 and ψ be a radial solution to (1.1) with $\psi|_{t=0} = \psi_0 \in \mathcal{BA}_\omega$ for $\omega \in (0, \omega_*)$. Then, the following holds:*

(i) If $\psi_0 \in \mathcal{BA}_{\omega,-}$*, then either* ψ *blows up in finite time or* $\psi = Q_\omega^-$ *up to the symmetries.*

(ii) If $\psi_0 \in \mathcal{BA}_{\omega,0}$*, then* $\psi = e^{i\omega t}Q_\omega$ *up to the symmetries.*

(iii) If $\psi_0 \in \mathcal{BA}_{\omega,+}$*, then either* ψ *scatters or* $\psi = Q_\omega^+$ *up to the symmetries.*

Remark 1.2. *(i) Nakanishi and Schlag [35] proved that the solutions whose energy is slightly larger than that of the ground state is classified into 9 sets (combination of blows up, scattering and trapped by the ground state generated by the phase for* $t > 0$ *and* $t < 0$*). See also [1, 15, 26, 33, 34] for the global dynamics above the ground state. In particular, it was studied in [1] that the behavior of the solutions* ψ *to* (1.1) *with* $\psi|_{t=0} = \psi_0$ *satisfying* $\mathcal{S}_\omega(\psi_0) < m_\omega + \varepsilon$ *are also classified into the 9 sets. However, it seems that from the result of [1], we could not determine the behavior of solutions by the initial data as in Theorem 1.3. Another difference between the result of [1] and ours is that we obtain a kind of uniqueness of solution which converges to the orbit of the ground state*

(ii) We may extend our results to general dimensions and power nonlinearities by using the argument of [9]. However, for simplicity, we restrict ourselves to three space dimension and cubic-quintic nonlinearity.

The proof of Theorem 1.3 is based on that of [13, 14]. However, it seems that due to the cubic and quintic nonlinearities, some part of the argument in [13, 14] does not work for our equation (1.1). For example, in [14], a Cauchy-Schwarz type inequality plays an important role (see [14, Claim 5.4] in detail). In contrast, it seems difficult to obtain a corresponding inequality for our equation (1.1). To overcome the difficulty, we employ the one-pass theorem (no return theorem) which was introduced by Nakanishi and Schlag [35] for the equation (1.7) with $d = p = 3$. Roughly speaking, one-pass theorem states that if a solution moves away from a neighborhood of the ground states, then the solution never return to the neighborhood. We employ the one-pass theorem to prove that if a threshold solution neither blows up nor scatters, the solution converges to the ground state exponentially.

Remark 1.3. *(i) The reason why we need the radially symmetry for solutions is due to the one-pass theorem. Indeed, a kind of Ogawa-Tsutsumi's saturated virial identity was used for the proof of the one-pass theorem. Except for the theorem, we do not require the condition.*

(ii) Recently, Ardila and Murphy [6] studies the threshold solutions to the following cubic-quintic nonlinear Schrödinger equation:

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi - |\psi|^4\psi = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^3. \tag{1.11}$$

Note that the quintic power nonlinearity is defocusing, which is different from our equation (1.1) *and any solutions to* (1.11) *are global* ($T^{\pm}_{\max} = \infty$)*. They also classified the threshold solutions, which are not necessary radially symmetric, to* (1.11)*. Let* ψ *be the threshold solution whose sign of the virial functional (the one corresponding to* $\mathcal{K}$*) is positive. Then, they showed that the solution either* ψ *scatters in both time directions and coincide with a special solution. To this end, they employed the modulation analysis and the concentration-compactness method. Their method might work for our equation* (1.1)*. However, we would like to stress that we can study the threshold solutions which blow up or scatter in a unified way by using the one-pass theorem.*

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