ON WEAK ELIMINATION OF IMAGINARIES, AN APPENDIX OF CASANOVAS AND FARRE' S PAPER AND BASIC EXAMPLES

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Abstract. We give an explicit proof of Fact 2.2.2 in [CF]. And we give basic examples related to (full)/weak/geometric elimination of imaginaries.

1. The characterization of weak elimination of imaginaries

Let $\mathcal{M}$ be a sufficiently saturated model of T. $\bar{a}, \bar{b}, \bar{c}, \dots$ denote finite tuples in $\mathcal{M}$ and $a, b, c, \dots$ denote elements of $\mathcal{M}$. $L(\bar{a})$ denotes the set of L-formulas with parameter $\bar{a}$. For $\varphi(\bar{x}, \bar{a}) \in L(\bar{a})$, $\varphi(\bar{x}, \bar{a})^{\mathcal{M}} := \{\bar{m} \subset \mathcal{M} : \mathcal{M} \models \varphi(\bar{m}, \bar{a})\}$. We work in $\mathcal{M}^{\mathrm{eq}} := \{\bar{a}/E : \bar{a}/E$ is the E-class of $\bar{a}$, where $\bar{a}$ is a finite tuple of $\mathcal{M}$ and $E(\bar{x}, \bar{y})$ is a ϕ-definable equivalence relation with $\mathrm{lh}(\bar{x}) = \mathrm{lh}(\bar{y}) = \mathrm{lh}(\bar{a})$ $\}$. Let $A \subset \mathcal{M}^{\mathrm{eq}}$. For $\bar{a}/E \in \mathcal{M}^{\mathrm{eq}}$, we write $\bar{a}/E \in \mathrm{acl}^{\mathrm{eq}}(A)$ if the orbit of $\bar{a}/E$ by automorphisms fixing A pointwise is finite, and $\bar{a}/E \in \mathrm{dcl}^{\mathrm{eq}}(A)$ if $\bar{a}/E$ is fixed by automorphisms fixing A pointwise. For $\bar{a} \subset \mathcal{M}$, we write $\bar{a} \in \mathrm{acl}(A)$ if the orbit of $\bar{a}$ by automorphisms fixing A pointwise is finite, and $\bar{a} \in \mathrm{dcl}(A)$ if $\bar{a}$ is fixed by automorphisms fixing A pointwise.

Definition 1.1. We say that T admits weak elimination of imaginaries (WEI) in sense of B.Poizat (See pp.321-322 in [Po]), for any $\varphi(\bar{x}.\bar{a}) \in L(\bar{a})$ we have the smallest algebraically closed set B such that $\varphi(\bar{x}.\bar{a})$ is definable over B.

Fact 1.2. *The following are equivalent: Fact 2.2.2 in* [CF]

1. *T admits WEI in sense of B.Poizat.*
2. *For any $\varphi(\bar{x}, \bar{a}) \in L(\bar{a})$ there exists an $\emptyset$-definable formula $\psi_{\bar{a}}(\bar{x}, \bar{z})$ such that $1 \leq |\{\bar{b} \subset \mathcal{M} : \varphi(\bar{x}, \bar{a})^{\mathcal{M}} = \psi_{\bar{a}}(\bar{x}, \bar{b})^{\mathcal{M}}\}| < \omega$. Note that $1 = |\{\bar{b} \subset \mathcal{M} : \varphi(\bar{x}, \bar{a})^{\mathcal{M}} = \psi_{\bar{a}}(\bar{x}, \bar{b})^{\mathcal{M}}\}|$ is equivalent to elimination of imaginaries.*
3. *T admits WEI in sense of A.Pillay (See* [Pi]*); for any $\bar{a}/E \in \mathcal{M}^{\mathrm{eq}}$ there exist a finite tuple $\bar{b} \subset \mathcal{M}$ such that $\bar{a}/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(\bar{a}/E)$*
4. *For any $\varphi(\bar{x}, \bar{a}) \in L(\bar{a})$, there exists non-empty finite set $B = \{\bar{b}_i : 1 \leq i \leq n\}$ of real tuples of the same length such that σ fixes B setwise if and only if $\sigma(e) = e$ for any $\sigma \in \mathrm{Aut}(\mathcal{M})$, where $e (\in \mathcal{M}^{\mathrm{eq}})$ is the canonical parameter of $\varphi(\bar{x}, \bar{a})$*

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Proof. (1) $\Leftrightarrow$(2): Theorem 16.15 in [Po]. (2)$\Rightarrow$ (3): Clear. (3)$\Rightarrow$ (1): Let $e \in \mathcal{M}^{\mathrm{eq}}$ be the canonical parameter of $\varphi(\bar{x}, \bar{a})$ and let $\bar{b} \in \mathcal{M}$ be such that $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(e)$. Then we can show that $\mathrm{acl}(\bar{b})$ is the smallest algebraically closed set which over $\varphi(\bar{x}, \bar{a})$ is definable, for details see the proof of Proposition 1.3 in [Y]. (2)$\Rightarrow$(4) : Clear. (4)$\Rightarrow$ (3) : Clear. □

2. Basic (stable) examples not related to infinite groups

We say T admit (full) elimination of imaginaries (EI) if for any $\bar{a}/E \in \mathcal{M}^{\mathrm{eq}}$ there exist a finite tuple $\bar{b} \subset \mathcal{M}$ such that $\bar{a}/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(\bar{a}/E)$.
We say T admit geometric elimination of imaginaries (GEI) if for any $\bar{a}/E \in \mathcal{M}^{\mathrm{eq}}$ there exist a finite tuple $\bar{b} \subset \mathcal{M}$ such that $\bar{a}/E \in \mathrm{acl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(\bar{a}/E)$.
We show that the implication EI$\Rightarrow$WEI$\Rightarrow$GEI is strict by basic examples for researchers who are not familiar with these issues. We pose problems that these examples can be found as infinite group structures.
Two examples of admitting WEI without admitting EI
(1) Let E be an equivalence relation with two classes such that one class is infinite and the other class is finite. Let a be such that $E(x, a)$ is infinite and let $b_1, \ldots, b_j, \ldots, b_n$ be distinct elements such that $\neg E(a, b_j)$ for each $1 \leq j \leq n$. Suppose that $n \geq 2$ and let $\bar{b} \subseteq b_1, \ldots, b_j, \ldots, b_n$ be arbitrary. Now we have $a/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(a/E)$ but $\bar{b} \notin \mathrm{dcl}^{\mathrm{eq}}(a/E)$. (When we take $\bar{c}$ from the E-class of a, then we have $\bar{c} \notin \mathrm{acl}^{\mathrm{eq}}(a/E)$.)
(2) Let $U = \mathbb{Z} \times \mathbb{Z}, V = \mathbb{Z}$. We define an equivalence relation E on U such that $(x, y)E(u, v)$ iff $x = u$.
We also define a relation R on $U \times V$ such that $R((n, x), n) \wedge R((n, y), n + 1)$ for any $n, x, y \in \mathbb{Z}$. When we have $R(u, v)$ we may write $R(u/E, v)$.
For $u/E \in V/E$ and $v \in V$ we define the R-distance $d_R(u/E, v)$ between u/E and v such that $d_R(u/E, v) = m$, where m is a positive integer iff there exist distinct $v_1, \ldots, v_{m-1}, v_m = v \in V$ and distinct $u/E = u_0/E, u_1/E, \ldots u_{m-1}/E \in U/E$ such that $R(u/E, v_1) \wedge R(u_1/E, v_1) \wedge R(u_1/E, v_2) \wedge \ldots \wedge R(u_i/E, v_i) \wedge R(u_i/E, v_{i+1}) \wedge \ldots \wedge R(u_{m-1}/E, v_{m-1}) \wedge R(u_{m-1}/E, v)$.
If $m = 1$, then we have $R(u/E, v)$, we put $d_R(u/E, v) = 1$.
Let $M = (U \sqcup V; U, V, E, R)$.
Claim 1: $\mathrm{Th}(M)$ does not admit EI.
For $v, v' \in V$ and $u/E \in U/E$ we have

$$\mathrm{tp}(v/(u/E)) = \mathrm{tp}(v'/(u/E)) \Leftrightarrow d_R(u/E, v) = d_R(u/E, v')$$

In a saturated model of $\mathrm{Th}(M)$ it is possible that $d_R(u/E, v) = \infty$.
If $d_R(u/E, v) < \infty$, then $\mathrm{tp}(v/(u/E))$ is algebraic with two solutions by symmetric R-paths from u/E. If $d_R(u/E, v) = \infty$, then $\mathrm{tp}(v/(u/E))$ is non-algebraic. So there is no tuple $\bar{v} \subset V$ such that $\bar{v} \in \mathrm{dcl}^{\mathrm{eq}}(u/E)$. $\mathrm{Th}(M)$ does not admit EI.
Claim 2: $\mathrm{Th}(M)$ admits WEI.
We have $(n, x)/E \in \mathrm{dcl}^{\mathrm{eq}}((n, n+1))$. We also have $(n, x)/E \in \mathrm{dcl}^{\mathrm{eq}}((n+1, n+2))$ because $(n, x)/E$ is the unique solution of $R(*, n+1) \wedge \neg R(*, n+2)$. Similarly we have $(n, x)/E \in \mathrm{dcl}^{\mathrm{eq}}((n-1, n))$.
The solutions of $\mathrm{tp}_M((n, n+1)/((n, x)/E)))$ are $(n, n+1)$ and $(n+1, n)$.
The solutions of $\mathrm{tp}_M((n+1, n+2)/((n, x)/E)))$ are $(n+1, n+2)$ and $(n, n-1)$.
The solutions of $\mathrm{tp}_M((n-1, n)/((n, x)/E)))$ are $(n-1, n)$ and $(n+2, n+1)$.
Therefore we have $(n, n+1), (n+1, n+2), (n-1, n) \in \mathrm{acl}^{\mathrm{eq}}((n, x)/E)$.

Three examples admitting GEI without admitting WEI

(1)(2) are due to Byunghan kim and (3) is due to Akito Tsuboi.

(1) Consider the theory of n-many E-classes such that each class is infinite, where E is an equivalence relation and $2 \leq n < \omega$. Then the empty sequence witnesses for GEI of a/E. But if $a/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$, then $\bar{b} \notin \mathrm{acl}^{\mathrm{eq}}(a/E)$.

(2) Let E be an equivanlence relation such that $a/E \neq b/E$ and $c/E = \{c\}$ where both $E(x,a)$ and $E(x,b)$ are infinite. Set two predicate $P = E(x,a) \sqcup E(x,b)$ and $Q = \{c\}$. Then c witnesses GEI for a/E and b/E in $(P \sqcup Q; P, Q, E)$. But if $a/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{d})$, then $\bar{d} \notin \mathrm{acl}^{\mathrm{eq}}(a/E)$.

(3) Consider two unary predicates $U = \mathbb{N} \times \mathbb{N}$ and $V = \mathbb{N}$. Let E be the equivalence relation on U such that $E((x,y),(u,v))$ iff $x = u$. Let $R \subset U \times V$ be such that $((2n,x),2n), ((2n+1,y),2n), ((2n,u),2n+1), ((2n+1,v),2n+1) \in R$ for any $n,x,y,u,v \in \mathbb{N}$. We work in $(U \sqcup V; U, V, E, R)$. We have $(2n,x)/E$ is interalgebraic $2n$ and $(2n+1,y)/E$ is interalgebraic with $2n+1$. But there is no V-tuple $\bar{v}$ such that $u/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{v})$ for any $u \in U$.

Two examples without admitting GEI

(1) Let $U = \mathbb{N} \times \mathbb{N}, V = \mathbb{N}$. We define an equivalence relation E on U such that $(x,y)E(u,v)$ iff $x = u$. Let $M = (U \cup V; U, V, E)$.

For any $\bar{v}, \bar{v'} \subset V$ with $\mathrm{lh}(\bar{v}) = \mathrm{lh}(\bar{v'})$ and any $u \in U$, we have $\mathrm{tp}_M(\bar{v}/(u/E)) = \mathrm{tp}_M(\bar{v'}/(u/E))$. So there is no tuple $\bar{v} \subset V$ such that $\bar{v} \in \mathrm{acl}^{\mathrm{eq}}(u/E)$.

(2) The theory T of infinitely many E-classes where each E-class is infinite does not admit GEI: Fix $a \in M \models T$. Note that $a/E \notin \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ and $\emptyset \in \mathrm{dcl}^{\mathrm{eq}}(a/E)$. Suppose that $a/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{a}\bar{b})$ with $\bar{a} = a_1, \ldots, a_i, \ldots, a_n$, $\bar{b} = b_1, \ldots, b_j, \ldots, b_m$, $E(a, a_i)$ for each $1 \leq i \leq n$ and $\neg E(a, b_j)$ for each $1 \leq j \leq m$. Then $\mathrm{tp}_M(\bar{a}/(a/E))$ is non-algebraic and $\mathrm{tp}(\bar{b}/\bar{a}(a/E))$ is non-algebraic, we have $\mathrm{tp}(\bar{a}\bar{b}/(a/E))$ is non-algebraic. So we have $\bar{a}\bar{b} \notin \mathrm{acl}^{\mathrm{eq}}(a/E)$.

3. Questions on WEI

Question 3.1. (1) *If T is stable, do we have that T admits WEI if and only if $< \mathrm{Aut}(\mathcal{M}/A), \mathrm{Aut}(\mathcal{M}/B) >= \mathrm{Aut}(\mathcal{M}/A \cap B)$ for $A = \mathrm{acl}(A), B = \mathrm{acl}(B) \subset \mathcal{M}$?*

(2) *Find a stable structure which is interpretable in an infinite group and admitting GEI without admitting WEI. Moreover find a stable group admitting GEI without admitting WEI. The beautiful pairs $(K, P(K))$ of ACF_p admit WEI in some additional sorts: algebraic principal homogeneous spaces $(G(P(K)), G(P(K)) \cdot v)$, where $v \in V(K)$ is some generic over $P(K)$. (See chapter 7 in* [B]*.)*

(3) *Hrushovski's new strongly minimal set admits WEI (See* [Hr]*) but does not have finite set property (See* [BV]*), so does not have EI. Find a new strongly minimal set D admitting GEI without admitting WEI, and determine the natural number n that D is n-ample but not $(n+1)$-ample. (If strongly minimal set D admits EI, then D is non-trivial and 1-ample by* [Pi]*.)*

(4) $\mathrm{SCF_e}$ *for each $e \in \omega \cup \{\infty\}$ in the language of field does not admit EI (See Remark 5.3 in* [M]*) and has finite set property, so it does not have WEI. Is* $\mathrm{SCF_e}$ *for each $e \in \omega \cup \{\infty\}$ in the language of field stable? Does* $\mathrm{SCF_e}$ *for each $e \in \omega \cup \{\infty\}$ in the language of field admit GEI?*

(5) [W1], [W2]. *Any infinite Boolean algebra does not admit EI. Is there an infinite Boolean algebra admitting GEI without admitting WEI?*

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References

[B] G.J.Boxall, Lovely pairs and dense pairs of real closed fields, Ph.D thesis, The university of Leeds, July 20, 2009.

[BV] J.T.Baldwin, V. V. Verbovskiy, Towards a finer classification of strongly minimal sets, (2021), preprint.

[CF] E.Casanovas, R.Farré, Weak forms of elimination of imaginaries, Math.Logic Quarterly 50 Issue 2 (2004), 126-140.

[Hr] E.Hrushovski, A new strongly minimal set, Ann. Pure Appl. Logic 62 (1993), 147-166.

[M] M.Messmer, Some model theory of separably closed fields, Model theory of fields, Lecture note in logic 5 (1996).

[Pi] A.Pillay, Some remarks on modular regular types, J. Symbolic Logic 56 (1998), 1003-1011.

[Po] B.Poizat, A Course in Model Theory, Universitext, Springer-Verlag (2000)

[Y] I. Yoneda, Pillay's alternative definition for weak elimination of imaginaries coincides with Poizat's original one, RIMS kokyuroku 2218 (2022), 59-63.

[W1] R.Wencel, Weak elimination of imaginaries for Boolean algebra, Ann. Pure Appl. Logic 132 (2005), 247-270.

[W2] R.Wencel, Imaginaries in Boolean algebra, Math.Logic Quarterly 58 Issue 3 (2012), 217-235.

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