On indivisible structures

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Abstract

An L-structure M is said to be invisible if for any partition $M = X \sqcup Y$, X or Y contains a copy of M as a substructure. In this note we discuss some examples of indivisible structures and their common properties.

1 Introduction

Structural Ramsey properties have been studied by many researchers in combinatorics and also in model theory. One of the most famous and important result is Kechris-Pestov-Todorcevic's result [5] in 2005 which connects Ramsey theory, Fraïssé theory and topological dynamics. In their work, colorings on finite structures induced from a countable homogeneous structure are investigated and such classes of finite structures are called Ramsey classes.

On the other hand, it seems that coloring on infinite structures haven't been studied so much compared to Ramsey classes. On this topic, the series of studies by Sauer is one of the most successful works. Here, we only mention about his recent work on the indivisibility of infinite homogeneous structures. In [4], he gave a characterization of indivisibility of countable homogeneous structure M where M is ω-categorical and $Age(M)$ has free amalgamation property. Note that, recently, not only relational structures but also metric structures become a target of the study of indivisibility. See, for example, [1] and [2].

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In this note, we will focus on several examples of countable and uncountable structures and investigate them by considering a "good" enumeration.

2 Indivisible structures

Definition 1. Let α and β be infinite ordinal numbers. We consider a subset of α as a subsequence of α by the natural ordering on α. Let M be an L-structure.

1. For (possibly infinite) tuples $\bar{a}, \bar{b} \in M$, $\bar{a} \cong \bar{b}$ means that $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$ in M.
2. An enumeration of M is a bijection $f : \alpha to M$ for some ordinal number α. We say $M = \{a_i \mid i < \alpha\}$ is an enumeration, we consider an enumeration $f : \alpha \to M$ with $f(i) = a_i$.
3. Let $M = \{a_i \mid i < \alpha\}$ be an enumeration of M.
   (a) For $j < \alpha$, $a_{<j} := \{a_i \mid i < \alpha\}$.
   (b) For a subset $I \subset \alpha$, $a_I := \{a_i \mid i \in I\}$, which is considered as a subsequence of the enumeration $\{a_i \mid i < \alpha\}$.
4. We say an enumeration $M = \{a_i \mid i < \alpha\}$ is homogeneous if the following condition holds: For any bounded subset $I, J \subset \alpha$ with $a_I \cong a_J$ and for any $i > I$, there is $j > J$ such that $a_I a_i \cong a_J a_j$.
5. We say an enumeration $M = \{a_i \mid i < \alpha\}$ is good if for any $i < \alpha$, the set $\mathrm{qftp}(a_i/a_{<i})^M = \{a \in M \mid a \models \mathrm{qftp}(a_i/a_{<i})\}$ contains a copy of M.

Example 2. 1. Every enumeration of order type ω of the following structures is homogeneous and good.
   (a) A countable infinite set with the empty language.
   (b) $(\mathbb{Q}; <)$
   (c) The (countable) random graph $(G; R(x, y))$
2. For an infinite cardinal κ, the natural enumeration $\{\alpha \mid \alpha < \kappa\}$ is a good and homogeneous enumeration for $(\kappa, <)$.
3. Let M be a countable saturated extension of $(\omega, <)$ or $(\mathbb{Z}, <)$. We say $a < b \in M$ have finite distance if there are only finitely many $x \in M$ such that $a < x < b$. Let $\{a_i \mid i \in \omega \times \omega\}$ be an enumeration of M such that if $a_i < a_j$ have finite distance, then there is no $k > \max(i, j)$ such that $a_i < a_k < a_j$. Then this enumeration is good and homogeneous.

4. Let $<$ be the lexicographic order on ω^n. Then the natural enumeration of ω^n by $<$ is a good and homogeneous enumeration for $(\omega^n, <)$.
5. Let $M = (\omega^{<\omega}; <_{lex}, \subset, \cap)$ be an infinitely branching tree with the lexicographic order $<_{lex}$, subsequence relation $\subset$, and the intersection function $\cap$. (For example, $(0,1,1) \subset (0,1,1,2,0)$, $(0,1,1) <_{lex} (0,2)$, and $(0,1,0) \cap (0,1,2,3) = (0,1)$.) Consider an ordering $<$ on M such that for any $\eta, \nu \in M$, $\eta < \nu$ if and only if
   - $len(\eta) < len(\nu)$, or
   - $len(\eta) = len(\nu)$ and $\eta <_{lex} \nu$.

   Let $\{a_i \mid i < \alpha\}$ be the enumeration of M by $<$. Then it is good and homogeneous.
6. Let E be an equivalence class on ω which divides ω into infinitely many infinite sets. The structure $M = (\omega; E)$ is indivisible, but there is no good enumeration of M.
7. (Zahar and Sauer [6]) Let $M_n = (G_n, R)$ be a countable K_n-free random graph with $n \geq 3$. M_n has no good enumeration, though M_n is indivisible.

Remark 3. Suppose that $M = \{a_i \mid i < \omega\}$ is a countable saturated structure admitting QE.

1. Suppose that for any $n < \omega$, $\mathrm{acl}(a_{<n}) = a_{<n}$. Then $\{a_i \mid i < \omega\}$ is a homogeneous enumeration.

   To show it, let $I, J \subset \omega$ are subsequences with the same length and $i > I$. If $a_I \cong a_J$, since M is saturated, by letting $p(x, a_I) = \mathrm{qftp}(a_i/a_I)$, $p(x, a_J)$ has infinitely many solution in M. (Notice that p is non-algebraic.) Hence we can choose $j > J$ such that $a_j \models p(x, a_J)$.
2. If M has a good enumeration, then every enumeration is good.

Theorem 4. Let M be an L-structure. (M is possibly uncountable.) If M has a good homogeneous enumeration, then M is indivisible.

Proof. Let $\{a_i \mid i < \alpha\}$ be a good homogeneous enumeration of M and let $M = X \sqcup Y$ be a partition of M. Suppose that Y has no copy of M. We will inductively construct $b_i \in X$ $(i < \alpha)$ such that

- $\{b_i \mid i < \alpha\}$ is a subsequence of $\{a_i \mid i < \alpha\}$,
- $\{b_j \mid j \leq i\} \cong \{a_j \mid j \leq i\}$ for all $i < \alpha$.

Suppose that $b_j \in X$ $(j < i)$ have been chosen. Let $p(x, a_{<i}) = \text{qftp}(a_i/a_{<i})$. Since the enumeration is homogeneous, $p(x, b_{<i})$ has a solution in M. Also, since the enumeration is good, $p(x, b_{<i})^M$ contains a copy of M. Hence, there must be a solution $b_i \in X$ of $p(x, b_{<i})$ by the assumption that Y has no copy of M. □

Corollary 5. The structures in 1-5 in Example 2 are all indivisible.

However, we cannot apply the above theorem to equivalence classes and K_n-free random graphs in Example 2.

Next we will see that some elementary extensions are indivisible.

Theorem 6. Let L be finite relational and M a countable L-structure admitting QE. Suppose that $\text{acl}(A) = A$ on M. If M has a good enumeration of order type ω, then every saturated elementary extension N of M is indivisible.

Proof. Let $\kappa = |N|$ and $\{a_i \mid i < \kappa\}$ be an enumeration of N.

Claim A. $\{a_i \mid i < \kappa\}$ is a homogeneous enumeration of N.

Let $I, J \subset \kappa$ be bounded subsequences of the same order type with $a_I \cong a_J$ and let $i > I$. Because N is saturated, admits QE, and acl is trivial, by letting $p(x, a_I) = \text{qftp}(a_i, a_I)$, $p(x, a_J)$ has κ-many solutions in N. Hence there is $j > J$ such that $a_j \models p(x, a_J)$. (End of the proof of Claim A)

Next we will show that $\{a_i \mid i < \kappa\}$ is a good enumeration of N.

Fix $i < \kappa$ and let $p(x) = \text{qftp}(a_i/a_{<i})$. Let X be the set $\{x_j \mid j < \kappa\}$ of variables. Consider the following set of formulas:

$$\Sigma(X) = \{x_{<j} \cong a_{<j} \mid j < \kappa\} \cup \bigcup_{j<\kappa} p(x_j).$$

Claim B. $\Sigma(X)$ is finitely satisfiable.

Let $\varphi(x, \bar{a}) \in p(x)$. Since M admits QE and acl is trivial, we can find $\bar{b} \in M$ such that $M \models \exists^{=\infty} x \varphi(x, \bar{b})$. Because M has a good enumeration of order type ω, $\varphi(M, \bar{b})$ contains a copy of M. This means $\{x_{<j} \cong a_{<j} \mid j < \kappa\} \cup \{\varphi(x, \bar{a})\}$ is finitely satisfiable. (End of Proof of Claim B)

Notice that every parameter appeared in $\Sigma(X)$ is contained in $a_{<i}$. Hence, by the saturation of N, there is a solution of $\Sigma(X)$ in N, which shows that $\{a_i \mid i < \kappa\}$ is a good enumeration. □

Corollary 7. Let M be an uncountable saturated elementary extension of a random graph. (At least under the assumption of Continuum Hypothesis, such extension exists.) Then M is indivisible.

At the end of this note, we see an example of elementarily indivisible structures.

Definition 8. An L-structure M is said to be elementarily indivisible if for any partition $M = X \sqcup Y$ of M, there is an isomorphic elementary substructure $M \cong M' \prec M$ such that M' is a subset of X or Y.

In [3], Hasson et. al. asked that if there is a (countable) elementarily indivisible structure which is not homogeneous. In uncountable case, it is easy to find such an example. (Recall that, for uncountable case, M is homogeneous if it is $|M|$-homogeneous.)

Proposition 9. There is uncountable elementarily indivisible structure which is not homogeneous.

Proof. Let $M = (\omega_1 \times \mathbb{Q}, <)$, where $<$ is the lexicographic order on $\omega_1 \times \mathbb{Q}$. Then $M \models DLO$, so M admits QE. It is easy to check that M is indivisible, hence elementarily indivisible, and M is not ω_1-homogeneous. □

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