ALGEBRAIC TOPOLOGY AND PHYSICS

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1. Introduction

Algebraic topology plays an important role to classify geometric objects. We study geometric objects by extracting algebraic informations, such as generalized cohomology groups and invariants. On the other hand, in theoretical physics the classification of physical systems is the fundamental problem. It has been realized that algebraic topology is useful for such classification problems in physics. The relation between algebraic topology and physics is becoming more and more popular recently. In this talk, I give an introductory account of this relation, including my related works.

2. classification of invertible QFTs via algebraic topology

2.1. **QFTs.** Quantum field theories (QFTs) are the ways to describe physical systems. There are various formulations. In relation with topology, it is convenient to use the *cobordism picture*.

The idea is to allow the spacetime to have nontrivial topology. In this picture, a $\langle d-1, d\rangle$-dimensional QFT for $\mathcal{S}$-manifolds is defined as a symmetric monoidal functor

$$T\colon \mathrm{Bord}^{\mathcal{S}}_{\langle d-1,d\rangle} \to \mathrm{sVect}_{\mathbb{C}},$$

from the bordism category for $\mathcal{S}$-manifolds to the category of $\mathbb{Z}/2$-graded complex vector spaces. Here, $\mathcal{S}$ is a structure on manifolds, such as orientation, Riemannian metric, tangential G-structure, connections on them. Objects of $\mathrm{Bord}^{\mathcal{S}}_{\langle d-1,d\rangle}$ model the space in physics, and the morphisms model the spacetime in physics. It is important that we are allowing *non-topological* QFTs. We also note that actually what is written above is not the end of the definition.

An important quantity we get from a QFT is the *partition function*. For a d-dimensional closed $\mathcal{S}$-manifold (M^d, g), T gives a complex number

$$Z_T(M^d, g) := T(M^d, g) \in \mathrm{Hom}_{\mathbb{C}}(T(\varnothing), T(\varnothing)) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C},$$

called the partition function for (M^d, g).

A QFT is called *invertible* if the image is contained in the subcategory $\mathrm{sLine}_{\mathbb{C}} \subset \mathrm{sVect}_{\mathbb{C}}$ of 1-dimensional $\mathbb{Z}/2$-graded complex vector spaces. Invertible QFTs are the easiest but an important class of QFTs. For example they arise as "symmetry protected topological phases" in condensed matter physics. They are also important in the study of anomaly. Let us look at the easiest example of invertible QFTs.

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Example 2.1 (The holonomy theory). Let us fix a manifold X and line bundle with connection (L, ∇) over X. Set $d = 1$ and the structure $\mathcal{S}$ to be orientation and a map to X. Then we can construct a functor

$$\mathrm{Hol}_{(X,L,\nabla)} \colon \mathrm{Bord}_{\langle 0,1 \rangle}^{\mathrm{Ori} \times X} \to \mathrm{sLine}_{\mathbb{C}},$$

by assigning the pullback of L to an object and the parallel transport to a morphism. The partition function is given by the holonomy along closed curves on X. This is a non-topological invertible theory.

2.2. **Classification by generalized cohomology theories.** We want to classify invertible QFTs up to deformation equivalence. Invertible QFTs are known to be nicely classified by generalized cohomology theories.

To get the feeling, let us continue in the setting of Example 2.1. Recall that the partition function for $\mathrm{Hol}_{(X,L,\nabla)}$ is given by the holonomy function. It is classical that the holonomy function recovers the gauge equivalence class of line bundle with connection over X. The the deformation classification of line bundles with connections over X is given by $H^2(X;\mathbb{Z})$. This explains the heuristic idea that, restricted to the holonomy theories with target X, the deformation classification is given by $H^2(X;\mathbb{Z})$. But a general invertible QFTs for $d = 1$ and $\mathcal{S} = \mathrm{Ori} \times X$ is not necessarily of the form $\mathrm{Hol}_{(X,L,\nabla)}$. In order to give a classification in general, we need to use more sophisticated generalized cohomology theory. In this case the correct one turns out to be $(I\Omega^{\mathrm{SO}})^2(X)$, the Anderson dual to oriented bordisms.

In general, the classification of d-dimensional QFTs for $\mathcal{S}$-manifolds is conjectured to be given by a generalized cohomology theory $(I\Omega^{\mathcal{S}})^{d+1}$, the Anderson dual to the $\mathcal{S}$-bordism homology theory ([FH21]).

3. My works

I have recently been working on this subject. My works are mainly divided into two types.

The first one is aimed at verifying the conjectural correspondence between algebraic topology and QFTs. In the collaboration with Kazuya Yonekura, a physicist, we construct a new model of the Anderson duals ([YY21]). One difficulty of the conjecture is that the Anderson duals are defined in an abstract way, so it is difficult to relate them to physics. We constructed a model which abstractize the known properties of invertible theories, and this resolve the difficulty. This result supports the conjecture.

The second one is to apply the classification by algebraic topology to actual physical problems. In the work with Yuji Tachikawa, also a physicist, we showed the vanishing of the anomaly in heterotic string theories ([TY21]). We translated the physical problem into a purely mathematical problem, and proved the result by algebraic topological arguments.

References

[FH21] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, Geom. Topol. **25** (2021) 1165–1330.

[TY21] Y. Tachikawa and M. Yamashita, *Topological modular forms and the absence of all heterotic global anomalies*, 2021. arXiv:2108.13542 [hep-th].

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