ON WEAK INDEPENDENCE THEOREM

DANIEL MAX HOFFMANN†

Institut für Geometrie, Technische Universität Dresden
Instytut Matematyki, Uniwersytet Warszawski

1. Introduction

This text is based on my talk at the RIMS Model Theory Workshop, taking place in December 2022. In October 2022, [3] was accepted for publication in the Journal of Mathematical Logic and it is a quite long paper achieving various results, some of them being technical and so difficult to read separately. My talk at the RIMS was thought as a transparent exposition of one of the main results from [3], which is of a more geometric flavor and so easier to extract from the whole text.

I thank organizers of the RIMS Model Theory Workshop 2022. Moreover, I take - again - the opportunity to express my gratitude to Hirotaka Kikyo for his support and enormous kindness.

2. Basics

As usually, we fix a language $\mathcal{L}$, an $\mathcal{L}$-theory T and a monster model $\mathfrak{C} \models T$ (i.e. a model of T which is κ-saturated and strongly κ-homogeneous for some big cardinal κ). We start with a general definition:

Definition 2.1. Let $\mathfrak{M}$ be somehow saturated $\mathcal{L}$-structure and let $\mathop{\smile\!\!\!\!\!\!\vert}^{\circ}$ be a ternary relation on small subsets of $\mathfrak{M}$ (we write $A \mathop{\smile\!\!\!\!\!\!\vert}^{\circ}_{B} C$, where $A, B, C \subseteq \mathfrak{M}$ are of size smaller than the saturation of $\mathfrak{M}$). We say that $\mathop{\smile\!\!\!\!\!\!\vert}^{\circ}$ satisfies the **Independence Theorem over a Model** if the following holds:

$$\text{IF: } M \preceq \mathfrak{M},\quad A, B \subseteq \mathfrak{M},\quad c_1, c_2 \subseteq \mathfrak{M} \text{ and } c_1 \equiv_M c_2,$$

$$(*) \qquad A \mathop{\smile\!\!\!\!\!\!\vert}\limits^{\circ}_{M} B,\quad c_1 \mathop{\smile\!\!\!\!\!\!\vert}\limits^{\circ}_{M} A,\quad c_2 \mathop{\smile\!\!\!\!\!\!\vert}\limits^{\circ}_{M} B$$

$$\text{THEN: there exists } c \subseteq \mathfrak{M} \text{ such that } c \equiv_{MA} c_1,\quad c \equiv_{MB} c_2 \text{ and } c \mathop{\smile\!\!\!\!\!\!\vert}\limits^{\circ}_{M} AB.$$

The above condition $(*)$ has geometric flavor. Therefore we have the following intuition: if $(*)$ holds in some structure $\mathfrak{M}$ then we should observe a bit of "geometric behavior" in $\mathfrak{M}$. This intuition becomes more transparent if we start to work with a natural notion of independence, for example if we set $\mathop{\smile\!\!\!\!\!\!\vert}^{\circ} = \mathop{\smile\!\!\!\!\!\!\vert}$ (the forking independence). Let us recall that for a tuple a and small subsets $A, B \subseteq \mathfrak{C}$ we set

$$a \mathop{\smile\!\!\!\!\!\!\vert}\limits_{A} B \qquad \iff \qquad \operatorname{tp}(a/BA) \text{ does not fork over } A.$$

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Remark 2.2. If T is simple (e.g. $\mathop{\smile\mkern-11mu\vert}$ is symmetric) then $\mathop{\smile\mkern-11mu\vert}$ satisfies $(*)$.

Definition 2.3. A ternary relation $\mathop{\smile\mkern-11mu\vert}^{\circ}$ on small subsets of $\mathfrak{C}$ is called a notion of independence if

1. (1) $\mathop{\smile\mkern-11mu\vert}^{\circ}$ is $\mathrm{Aut}(\mathfrak{C})$-invariant.
2. (2) (local character) For every small $a, B \subseteq \mathfrak{C}$ there exists $A \subseteq B$ such that $|A| \leqslant |T|$ and $a \mathop{\smile\mkern-11mu\vert}^{\circ}_{A} B$.
3. (3) (finite character) $a \mathop{\smile\mkern-11mu\vert}^{\circ}_{A} B$ if and only if for every finite $B_0 \subseteq B$ we have $a \mathop{\smile\mkern-11mu\vert}^{\circ}_{A} B_0$.
4. (4) (extension) For every small $a, A, B \subseteq \mathfrak{C}$ there exists $a' \equiv_A a$ such that $a' \mathop{\smile\mkern-11mu\vert}^{\circ}_{A} B$.
5. (5) $\mathop{\smile\mkern-11mu\vert}^{\circ}$ is symmetric.
6. (6) (transitivity) For any small $A \subseteq B \subseteq C \subseteq \mathfrak{C}$ and $a \subseteq \mathfrak{C}$ we have
$$a \mathop{\smile\mkern-11mu\vert}^{\circ}_{A} C \qquad \iff \qquad a \mathop{\smile\mkern-11mu\vert}^{\circ}_{A} B \text{ and } a \mathop{\smile\mkern-11mu\vert}^{\circ}_{b} C.$$

The following well-known theorem shows that $(*)$ is a meaningful assumption, even for an abstract notion of independence and might be used to characterize the class of simple theories.

Theorem 2.4 ([5])**.** *T is simple if and only if there exists a notion of independence $\mathop{\smile\mkern-11mu\vert}^{\circ}$ in $\mathfrak{C}$ which satisfies $(*)$. If this is the case, then moreover $\mathop{\smile\mkern-11mu\vert}^{\circ} = \mathop{\smile\mkern-11mu\vert}$.*

After exchanging "simple" with "NSOP_1" (No Strict Order Property of the first kind) and "$\mathop{\smile\mkern-11mu\vert}$" with "$\mathop{\smile\mkern-11mu\vert}^{K}$" (Kim-independence), a similar theorem to the above one holds, so importance of $(*)$ is more evident:

Theorem 2.5 ([4])**.** *T is $NSOP_1$ if and only if $\mathop{\smile\mkern-11mu\vert}^{K}$ satisfies $(*)$.*

Let us shortly explain the ingredients of the above theorem. There are several ways of defining NSOP_1, but we like to do it via so called Kim's Lemma for Kim-dividing.

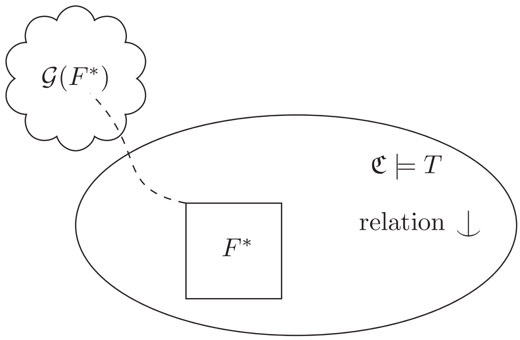
Definition 2.6. An $\mathcal{L}$-formula $\varphi(x, b)$ Kim-divides over A if there exists $k < \omega$ and a Morley sequence $(b_i)_{i<\omega}$ in $\mathrm{tp}(b/A)$ such that $\{\varphi(x, b_i) \mid i < \omega\}$ is k-inconsistent.

Under the assumption of the existence axiom for the forking independence in T, we have that T is NSOP_1 if and only if the above definition of Kim-dividing does not depend on the choice of the Morley sequence in $\mathrm{tp}(b/A)$ (in other words, if one Morley sequence witnesses Kim-dividing, any other Morley sequence is a witness as well). This not depending on the choice of the Morley sequence is the statement of the Kim's lemma. The assumption about existence axiom for the forking independence is only to work here with a simpler notion of dividing (otherwise we would need to work with "q-dividing").

Putting everything together, we can see Theorem 2.5 as an equivalence between Kim's Lemma for Kim-dividing and property $(*)$ for Kim-independence. Thus we see that the property $(*)$ corresponds to important properties of the theory and so we will study how we can obtain property $(*)$ in some general setting, called *Weak Independence Theorem*. There are applications of the upcoming Weak Independence Theorem in the NSOP_1 context (e.g. to show that some structures are NSOP_1), but we will not discuss these here.

3. Set-up

Recall that $\mathfrak{C} \models T$ is our monster model of stable theory T. Let moreover $F^* \subseteq \mathfrak{C}$ be a small substructure of $\mathfrak{C}$ (not necessarily elementary substructure, typically not a model of T). Here, we are interested in understanding the theory of F^*, by finding a ternary relation on subsets of F^*, which should look like a good notion of independence. We have the following objects to be used in our construction.



The main idea is to combine the structure of $\mathcal{G}(F^*)$ with the forking independence relation $\mathop{\smile\!\!\!\!\!\mid}$ from $\mathfrak{C}$ to define a desired ternary relation on F^*. To do it, we will need to assume several technical conditions. To make it more transparent let us list all of them at once:

- T is stable, even more T has nfcp (*no finite cover property*)
- T has the property $B(3)$: Let $\{a_0, a_a, a_2\}$ be an A-independent set. Then for every $\sigma \in \mathrm{Aut}(\mathrm{acl}(a_0 a_1 A)/\mathrm{acl}(a_0 A)\,\mathrm{acl}(a_1 A))$ there exists an extension $\sigma \subseteq \bar{\sigma}$ being an element of $\mathrm{Aut}(\mathfrak{C}/\mathrm{acl}(a_0 a_2 A)\,\mathrm{acl}(a_1 a_2 A))$.
- $T = (T_0^{\mathrm{eq}})^m$ for some $\mathcal{L}_0$-theory T_0, i.e. we add imaginary sorts and then do the Morleyization.

Now, let us discuss the choice of F^*. As we already mentioned, F^* is a substructure of $\mathfrak{C}$, but as we aim to understand the theory of F^*, we need to choose it to be a more generic one and therefore we assume that F^* is already somehow saturated in the sense of $\mathrm{Th}(F^*)$ (in other words F^* is a monster model for the complete $\mathcal{L}$-theory $\mathrm{Th}(F^*)$). Moreover, we will assume that F^* is a PAC substructure, and so we will be able to deploy some machinery from the model-theoretic Galois theory.

Definition 3.1. (1) Let $A \subseteq B$ be substructures of $\mathfrak{C}$. We say that $A \subseteq B$ is a regular extension if $\mathrm{dcl}(B) \cap \mathrm{acl}(A) = \mathrm{dcl}(A)$.
(2) A small substructure A of $\mathfrak{C}$ is *pseudo-algebraically closed* (PAC) if for every regular extension $A \subseteq A'$, $A' \subseteq \mathfrak{C}$, we have that A is existentially closed in A'.

To summarize, F^* is a somehow saturated PAC substructure of $\mathfrak{C}$. Let us fix also $F \preceq F^*$, a small model playing the role of "M" in the property $(*)$. A question which was leading us to the results of [3], was "What is $\mathop{\smile\!\!\!\!\!\mid}^K$ (the Kim-forking independence) for $\mathrm{Th}(F^*)$?". We mention this, as the answer was provided after proving the *Weak Independence Theorem*, which we discuss here.

4. Structure of Galois group

As we already pointed out, the main idea is to combine the forking independence relation from $\mathfrak{C}$ with the structure of the Galois group $\mathcal{G}(F^*)$. Now, the natural question is what do we mean by "the structure of the Galois group". To make it precise, let us recall some standard definitions and then define a proper first order structure on $\mathcal{G}(F^*)$.

Definition 4.1. Let $A \subseteq B \subseteq \mathfrak{C}$ and let b be a tuple from $\mathfrak{C}$.

- $\mathcal{G}(B/A) := \operatorname{Aut}(\operatorname{dcl}(B)/\operatorname{dcl}(A))$,
- $\mathcal{G}(A) := \mathcal{G}(\operatorname{acl}(A)/A)$,
- $\mathcal{G}(b/A) := \mathcal{G}(\operatorname{dcl}(Ab)/A)$,
- $[B : A] := |\mathcal{G}(B/A)|$,
- $A \subseteq B$ is normal if $\mathcal{G}(\mathfrak{C}/A) \cdot B \subseteq B$,
- $A \subseteq B$ is Galois if $A \subseteq B$ is normal, $\operatorname{dcl}(A) = A$ and $\operatorname{dcl}(B) = B$.

Assume that $A \subseteq B$ is a Galois extension and consider a being a tuple of elements from B. We call a a *primitive element of* $A \subseteq B$ if $B = \operatorname{dcl}(A, a)$. A collection of all primitive elements of $A \subseteq B$ forms a set denoted by $e(B/A)$. An interesting theorem, which follows easily from the elimination of imaginaries and which is a general version of the classical Primitive Element Theorem, is the following:

Theorem 4.2. *If $A \subseteq B$ is Galois and $[B : A] < \omega$ then $e(B/A) \neq \emptyset$.*

The primitive elements will come back in a moment, but first let us recall the way of turning profinite groups, being some Galois groups, into a first order structures which appeared in [1]. Consider a profinite group G and let us work in a language $\mathcal{L}_G$ living on sorts $(m(k))_{k<\omega}$ and having two binary relations: $\leqslant$ and C; and one ternary relation P (i.e. infinitely many variants of each of these relations, depending on the choice of sorts, e.g. $\leqslant_{k,k'}$ with arguments from $m(k) \times m(k'))$. We define

$$\begin{aligned} m(k) &:= \{gH \mid g \in G,\ H \trianglelefteq G,\ H \text{ is open and } [G:H] \leqslant k\}, \\ gH \leqslant g'H' &\iff H \subseteq H', \\ C(gH, g'J') &\iff H \subseteq H' \text{ and } gH = g'H', \\ P(g_1H_1, g_2H_2, g_3H_3) &\iff H_1 = H_2 = H_3 \text{ and } g_1g_2H_1 = g_3H_1. \end{aligned}$$

The authors of [1] developed the above first order structure of a profinite group to tame the theory of PAC fields, and fields are one sorted structures. Our goal is to work in a general model-theoretic setting, so many sorted structures may appear (and actually do appear as we work in $T = (T_0^{\mathrm{eq}})^m)$). This causes a new obstacle as was noticed in [2] - if we turn a Galois group of a substructure into a first order structure as described above, we will lose all information about on which sort a given primitive element is "living". Some proofs depend on constructing automorphisms by sending one primitive element to another primitive element - if there is only one sort, there is no choice, but if our structure is many sorted - a structure of a Galois group as structure of a pure profinite group does not contain information about sorts and proofs does not work in this setting. In [2], Galois groups of substructures were equipped with "sorting data" - an additional structure which codes whether a finite Galois extension has a primitive element in a given tuple of sorts. We use this to define a modification of the first order structure for profinite groups from [1].

First, we define a functor:

$$\left\{\begin{array}{c}\text{definably closed substr. of } \mathfrak{C}\\ \text{with morphisms } \phi: A \to B \text{ being}\\ \mathcal{L}\text{-embeddings } \phi: \operatorname{acl}(A) \to \operatorname{acl}(B)\\ \text{such that } \phi[A] \subseteq B \text{ is regular}\end{array}\right\} \xrightarrow{\ \mathcal{G}\ } \left\{\begin{array}{c}\text{sorted profinite}\\ \text{groups}\end{array}\right\}$$

We set $\operatorname{dcl}(A) = A \mapsto (\mathcal{G}(A), \bar{\mathcal{F}}_A)$, where naturally $\mathcal{G}(A)$ is the profinite group $\operatorname{Aut}(\operatorname{acl}(A)/A)$ and $\bar{\mathcal{F}}_A$ is the so called *sorting data* defined as follows. Let $\mathcal{N}(\mathcal{G}(A))$ denote the collection of all open normal subgroups of $\mathcal{G}(A)$. For every $N \in \mathcal{N}(\mathcal{G}(A))$, we set $\mathcal{F}_A(N)$ to be the set of all finite tuples J of sort indices (sorts of $\mathcal{L}$) such that the Galois extension $A \subseteq \operatorname{acl}(A)^N$ has a primitive element in the tuple of sorts indexed by J. Finally, $\bar{\mathcal{F}}_A := (\mathcal{F}_A(N))_{N\in\mathcal{N}(\mathcal{G}(A))}$.

In the next step, we explain what is our first order structure for a sorted profinite group coming from a Galois group. Again, consider $A = \operatorname{dcl}(A) \subseteq \mathfrak{C}$ and its Galois group presented as the above sorted profinite group, $(\mathcal{G}(A), \bar{\mathcal{F}}_A)$. As in the case of [1], we work in a many sorted language, but this time we also parameterize sorts via occurrences of the primitive elements - our new language $\mathcal{L}_{G,S}$ lives on sorts $m(k, J)$, where $k < \omega$ and J is a finite tuple of sort indices in $\mathcal{L}$. As previously, in the language $\mathcal{L}_{G,S}$ we have two binary relations: $\leqslant$ and C; and one ternary relation P (again - each of these relations exists in a copy for each combination of sorts for their arguments). The $\mathcal{L}_{G,S}$-structure derived from $(\mathcal{G}(A), \bar{\mathcal{F}}_A)$ is denoted by $S\mathcal{G}(A)$ and given as follows:

$$\begin{aligned} m(k,J) &:= \{gH \mid g \in G,\ H \in \mathcal{N}(\mathcal{G}(A)),\\ &\qquad\qquad [G:H] \leqslant k \text{ and } J \in \mathcal{F}_A(H)\},\\ gH \leqslant g'H' &\iff H \subseteq H',\\ C(gH, g'J') &\iff H \subseteq H' \text{ and } gH = g'H',\\ P(g_1H_1, g_2H_2, g_3H_3) &\iff H_1 = H_2 = H_3 \text{ and } g_1g_2H_1 = g_3H_1. \end{aligned}$$

5. Weak Independence Theorem

Finally, we come to the main theorem. Assume that $\mathop{\smile\!\!\!\!\!\mid}^{S\mathcal{G}}$ is a ternary relation on small subsets of $S\mathcal{G}(F^*)$ such that

$$S_1 \mathop{\smile\!\!\!\!\!\mid}_{S_0}^{S\mathcal{G}} S_2 \qquad \iff \qquad S_1' \mathop{\smile\!\!\!\!\!\mid}_{S_0}^{S\mathcal{G}} S_2'$$

whenever $S_1S_2 \equiv_{S_0} S_1'S_2'$ (with respect to the many sorted structure we defined previously). Moreover, we assume that $\mathop{\smile\!\!\!\!\!\mid}^{S\mathcal{G}}$ satisfies the Extension over a Model

Definition 5.1. Let $\mathfrak{M}$ be somehow saturated $\mathcal{L}$-structure and let $\mathop{\smile\!\!\!\!\!\mid}^{\circ}$ be a ternary relation on small subsets of $\mathfrak{M}$ (we write $A \mathop{\smile\!\!\!\!\!\mid}^{\circ}_B C$, where $A, B, C \subseteq \mathfrak{M}$ are of size smaller than the saturation of $\mathfrak{M}$). We say that $\mathop{\smile\!\!\!\!\!\mid}^{\circ}$ satisfies the **Extension over a Model** if the following holds:

$$\text{IF: } M \preceq \mathfrak{M},\ \ a, b, c \subseteq \mathfrak{M} \text{ such that } a \mathop{\smile\!\!\!\!\!\mid}_{M}^{\circ} b$$

$$\text{THEN: there exists } a' \equiv_{Mb} a \text{ such that } a' \mathop{\smile\!\!\!\!\!\mid}_{M}^{\circ} bc$$

We define a new ternary relation, on the subsets of F^*, which combines $\mathop{\downarrow}^{S\mathcal{G}}$ from $S\mathcal{G}(F^*)$ and $\mathop{\downarrow}$ from $\mathfrak{C}$:

$$A \mathop{\downarrow}_B^{\partial} C \iff A \mathop{\downarrow}_B C \text{ and } S\mathcal{G}(\mathrm{acl}(A) \cap F^*) \mathop{\downarrow}^{S\mathcal{G}}_{S\mathcal{G}(\mathrm{acl}(B) \cap F^*)} S\mathcal{G}(\mathrm{acl}(BC) \cap F^*)$$

Theorem 5.2 (Weak Independence Theorem)**.** *If $\mathop{\downarrow}^{S\mathcal{G}}$ satisfies $(*)$ then $\mathop{\downarrow}^{\partial}$ satisfies $(*)$.*

Corollary 5.3. *If $S\mathcal{G}(F^*)$ is NSOP$_1$ then F^* is NSOP$_1$.*

Proof. It is enough to consider the Kim-independence in $S\mathcal{G}(F^*)$ in the place of $\mathop{\downarrow}^{S\mathcal{G}}$ and then use the Weak Independence Theorem. □

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† Institut für Geometrie, Technische Universität Dresden, Dresden, Germany
and
Instytut Matematyki, Uniwersytet Warszawski, Warszawa, Poland

E-mail address: `daniel.max.hoffmann@gmail.com`

URL: `https://sites.google.com/site/danielmaxhoffmann/`