Problems on Twin Primes, Goldbach's Conjecture, the Riemann Hypothesis and zeros of L-functions in Number Theory

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1 Problems on twin primes

Number Theory has a very long history that dates back thousands of years. The main goal of this study is to understand properties of numbers which essentially can be reduced to understanding prime numbers. Although we have the outstanding *Prime Number Theorem*, more precise information about the distribution of prime numbers is mostly unknown. It is also not known if there are infinitely many pairs of prime numbers having difference 2, the so-called twin prime pairs. In other words, the *Twin Prime Conjecture* states that there are infinitely many prime numbers p such that $p+2$ is also a prime number.

Recent breakthroughs in Analytic Number Theory have succeeded in showing the infinitude of prime pairs with small gaps. Yitang Zhang [Zha14] was the first ever to obtain a quantitative bounded gap between two prime numbers. He proved that there are infinitely many prime pairs (p_1, p_2) such that $|p_1 - p_2| \leq 70\,000\,000$. In a project led by a 2006 Fields Medalist, Terence Tao, called D. H. J. Polymath, it was proven [Pol14a] that Zhang's bound can be drastically improved to 4680. James Maynard, one of this year's Fields Medalists, showed independently in [May15] that there are infinitely many prime pairs (p_1, p_2) with gaps at most 600, that is, such that $|p_1 - p_2| \leq 600$. Under a certain conjecture, the above bound can be improved to 12. In Polymath's later work [Pol14b], Maynard's unconditional bound 600 is improved to 246, and conditionally to 6. These are currently the best known results in this direction.

2 Goldbach's conjecture and relation to twin primes

The 280-year-old *Goldbach's conjecture* is one of the most famous unsolved problems in Number Theory. It conjectures that all even integers strictly greater than 2 can be written as a sum of two prime numbers, such as

$$4 = 2+2, \qquad 10 = 3+7 = 5+5, \qquad 30 = 7+23 = 11+19 = 13+17.$$

This conjecture immediately implies that any integer $n \geq 6$ is a sum of three prime numbers: If Goldbach's conjecture is true, then for any integer $m \geq 2$, there exist prime numbers p_1 and p_2 such that $2m = p_1 + p_2$. Hence,

$$2m+2 = p_1 + p_2 + 2, \qquad 2m+3 = p_1 + p_2 + 3,$$

or in other words, any integer $n \geq 6$ is a sum of three primes.

Remark. Note that a positive integer $n < 6$ is either 1, a prime number or a sum of two primes:

$$1, \qquad 2 \text{ (prime)}, \qquad 3 \text{ (prime)}, \qquad 4 = 2+2, \qquad 5 = 2+3,$$

thus 6 is the smallest positive integer that can be written as a sum of three primes.

Since odd numbers such as $11, 17, 23$ cannot be written as a sum of two primes, Goldbach's conjecture cannot be extended to all integers $n > 3$. In other words, Goldbach's conjecture asserts the minimal possible additive representation of any positive integers. For odd integers $n > 6$, it has been proven by Harald Helfgott in 2013 (to be published in a book) that n can always be written as a sum of three primes. As seen in the previous paragraph, this is a

2020 *Mathematics Subject Classification*: 11M26, 11N05, 11N37, 11P32.
Keywords: prime numbers, Goldbach's conjecture, Riemann Hypothesis, zeta function, L-functions, zeros.
Acknowledgement: The author's research at the time of the workshop was supported by JSPS KAKENHI Grant Numbers 18K13400 and 22K13895, and also MEXT Initiative for Realizing Diversity in the Research Environment.

consequence of, and thus is weaker than, Goldbach's conjecture. The conjecture for even integers remains to date a mystery.

Now for odd positive integers $2m+1$ which can be written as a sum of two primes, one of the primes must be even and the other is odd. Since 2 is the only even prime number, we must have

$$2m+1 = 2+p, \qquad \text{for some prime } p.$$

If $2m+1$ is also a prime number, then the twin prime pair $(p, 2m+1)$ satisfies a *Goldbach representation*, and thus the study of such representations is also important in understanding twin prime pairs.

3 The Riemann Hypothesis - zeros of L-functions

The Riemann hypothesis, proposed over 160 years ago, is yet another important unsolved problems in Mathematics. The Riemann Hypothesis is a conjecture about the location of zeros of the *Riemann zeta function* $\zeta(s)$, also generalized to more general *L-functions* as *Generalized Riemann Hypothesis* for a family called *Dirichlet L-functions*, or for the "largest" considered family of *L-functions*, the *Grand Riemann Hypothesis*. The hypothesis asserts that the "critical" zeros, known as *nontrivial zeros*, of these functions all lie on the same straight line $\{s \in \mathbb{C} : \mathrm{Re}(s) = 1/2\}$. The importance of this problem extends to not only Number Theory but also many other areas of Mathematics and even Physics as reflected in many known equivalent statements. In Analytic Number Theory alone, we know the equivalence between the Riemann Hypothesis and many prime related problems, and even Goldbach related problems. It is important to note that the Twin Prime Conjecture and Goldbach's conjecture are independent problems to the Riemann Hypothesis and neither is stronger than the others.

Among the above mentioned L-functions, there is a class of L-functions which may have the so-called *exceptional zeros*. D. R. Heath-Brown [Hea83] in 1983 showed if there are infinitely many such L-functions having exceptional zeros, then there are infinitely many twin primes. Nevertheless, with John B. Friedlander, Daniel A. Goldston and Henryk Iwaniec [FGIS21], we proved that a *weak* form of *Goldbach-counting conjecture* implies the <u>non-existence of exceptional zeros of Dirichlet L-functions</u>. Goldbach's conjecture can also be stated in terms of counting the number of additive representations satisfying Goldbach's conjecture and the weak conjecture we used here is a weaker form of this counting conjecture. This, unfortunately, is against Heath-Brown's approach to prove the Twin Prime Conjecture. In particular, regardless of the truth of the Twin Prime Conjecture, we cannot at the same time "believe" both Goldbach's conjecture and the existence of exceptional zeros of Dirichlet L-functions.

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