On prime models of definably complete locally o-minimal theories

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Abstract

Pillay and Steinhorn showed that every o-minima theory has a prime model and it is unique up to isomorphism. We consider whether its analogy holds in generalization of o-minimal theory. We give a partial answer of this question in case of definably complete locally o-minimal theory.

1 Introduction

We are interested in definably complete locally o-minimal structures and its theories. Pillay and Steinhorn showed the following:

Theorem 1 ([1]). *Every o-minimal thoery has prime model over any sets, that are unique up to isomorphism.*

On the other hand, we gave an example of definably complete locally o-minimal theory which does not have any prime models. In this note, we consider uniqueness of prime models in definable complete locally o-minimal theories.

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2 Preliminaries

First, we define o-minimality and its generalization. Let a structure $\mathcal{M} = (M, <, \cdots)$ be expansion of a linear ordered set without endpoints.

Definition 2. $\mathcal{M}$ is o-minimal if every definable subset of M is a finite union of points and open intervals.

$\mathcal{M}$ is locally o-minimal if for any points $a \in M$ and definable subset A of M, there is an open interval I containing a such that $A \cap I$ is a finite union of points and open intervals. $\mathcal{M}$ is definably complete if every definable subset of M has supremum and infimum in $M \cup \{+\infty, -\infty\}$.

Example 3. Dense linear ordered sets without endpoints, real closed fields and $(\mathbb{R}, 0, 1, +, -, \cdot, \exp(-))$ are o-minimal.

$(\mathbb{R}, <, 0, +, \sin(-))$ is not o-minimal, but it is definably complete locally o-minimal.

We say that an A-formula is a formula with parameters from A. An A-definable set is a set defined by A-formula. If it is an interval, it is called A-interval.

In definably complete locally o-minimal structure, analogy of monotonicity theorem holds as following (called strong local monotonicity theorem).

Theorem 4 ([2]). *Let I be a A-interval and $f : I \to M$ be a A-definable function. Then there exists a mutually disjoint A-definable partition $I = X_d \cup X_c \cup X_+ \cup X_-$ satisfying the following conditions:*

(1) the A-definable set X_d is discrete and closed;

(2) the A-definable set X_c is open and f is locally constant on X_c;

(3) the A-definable set X_+ is open and f is locally strictly increasing and continuous on X_+;

(4) the A-definable set X_- is open and f is locally strictly decreasing and continuous on X_-.

In this note, a language $\mathcal{L}$ contains $<$ and we assume $<$ is linear ordering without endpoints in $\mathcal{L}$-structures.

Definition 5. $\mathcal{L}$-theory T is o-minimal (resp. definably complete locally o-minimal) if every model of T is o-minimal (resp. definably complete locally o-minimal) structure.

Remark 6. If $\mathcal{M}$ is o-minimal, its complete theoery $\mathrm{Th}(\mathcal{M})$ is o-minimal by uniform finiteness theorem. On the other hand, definable completeness and local o-minimality are represented by $\mathcal{L}$-sentences. Therefore, if $\mathcal{M}$ is definably complete locally o-minimal, its complete theoery $\mathrm{Th}(\mathcal{M})$ is definably complete locally o-minimal.

We only consider complete theory and we work in monster model $\mathbb{M}$. Next, we define completeness fo formulas and definable set.

Definition 7. Let A be a subset of $\mathbb{M}$. An A-complete formula is an A-formula which isolates some type over A. An A-complete set is A-definable set which is definable by some A-complete formula. Complete means $\emptyset$-complete.

On existence and uniqueness of prime models, the following fact is well-known.

Fact 8 ([1][4])**.** *Let a complete theory T be given.*

(1) Let $A \subset \mathcal{M} \models T$. Any model $\mathcal{M}$ that is constructible over A also is prime over A.

(2) Suppose that for any subset A of a model $\mathcal{M}$ of T and any formula φ having parameters A, there is a complete formula with parameters from A which, relative to $Th(\mathcal{M}, a)_{a\in A}$, implies φ. Then, for any $A \subset \mathcal{M}$, there is a model $\mathcal{M}$ of T that is constructible over A. (it is enough to show the assumption when φ has just one free variable)

(3) Let $A \subset \mathcal{M} \models T$. Then any two models that are constructible over A are isomorphic over A.

3 Main theorem

In this section, T is a definably complete locally o-minimal theory. We can show following theorems in a similar discussion in o-minimal theory (cf.[1])

Theorem 9. *If $dcl(\emptyset)$ is nonempty, then T has constructible model over emptyset.*

Theorem 10. *We assume that T has constructible model over emptyset and one variable complete formulas are points or open intervals. Then prime models of T are isomorphic.*

By using this theorem, it is easy to show the following corollaries.

Corollary 11. *If $dcl(\emptyset)$ is nonempty, then T has prime model over emptyset and it is unique up to isomorphism.*

Corollary 12. *Let A be a nonempty set. Then T has prime model A and it is unique up to isomorphism.*

The following is the main theorem.

Theorem 13. *We assume that a definably complete locally o-minimal $\mathcal{L}$-theory T has a complete formula. Then, if T has a prime model over $\emptyset$, any prime models of T are isomorphic.*

sketch of proof. We can assume T has an one variable complete formula $\theta_0(v)$ by taking existence symbol several times and we define $\mathcal{L}_c = \mathcal{L} \cup \{c\}$ where c is a new constant symbol.

If $\theta_0(v)$ is $v = v$, one variable complete formula is open interval and isolated types are dense in $S^1(T)$. So T has the constructible model. By theorem 10, we get theorem in this case.

We assume T has non-trivial complete formula $\theta_0(v)$. Then the boundary of $\theta_0(\mathbb{M})$ is complete set or union of two complete sets. So we take complete formula $\theta(v)$ that defines a discrete set. Let $\mathcal{N} \models T$ be prime over $\emptyset$ and take $a \in \theta(\mathcal{N})$. It is sufficient to show the following calim.

Claim A. *there are exists $a' \in \mathcal{N}$ such that $tp_{\mathcal{N}}(a) = tp_{\mathcal{N}}(a')$ and $(\mathcal{N}, a')$ is prime model of T'.*

By corollary 12, we get constructible model $\mathcal{M} \models T$ over a, see $\mathcal{M}$ as $\mathcal{L}_c$-structure $(\mathcal{M}, a)$ and define $T' = \mathrm{Th}_{\mathcal{L}_c}(\mathcal{M}, a)$. Then $(\mathcal{M}, a)$ is prime model of T'. Since $\mathcal{N}$ is prime, we get an elementary embbeding $j : \mathcal{N} \hookrightarrow \mathcal{M}$. If $j(a) = a$, $(\mathcal{N}, a)$ is prime model of T' since $(\mathcal{M}, a)$ is prime.

We consider in case $j(a) \neq a$. Since $\mathcal{M}$ is constructible over a and $\theta(v) \in \mathrm{tp}_{\mathcal{M}}(j(a)/a)$, $j(a)$ is defined by some formula $\varphi(v, a)$. It means $\mathcal{M} \models \exists b(\varphi(\mathcal{M}, b) = \{j(a)\})$ and this is represented by formulas.

We remark that points from $\theta(\mathcal{N})$ have same type over $\emptyset$ and they are isolated by points. Since $j(a)$ and a have same type, a is defained by $\varphi(v, b')$ for some $b' \in \theta(\mathcal{M})$. Similary, b' is defined by $\psi(v, a)$. So $\mathcal{M} \models \forall b[\varphi(\mathcal{M}, b') = \{b\} \to \psi(\mathcal{M}, b) = \{b'\}]$. Since b' and a have same type, we get $\mathcal{M} \models \forall b[\varphi(\mathcal{M}, a) = \{b\} \to \psi(\mathcal{M}, b) = \{a\}]$. So a is defined by $\psi(v, j(a))$. Since j is elementary embbeding, $\psi(v, b)$ defines some element $c' \in \mathcal{N}$ and then $j(c') = a$. Since $\mathcal{M}$ is prime over $\{a\}$, $(\mathcal{N}, c')$ is prime model of T'. $\square$

References

[1] Pillay Anand, and Charles Steinhorn, Definable sets in ordered structures. I., Transactions of the American Mathematical Society 295.2 (1986): 565-592.

[2] Fujita Masato, Kawakami Tomohiro, Komine Wataru, Tameness of definably complete locally o-minimal structures and definable bounded multiplication, Mathematical Logic Quarterly, Volume 68, Issue 4, November 2022, Pages 496-515.

[3] Lou van den Dries, Tame Topology and O-minimal Structures, Cambridge University Press (1998)

[4] David Marker, Model Theory: An Introduction, Springer (2002)