Scaling limits for random processes from the point of view of group cohomology

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Women in Mathematics
Based on joint work with Kenichi Bannai and Yukio Kametani

Motivation

Problem : Derive macroscopic dynamics from a microscopic stochastic process

General story

- $(S_t)_{t\geq 0}$: Microscopic stochastic process
- $(S_t^\varepsilon)_{t\geq 0}$: Properly scaled stochastic process in space and time with scaling parameter $\varepsilon > 0$
- $(\bar{S}_t)_{t\geq 0} = \lim_{\varepsilon\downarrow 0} (S_t^\varepsilon)_{t\geq 0}$: Macroscopic dynamics

Key ingredient for the convergence
Homogenization (averaging) in space and time : Microscopic state space has some homogeneity

Motivation

One particle model

- Example 1 : Random walk
  - $(S_t)_{t\geq 0}$: Discrete time/Continuous time simple random walk on $\mathbb{Z}^d$
  - $S_t^\varepsilon := \varepsilon S_{\varepsilon^{-2}t}$
  - $(\bar{S}_t)$: Brownian motion in $\mathbb{R}^d$ with the diffusion matrix $A = (a_{jk})_{j,k=1}^d$, $a_{jk} = \frac{1}{d}\delta_{jk}$
- Example 2 : Diffusion process in $\mathbb{R}^d$ with periodic coefficient
  - $G(x) = (g_{jk}(x))$: smooth positive definite, one-periodic $\Leftrightarrow$ Riemannian metric on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $m(x)dx$: Riemannian volume measure
  - $(S_t)_{t\geq 0}$: Diffusion process in $\mathbb{R}^d$ with the generator $L = \frac{1}{2}\frac{1}{m(x)}\frac{\partial}{\partial x_j}\left(m(x)g^{jk}(x)\frac{\partial}{\partial x_k}\right)$ (Laplace-Beltrami operator)
  - $S_t^\varepsilon := \varepsilon S_{\varepsilon^{-2}t}$
  - $(\bar{S}_t)$: Brownian motion in $\mathbb{R}^d$ with a constant diffusion matrix $A = (a_{jk})_{j,k=1}^d$ given by an implicit form and also a variational formula

Geometric interpretation

Example 1 : Random walk

- Microscopic geometric object : $(\mathbb{Z}^d, \mathbb{E}^d, p)$: weighted graph with the periodic weight $p : \mathbb{E}^d \to \mathbb{R}_{>0}$, $p(\pm e_j) = \frac{1}{2d}$
- Generator of the "Brownian motion" of the microscopic space : $Lf(x) = \sum_{e \in E_x} p(e)(f(te) - f(oe)) =$ $\frac{1}{2d} \sum_{j=1}^{d} (f(x + e_j) + f(x - e_j) - 2f(x))$
- Macroscopic geometric object : $(\mathbb{R}^d, G = (g_{jk}))$: Riemannian mannifold with the constant metric $g_{jk} = d\delta_{jk}$
- Generator of "Brownian motion" of the macroscopic space : $Lf = \frac{1}{2d} \sum_{j=1}^{d} \frac{\partial^2}{\partial u_j^2} f = \frac{1}{2} \Delta_G f$

Convergence of a geometric space with Riemannian structure and some homogeneity!

Motivation : Geometric interpretation

Example 2 : Diffusion process in $\mathbb{R}^d$ with periodic coefficient

- Microscopic geometric object : $(\mathbb{R}^d, G = (g_{jk}(x)))$: Riemannian mannifold with a periodic metric $g_{jk}(x)$
- Generator of the "Brownian motion" of the microscopic space : $L = \frac{1}{2} \frac{1}{m(x)} \frac{\partial}{\partial x_j} \left(m(x) g^{jk}(x) \frac{\partial}{\partial x_k} \right)$
- Macroscopic geometric object $(\mathbb{R}^d, \bar{G} = (\bar{g}_{jk}))$: Riemannian mannifold with a constant metric $\bar{g}_{jk}$
- Generator of "Brownian motion" of the macroscopic space : $Lf = \frac{1}{2} \sum_{j,k=1}^{d} \bar{g}^{jk} \frac{\partial^2}{\partial x_j \partial x_k} f = \frac{1}{2} \Delta_{\bar{G}} f$

Convergence of a geometric space with Riemannian structure and some homogeneity!

Period matrix and the macroscopic diffusion coefficient

For both examples, $G = \mathbb{Z}^d$ acts on the microscopic geometric object, and $(H^1(X,\mathbb{R}))^G \cong H^1(G,\mathbb{R}) \cong \mathbb{Z}^d$ holds.

Period matrix

- In the class of one-forms of microscopic geometric objects, there is a topological basis $d\theta_1, \dots d\theta_d \in (H^1(X,\mathbb{R}))^G$.
- Once we introduce a Riemannian structure, which induces an inner product $\langle \cdot, \cdot \rangle$ in the class of one-forms, there is a harmonic basis $H_1, \dots, H_d \in (H^1(X,\mathbb{R}))^G$ so that $\langle d\theta_j, H_k \rangle = \delta_{jk}$.
- The change-of-basis matrix from $d\theta_1, \dots d\theta_d$ to $H_1, \dots, H_d$ is called a period matrix.

Geometric interpretation of the macroscopic diffusion matrix

For these examples and more general random walks on periodic lattices, the macroscopic diffusion matrix A is the inverse of the period matrix. In other words, the period matrix is the "Riemannian" metric of the macroscopic geometric object.

Motivation

Our goal : Generalize these ideas to the case for a microscopic system with many particles!

Interacting particle systems

- Example 3 : Exclusion processes
  - $(\eta_t)_{t \geq 0}$: Continuous time Markov process on $\{0,1\}^{\mathbb{Z}^d}$
  - $Lf(\eta) = \sum_{x,y \in \mathbb{Z}^d} r_{x,y}(\eta)(f(\eta^{x,y}) - f(\eta))$ where $\eta^{x,y}$ is obtained from by exchanging η_x and η_y
  - $\pi : \{0,1\}^{\mathbb{Z}^d} \to \mathcal{M}(\mathbb{R}^d) : \langle \pi(\eta), f \rangle := \sum_{x \in \mathbb{Z}^d} \eta_x f(x)$
  - $\pi^\varepsilon : \{0,1\}^{\mathbb{Z}^d} \to \mathcal{M}(\mathbb{R}^d) : \langle \pi^\varepsilon(\eta), f \rangle := \varepsilon^d \sum_{x \in \mathbb{Z}^d} \eta_x f(\varepsilon x)$
  - $S_t^\epsilon := \pi^\varepsilon(\eta_{\varepsilon^{-2}t})$.
  - $(\bar{S}_t)$: Deterministic dynamic given by $\bar{S}_t = \rho(t,u)du$ where $\rho(t,u)$ is the solution of the diffusion equation

$$\partial_t \rho = \sum_{j,k=1}^{d} \partial_{u_j}(D_{jk}(\rho)\partial_{u_k}\rho).$$

Can we construct a good microscopic geometric object and understand $D_{jk}(\rho)$ as a period matrix? $\Rightarrow$ Yes!

Remarks

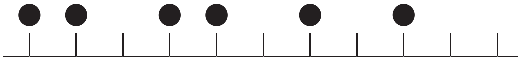
- The convergence of Markov processes $\fallingdotseq$ The convergence of the generator + tightness + the existence and uniqueness of the process with the generators
- The scaling limit of random walks on general periodic lattices (crystal lattices) are not trivial as the case for $\mathbb{Z}^d$ and the discrete harmonic analysis plays a role to describe the macroscopic diffusion matrix.
- The scaling limit like Example 2 is called the homogenization problem. There have been many studies on this topic.
- The scaling limit like Example 3 is called the hydrodynamic limit. There have been many studies on this topic too, but there was not a universal framework to unify different models.
- We introduced a universal framework for the microscopic geometric object.
- The role of the group action was not understood well in the theory of the hydrodynamic limit. (Even not for the one-particle case.)
- By introducing a general framework and its geometric interpretation, we also obtain new hydrodynamic limits for specific models.

Typical example : Exclusion process on $\mathbb{Z}^d$

- $\{0,1\}^{\mathbb{Z}^d}$: State space = Configuration space
- $\eta = (\eta_x) \in \{0,1\}^{\mathbb{Z}^d}$, η_x : number of particle at $x \in \mathbb{Z}^d$
- Exclusion process : Continuous time Markov process $\{\eta(t)\}_{t\geq 0}$ with the generator L

$$Lf(\eta) = \sum_{x,y\in\mathbb{Z}^d} r_{x,y}(\eta)\eta_x(1-\eta_y)\{f(\eta^{x,y}) - f(\eta)\}$$

- Jump rate : $r_{x,y} : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}_{\geq 0}$: "frequency of jump from x to y"



Typical example : Exclusion process on $\mathbb{Z}^d$

We always assume :

- Translation invariant : $r_{x,y}(\eta) = r_{0,y-x}(\tau_{-x}\eta)$
- Locality of interaction : $r_{x,y}$ are local functions
- Finite range : $\exists R > 0$ s.t. $r_{x,y} \equiv 0$ if $\|x-y\| := \sum_{i=1}^d |x_i - y_i| > R$
- Non degenerate $\Rightarrow$ the density of particles ρ characterizes the invariant measures $\{\nu_\rho\}$

Reversible or Mean-zero case : Expected HDL equation

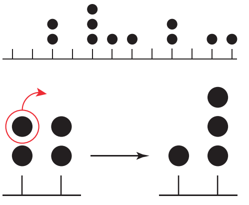
$$\partial_t \rho = \nabla \cdot D(\rho)\nabla\rho = \sum_{i,j=1}^d \partial_{u_i}\left(D_{ij}(\rho)\partial_{u_j}\rho\right)$$

Rigorous results :

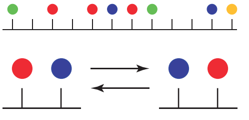
- Symmetric (not necessarily nearest neighbor) simple : $r_{x,y}(\eta) = r_{y,x}(\eta) = c_{x,y}$, $D_{ij}(\rho) = D_{ij} = \sum_{x\in\mathbb{Z}^d} c_{0,x}x_i x_j$
- Reversible and nearest neighbor (Funaki-Uchiyama-Yau (product measure), Varadhan-Yau (non-product measure with mixing condition)) : $D(\rho)$ is given by a variational formula

Other typical microscopic models

Generalized exclusion process : state space $\{0, 1, 2, \ldots, \kappa\}^{\mathbb{Z}^d}$



Multi-color (species) exclusion process : state space $\{0, 1, 2, \ldots, \kappa\}^{\mathbb{Z}^d}$



Open problems of hydrodynamic limits (before our work)

Specific models

- Multi-species exclusion process $\{0, 1, 2, \ldots, \kappa\}^{\mathbb{Z}^d}$
- Energy exchange model $\mathbb{R}_+^{\mathbb{Z}^d}$: Mesoscopic model obtained from some deterministic model

General extensions

- Finite range interaction (not nearest neighbor) models on $\mathbb{Z}^d$, where the underlying graph is $(\mathbb{Z}^d, \mathbb{E}_R^d := \{(x, y) : |x - y| \leq R\})$
- Models on crystal lattices, such as hexagonal lattice, diamond lattice...
- Stationary measures which are not product (except for the exclusion process)

Main result 1 : Framework of microscopic models

Microscopic models are defined by
geometric data and stochastic data

- Geometric (spatial/topological) data : the triple $(S, \phi, \mathcal{X})$
  - Local state space (Set S) (ex. $\{0,1\}, \{0,1,2\}, \mathbb{N}, \mathbb{R}, \mathbb{R}_+$)
  - Local interaction (Map $\phi : S \times S \to S \times S$) (ex. $\phi(s_1, s_2) = (s_2, s_1)$)
  - Underlying spatial space (Graph $\mathcal{X} = (X, E)$) (ex. $(\mathbb{Z}^d, \mathbb{E}^d), (\mathbb{Z}^d, \mathbb{E}^d_R)$, triangular lattice, diamond lattice)
- Stochastic (spatial/metric) data
  - Speed of local interaction $r : \Phi \to \mathbb{R}_{>0}$ (ex. $r_{x,y}(\eta)$))
  - Equilibrium measures : μ (ex. Bernoulli product measures ν_ρ)

Symmetry data also plays an essential role

- Symmetry data : G
  - Symmetry of the underlying space space (Group G acting on $\mathcal{X}$) (ex. $G \cong \mathbb{Z}^d$)

Topological structure constructed by geometric data

Suppose the triple $(S, \phi, \mathcal{X})$ is given.

- The data (S, ϕ) defines the space of conserved quantities $\mathrm{Consv}^\phi(S)$, which is a subspace of function $\{f : S \to \mathbb{R}\}$
- The data $(S, \phi, \mathcal{X})$ defines a graph structure (S^X, Φ), which we call a configuration space with transition structure :
  $\Phi = \{(\eta, \eta^e) \ : \ \eta \in S^X, e \in E\}$.

- We introduce a uniform cohomology on the graph (S^X, Φ)
  - $C^0_{\mathrm{unif}}(S^X)$: set of uniform functions
  - $C^1_{\mathrm{unif}}(S^X)$: set of uniform one forms
  - $\partial : C^0_{\mathrm{unif}}(S^X) \to C^1_{\mathrm{unif}}(S^X)$: differential (usual graph differential)
  - $Z^1_{\mathrm{unif}}(S^X)$: set of uniform closed forms
  - $\partial C^0_{\mathrm{unif}}(S^X)$: set of uniform exact forms
  - $H^0_{\mathrm{unif}}(S^X) := \ker \partial$
  - $H^1_{\mathrm{unif}}(S^X) := Z^1_{\mathrm{unif}}(S^X)/\partial C^0_{\mathrm{unif}}(S^X)$

Main result 2 : Characterization of "smooth" cohomology

- Assumption 1
  - (S, ϕ) is irreducibly quantified ($\sim$ the dynamics is non-degenerate)
  - $\mathcal{X}$ is transferable ($(\mathbb{Z}^d, \mathbb{E}^d_R)$, $d \geq 2$ satisfies the condition)
- Assumption 2
  - (S, ϕ) is simple ($\mathrm{Consv}^\phi(S)$ is the one-dimensional space, and some more)
  - $\mathcal{X}$ is weakly transferable ($(\mathbb{Z}^d, \mathbb{E}^d_R)$, $d \geq 1$ satisfies the condition)

Theorem (Bannai-Kametani-S)

Under the assumptions 1 *or* 2

$$H^0_{\mathrm{unif}}(S^X) \cong \mathrm{Consv}^\phi(S), \quad H^1_{\mathrm{unif}}(S^X) \cong \{0\}.$$

- $\mathcal{X}$ must be an infinite graph under the assumption.

Main result 3 : De Rham cohomology for S^X/G

Assume that a group G acts freely on the locale $\mathcal{X}$.

- Action of G on functions and forms are naturally induced.
  - $\mathcal{E} := \partial(C^0_{\mathrm{unif}}(S^X)^G)$: set of G-invariant uniform exact forms
  - $\mathcal{C} := Z^1_{\mathrm{unif}}(S^X)^G$: set of G-invariant uniform closed forms
  - $H^1(G, \mathrm{Consv}^\phi(S))$: the first group cohomology of G with coefficients in $\mathrm{Consv}^\phi(S)$

Theorem (Bannai-Kametani-S)

Under the assumptions 1 *or* 2

$$\mathcal{C}/\mathcal{E} \cong H^1(G, \mathrm{Consv}^\phi(S)).$$

In particular, if $G \cong \mathbb{Z}^d$, *then*

$$\mathcal{C} \cong \mathcal{E} \oplus \bigoplus_{k=1}^{d} \mathrm{Consv}^\phi(S).$$

Main result 4 : A version of Hodge-Kodaira theorem

- Using the stochastic data, an inner product is defined on $(C^1_{\mathrm{unif}}(S^X))^G$. (analogy to Riemannian metric)
  - $\mathcal{E}_{L^2} := \overline{\partial(C^0_{\mathrm{unif}}(S^X)^G)}$: completion of set of G-invariant uniform exact forms
  - $\mathcal{C}_{L^2} := Z^1_{L^2}(S^X)^G$: set of G-invariant L^2 closed forms
- Assume : S is a finite set and $G \cong \mathbb{Z}^d$, and the induced measure on S^X is product.

Theorem (Bannai-S)

Under the assumptions 1 or 2, and several essential assumptions including above,

$$\mathcal{C}_{L^2} \cong \mathcal{E}_{L^2} \oplus \bigoplus_{k=1}^{d} \mathrm{Consv}^{\phi}(S).$$

New interpretation of the macroscopic diffusion matrix

There are two natural decomposition of closed forms

$$\mathcal{C}_{L^2} \cong \mathcal{E}_{L^2} \oplus \bigoplus_{k=1}^{d} \mathrm{Consv}^{\phi}(S) :$$ topological (Varadhan's) decomposition

$$\mathcal{C}_{L^2} \cong \mathcal{E}_{L^2} \oplus \bigoplus_{k=1}^{d} \mathrm{Consv}^{\phi}(S) :$$ orthogonal decomposition

Diffusion matrix

Macroscopic diffusion matrix $D(\rho)$ = Transition matrix of two different decomposition under the measure ν_ρ = The inverse of the period matrix

Summary

- The decomposition of the space of $(H^1)^G(X)$ or $(H^1)^G(S^X)$ is the key to understand the macroscopic diffusion matrix
- The group cohomology of G play an essential role as $(H^1)^G(S^X)$ can be computed easily once we prove $H^1(S^X) = \{0\}$
- Uniform functions and uniform forms are "smooth functions" on S^X/G
- Hodge-Kodaira theorem is generalized to configuration spaces
- The macroscopic diffusion matrix is the inverse of the period matrix universally.