# GENERIC EXPANSIONS OF NATP THEORIES

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Abstract. We show that adding a generic predicate $P$ to an NATP theory preserves NATP, with the assumption of modular pregeometry and elimination of quantifiers and $\exists^\infty$.

## 1. Preliminaries

**Notation 1.1.** Let $\kappa$ and $\lambda$ be cardinals.

(i) By $\kappa^\lambda$, we mean the set of all functions from $\lambda$ to $\kappa$.
(ii) By $\kappa^{<\lambda}$, we mean $\bigcup_{\alpha<\lambda}\kappa^\alpha$ and call it a *tree*. If $\kappa = 2$, we call it a *binary tree*. If $\kappa \geq \omega$, then we call it an *infinite tree*.
(iii) By $\emptyset$ or $\langle\rangle$, we mean the empty string in $\kappa^{<\lambda}$, which means the empty set.

Let $\eta, \nu \in \kappa^{<\lambda}$.

(iv) By $\eta \trianglelefteq \nu$, we mean $\eta \subseteq \nu$. If $\eta \trianglelefteq \nu$ or $\nu \trianglelefteq \eta$, then we say $\eta$ and $\nu$ are *comparable*.
(v) By $\eta \perp \nu$, we mean that $\eta \ntrianglelefteq \nu$ and $\nu \ntrianglelefteq \eta$. We say $\eta$ and $\nu$ are *incomparable* if $\eta \perp \nu$.
(vi) By $\eta \wedge \nu$, we mean the maximal $\xi \in \kappa^{<\lambda}$ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$.
(vii) By $l(\eta)$, we mean the domain of $\eta$.
(viii) By $\eta <_{lex} \nu$, we mean that either $\eta \trianglelefteq \nu$, or $\eta \perp \nu$ and $\eta(l(\eta\wedge\nu)) < \nu(l(\eta\wedge\nu))$.
(ix) By $\eta^\frown\nu$, we mean $\eta \cup \{(i + l(\eta), \nu(i)) : i < l(\nu)\}$.

Let $X \subseteq \kappa^{<\lambda}$.

(x) By $\eta^\frown X$ and $X^\frown\eta$, we mean $\{\eta^\frown x : x \in X\}$ and $\{x^\frown\eta : x \in X\}$ respectively.

Let $\eta_0, \cdots, \eta_n \in \kappa^{<\lambda}$.

(xi) We say a subset $X$ of $\kappa^{<\lambda}$ is an *antichain* if the elements of $X$ are pairwise incomparable, *i.e.*, $\eta \perp \nu$ for all $\eta, \nu \in X$).

Let $\mathcal{L}_0 = \{\trianglelefteq, <_{lex}, \wedge\}$ be a language where $\trianglelefteq$, $<_{lex}$ are binary relation symbols and $\wedge$ is a binary function symbol. Then for cardinals $\kappa > 1$ and $\lambda$, a tree $\kappa^{<\lambda}$ can be regarded as an $\mathcal{L}_0$-structure whose interpretations of $\trianglelefteq, <_{lex}, \wedge$ follow Notation 1.1.

**Definition 1.2.** Let $\overline{\eta} = (\eta_0, \cdots, \eta_n)$ and $\overline{\nu} = (\nu_0, \cdots, \nu_n)$ be finite tuples of $\kappa^{<\lambda}$.

(i) By $\mathrm{qftp}_0(\overline{\eta})$, we mean the set of quantifier-free $\mathcal{L}_0$-formulas $\varphi(\overline{x})$ such that $\kappa^{<\lambda} \models \varphi(\overline{\eta})$.
(ii) By $\overline{\eta} \sim_0 \overline{\nu}$, we mean $\mathrm{qftp}_0(\overline{\eta}) = \mathrm{qftp}_0(\overline{\nu})$ and say they are *strongly isomorphic*.

Let $\mathcal{L}$ be a language, $T$ a complete $\mathcal{L}$-theory, $\mathbb{M}$ a monster model of $T$ and $(a_\eta)_{\eta\in\kappa^{<\lambda}}$, $(b_\eta)_{\eta\in\kappa^{<\lambda}}$ be tree-indexed sets of tuples from $\mathbb{M}$. For $\overline{\eta} = (\eta_0, \cdots, \eta_n)$, denote $(a_{\eta_0}, \cdots, a_{\eta_n})$ by $\overline{a}_{\overline{\eta}}$. By $\overline{a}_{\overline{\eta}} \equiv_{\Delta,A} \overline{b}_{\overline{\nu}}$ (or $\mathrm{tp}_\Delta(\overline{a}_{\overline{\eta}}/A) = \mathrm{tp}_\Delta(\overline{b}_{\overline{\nu}}/A)$), we mean that for any $\mathcal{L}_A$-formula $\varphi(\overline{x}) \in \Delta$ where $\overline{x} = x_0 \cdots x_n$, $\overline{a}_{\overline{\eta}} \models \varphi(\overline{x})$ if and only if $\overline{b}_{\overline{\nu}} \models \varphi(\overline{x})$.

(iii) We say $(a_\eta)_{\eta\in\kappa^{<\lambda}}$ is *strongly indiscernible* over $A$ if $\mathrm{tp}(\overline{a}_{\overline{\eta}}/A) = \mathrm{tp}(\overline{a}_{\overline{\nu}}/A)$ for any $\overline{\eta}$ and $\overline{\nu}$ such that $\mathrm{qftp}_0(\overline{\eta}) = \mathrm{qftp}_0(\overline{\nu})$.

(iv) We say $(b_\eta)_{\eta\in\kappa^{<\lambda}}$ is *strongly based* on $(a_\eta)_{\eta\in\kappa^{<\lambda}}$ over $A$ if for all $\overline{\eta}$ and a finite set of $\mathcal{L}_A$-formulas $\Delta$, there is $\overline{\nu}$ such that $\overline{\eta}\sim_0\overline{\nu}$ and $\overline{b}_{\overline{\eta}}\equiv_{\Delta,A}\overline{a}_{\overline{\nu}}$.

**Fact 1.3.** *Let $(a_\eta)_{\eta\in\omega^{<\omega}}$ be a tree-indexed set. Then there is a strongly indiscernible sequence $(b_\eta)_{\eta\in\omega^{<\omega}}$ which is strongly based on $(a_\eta)_{\eta\in\omega^{<\omega}}$.*

The proof of the above fact can be found in [KK11], [KKS14] and [TT12]. It is called the *modeling property* of strong indiscernibility (in short, we write it the *strong modeling property*).

**Definition 1.4.** Let $T$ be a first-order complete $\mathcal{L}$-theory. We say a formula $\varphi(x,y)\in\mathcal{L}$ has (or is) *k-antichain tree property* ($k$-ATP) if for any monster model $\mathbb{M}$, there exists a tree indexed set of parameters $(a_\eta)_{\eta\in 2^{<\omega}}$ such that

(i) for any antichain $X$ in $2^{<\omega}$, the set $\{\varphi(x,a_\eta):\eta\in X\}$ is consistent and
(ii) for any pairwise comparable distinct elements $\eta_0,\cdots,\eta_{k-1}\in 2^{<\omega}$, $\{\varphi(x;a_{\eta_i}):i<k\}$ is inconsistent.

We say $T$ has $k$-ATP if there exists a formula $\varphi(x,y)$ having $k$-ATP and

- If $k=2$, we omit $k$ and simply write ATP.
- If $T$ does not have ATP, then we say $T$ has (or is) NATP.
- If $T$ is not complete, then saying '$T$ is NATP' means that any completion of $T$ is NATP.

**Remark/Definition 1.5.**

(1) We say an antichain $X\subseteq\kappa^{<\lambda}$ is *universal* if for each finite antichain $Y\subseteq\kappa^{<\lambda}$, there exists $X_0\subseteq X$ such that $Y\sim_0 X_0$. A typical example of a universal antichain is $\kappa^{\lambda'}\subseteq\kappa^{<\lambda}$ where $\kappa>1$ and $\omega\le\lambda'<\lambda$.
(2) Let $\varphi(x;y)$ be a formula and $(a_\eta)_{\eta\in\kappa^{<\lambda}}$ be a tree indexed set of parameters where $\kappa>1$ and $\lambda$ is infinite. We say $(\varphi(x;y),(a_\eta)_{\eta\in\kappa^{<\lambda}}))$ witnesses ATP if for any $X\subseteq\kappa^{<\lambda}$, the partial type $\{\varphi(x,a_\eta)\}_{\eta\in X}$ is consistent if and only if $X$ is pairwise incomparable. Note that $T$ has ATP if and only if it has a witness for some $\kappa>1$ and infinite $\lambda$ by compactness.

**Remark 1.6.** By [AK20, Corollary 4.9] and [AKL21, Remark 3.6], if $\varphi(x;y)$ has ATP, then there is a witness $(\varphi(x;y),(a_\eta)_{\eta\in 2^{\le\omega}})$ with strongly indiscernible $(a_\eta)_{\eta\in 2^{\le\omega}}$.

**Fact 1.7.** [AKL21, Corollary 3.23(b)] *Let $\kappa$ and $\lambda$ be infinite cardinals with $\lambda<cf(\kappa)$, $f:2^\kappa\to X$ be an arbitrary function and $c:X\to\lambda$ be a coloring map. Then there is a monochromatic subset $S\subseteq 2^\kappa$ such that for any $k<\omega$, there is some tuple in $S$ strongly isomorphic to the lexicographic enumeration of $2^k$.*

**Fact 1.8.** [AKL21, Theorem 3.27] *Let $T$ be a complete theory and $2^{|T|}<\kappa<\kappa'$ with $cf(\kappa)=\kappa$. The following are equivalent.*

*(1) $T$ is NATP.*
*(2) For any strongly indiscernible tree $(a_\eta)_{\eta\in 2^{<\kappa'}}$ and a single element $b$, there are $\rho\in 2^\kappa$ and $b'$ such that*
  *(a) $(a_{\rho^\frown 0^i})_{i<\kappa'}$ is indiscernible over $b'$,*
  *(b) $b\equiv_{a_\rho}b'$.*

**Remark 1.9.** Let $\lambda=2^{|T|}<\kappa<\kappa'$ with $cf(\kappa)=\kappa$ and $c:2^\kappa\to\lambda$. If $T$ is a complete NATP theory, by Fact 1.8, for any strongly indiscernible tree $(a_\eta)_{\eta\in 2^{<\kappa'}}$ and a single element $b$, there are $\rho\in 2^\kappa$ and $b'$ satisfying conditions $(a)$ and $(b)$ of Fact 1.8. On the other hand, by Fact 1.7, there is a universal antichain $S\subseteq 2^\kappa$ such that $|c(S)|=1$.

Suppose that the length of each tuple $a_\eta$ is finite. Then identifying $\lambda$ with $S_x(b) =$ (the set of all complete types over $b$ with $|x| = |a_\eta|$) and letting $c(\eta) = \mathrm{tp}(a_\eta/b)$ for each $\eta \in 2^\kappa$, we obtain $S \subseteq 2^\kappa$ such that for all $a_\eta, a_{\eta'} \in 2^\kappa$, $\mathrm{tp}(a_\eta/b) = \mathrm{tp}(a_{\eta'}/b)$. In fact, the proof of [AKL21, Theorem 3.27] shows that for any $\rho$ in such $S$, there always exists $b'$ satisfying $(a), (b)$ of Fact 1.8.

**Remark 1.10.** Recall that if a complete theory $T$ has ATP, then there are $\varphi(x; y) \in \mathcal{L}$ and a strongly indiscernible tree $(a_\eta)_{\eta \in 2^{\leq \omega}}$ that witness ATP (Remark 1.6). For this witness, $\{\varphi(x, a_\eta) \mid \eta \in 2^\omega\}$ has infinitely many realizations.

*Proof.* Easy to verify using strong indiscernibility and compactness. □

**Remark 1.11.** Let $T$ be a complete theory having NATP. Let $(a_\eta)_{\eta \in 2^{\kappa'}}$ be a strongly indiscernible sequence over $\emptyset$ and let $(b_i)_{i \in \omega}$ be an indiscernible sequence over $A := \{a_\eta : \eta \in 2^{<\kappa'}\}$ with $b_i = (b_{i,0}, b_{i,1}, \ldots)$ such that $b_{i,0} \neq b_{j,0}$ for $i \neq j \in \omega$. Suppose there is a regular cardinal $\kappa < \kappa'$ such that $2^{|T|+|b_0|} < \kappa$. By Remark 1.9, there is an universal anticahin $S \subset 2^\kappa$ such that $a_\eta \equiv_{b_0} a_{\eta'}$ for all $\eta, \eta' \in S$. Take $\rho \in S$ arbitrary. Put $p(x, a_\rho) := \mathrm{tp}(b_0, a_\rho)$ for $x = (x_0, x_1, \ldots)$ and for each $n \in \omega$, put

$$p_n(x_0, \ldots, x_n) := \bigcup_{i \leq n} p(x_i, a_\rho) \cup \{x_{i,0} \neq x_{j,0} : i \neq j \leq n\},$$

which is consistent by $b_0, \ldots, b_n$. Then, for each $n \geq 0$,

$$p_n(x_1, \ldots, x_n, a_\rho) \cup p_n(x_1, \ldots, x_n, a_{\rho\frown 0})$$

is consistent. Thus, the type

$$p(x, a_\rho) \cup p(x, a_{\rho\frown 0})$$

has infinitely many solutions whose first components are distinct.

*Proof.* Suppose not. By compactness, there is a formula $\psi(x_0, \ldots, x_n, y) \in p_n(x_0, \ldots, x_n, y)$ such that

$$\psi(x_0, \ldots, x_n, a_\rho) \wedge \psi(x_0, \ldots, x_n, a_{\rho\frown 0})$$

is inconsistent. By strongly indiscerniblity, for any $\eta \trianglerighteq \nu \in 2^{<\kappa'}$,

$$\psi(x_0, \ldots, x_n, a_\eta) \wedge \psi(x_0, \ldots, x_n, a_\nu)$$

is inconsistent.

On the other hand, since $b_0, \ldots, b_n \models \psi(x_0, \ldots, x_n, a_\rho)$, by the choice of $S$,

$$b_0, \ldots, b_n \models \psi(x_0, \ldots, x_n, a_\eta)$$

for all $\eta \in S$. Since $S$ is a universal antichain, for any antichain $X$ in $2^{<\kappa'}$,

$$\{\psi(x_0, \ldots, x_n, a_\eta) : \eta \in X\}$$

is consistent. Therefore, $\psi(x_0, \ldots, x_n, y)$ witnesses ATP with $(a_\eta)_{\eta \in 2^{<\kappa'}}$, which contradicts the assumption that $T$ has NATP. □

**Definition 1.12** ([TZ])**.**

(1) A *pregeometry* $(X, \mathrm{cl})$ is a set $X$ with a closure operator $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$ and singletons $a, b \in X$,
  (a) (Reflexivity) $A \subseteq \mathrm{cl}(A)$;
  (b) (Finite character) $\mathrm{cl}(A) = \bigcup_{A' \subseteq A, A' \text{: finite}} \mathrm{cl}(A')$;
  (c) (Transitivity) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
  (d) (Exchange) If $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(Aa)$.

(2) Let $(X, \mathrm{cl})$ be a pregeometry and $A \subseteq X$.
   (a) $A$ is called *independent* if for all singleton $a \in A$, $a \notin \mathrm{cl}(A \setminus \{a\})$;
   (b) $A_0 \subseteq A$ is called a *generating set* for $A$ if $A \subseteq \mathrm{cl}(A_0)$;
   (c) $A_0$ is called a *basis* for $A$ if $A_0$ is an independent generating set for $A$.

**Definition 1.13.** Let $(X, \mathrm{cl})$ be a pregeometry and $A \subseteq X$. It is well-known that all bases of $A$ have the same cardinality([TZ]).

(1) The *dimension* of $A$, $\dim(A)$ is the cardinal of a basis for $A$.
(2) $A$ is called *closed* if $\mathrm{cl}(A) = A$.
(3) $(X, \mathrm{cl})$ is called *modular* if for any closed finite dimensional sets $B, C$,

$$\dim(B \cup C) = \dim(B) + \dim(C) - \dim(B \cap C).$$

We say $T$ *is a (modular) pregeometry with* acl or acl *defines a (modular) pregeometry in* $T$ if $(\mathcal{M}, \mathrm{acl})$ is a (modular) pregeometry.

**Remark 1.14.** Assume $T$ is a pregeomtry with acl. Let $A, B$ be algebraically closed and $c$ be a singleton not in $\mathrm{acl}(AB)$. Then for $D := A \cap B$, we have the following:

(1) $\mathrm{acl}(cD) \cap \mathrm{acl}(AB) = \mathrm{acl}(D)$.
(2) $\mathrm{acl}(cA) \cap \mathrm{acl}(AB) = \mathrm{acl}(A)$.
(3) $\mathrm{acl}(cB) \cap \mathrm{acl}(AB) = \mathrm{acl}(B)$.

Moreover, if $T$ is modular, then

(4) $\mathrm{acl}(Ac) \cap \mathrm{acl}(Bc) = \mathrm{acl}(cD)$.

*Proof.* (1)-(3) are easily obtained by exchange property.
(4): By finite character, we may assume that $\dim(A)$ and $\dim(B)$ are finite. It is enough to show that $\dim(\mathrm{acl}(Ac) \cap \mathrm{acl}(Bc)) = \dim(\mathrm{acl}(cD))$ because $\mathrm{acl}(cD) \subseteq \mathrm{acl}(Ac) \cap \mathrm{acl}(Bc)$ and they are algebraically closed.

By modularity and $c \notin \mathrm{acl}(AB)$,

$$\begin{aligned}
\dim(\mathrm{acl}(Ac) \cap \mathrm{acl}(Bc)) &= \dim(\mathrm{acl}(Ac)) + \dim(\mathrm{acl}(Bc)) - \dim(\mathrm{acl}(Ac) \cup \mathrm{acl}(Bc)) \\
&= (\dim(A) + 1) + (\dim(B) + 1) - (\dim(\mathrm{acl}(AB)) + 1) \\
&= \dim(A) + \dim(B) - \dim(AB) + 1 \\
&= \dim(D) + 1.
\end{aligned}$$

□

## 2. Adding a generic predicate

The generic predicate construction was introduced in [CP98], and it is known that each of $\mathrm{NTP}_2$ and $\mathrm{NTP}_1$ is preserved by such a construction, proved in [Che14] and [Dob18] respectively. We collect some necessary facts from [CP98] first, and then show that NATP is also preserved using similar ideas given in aforementioned papers.

Throughout, consider a complete theory $T$ in a first-order language $\mathcal{L}$, which contains some unary predicate $S$. The reader may note that the following fact is stated in [Che14] and [Dob18], but the location of brackets in the first item is corrected. The notation $\mathrm{tp}_T$, $\mathrm{acl}_T$ or $\mathrm{acl}_{T_{P,S}}$ shall mean in the same way as in the previous sections, whose intended meaning will be clear from the context.

**Fact 2.1.** *Assume that $T$ has elimination of quantifiers and elimination of $\exists^\infty$. Then*

*(1)* [CP98, Theorem 2.4] $T^0_{P,S}$ *has a model companion, denoted by* $T_{P,S}$*, which is axiomatized by* $T$ *together with*

$$\forall z \left[ \exists x \left( \varphi(x,z) \wedge (x \cap \mathrm{acl}_T(z) = \emptyset) \wedge \bigwedge_{i<n} S(x_i) \wedge \bigwedge_{i \neq j < n} x_i \neq x_j \right) \right.$$
$$\left. \rightarrow \exists x \left( \varphi(x,z) \wedge \bigwedge_{i \in I} P(x_i) \wedge \bigwedge_{i \notin I} \neg P(x_i) \right) \right]$$

*where* $x = (x_0, \cdots, x_{n-1})$ *and* $I$ *ranges over all subsets of the set* $\{0, \cdots, n-1\}$*. Indeed, above expression can be written in a first-order formula* [CP98, Lemma 2.3].

*For (2) and (3), let* $a, b$ *be tuples of* $(M, P) \models T_{P,S}$ *and* $A \subseteq M$*.*

*(2)* [CP98, Proposition 2.5 and Corollary 2.6(2)] $\mathrm{tp}_{T_{P,S}}(a) = \mathrm{tp}_{T_{P,S}}(b)$ *if and only if there exists an* $\mathcal{L}_P$*-isomorphism between substructures:*

$$f : (\mathrm{acl}(a), P \cap \mathrm{acl}_T(a)) \rightarrow (\mathrm{acl}(b), P \cap \mathrm{acl}_T(b))$$

*such that* $f(a) = b$*.*

*(3)* [CP98, Corollary 2.6(3)] $\mathrm{acl}_T(A) = \mathrm{acl}_{T_{P,S}}(A)$.

The following remark will be freely used.

**Remark 2.2.**

(1) Note that $T_{P,S}$ is not necessarily complete, so '$T_{P,S}$ is NATP' means that any completion of it is NATP (Definition 1.4).

(2) Due to Fact 2.1(3), we can say $T_{P,S}$ also has the exchange property for cl = acl if $T$ has this property. We will not distinguish between $\mathrm{acl}_T$ and $\mathrm{acl}_{P,S}$, so the subscripts for acl will be omitted.

(3) If it happens that $T \models S(x) \leftrightarrow x = x$, then we simply write $T^0_{P,S}$ for $T^0_P$ and $T_{P,S}$ for $T_P$.

**Theorem 2.3.** *Let* $T$ *be a modular pregeometry with* acl *and let* $T$ *have quantifier elimination and elimination of* $\exists^\infty$*. If* $T$ *is NATP, then* $T_P$ *is also NATP.*

*Proof.* Fix a monster model $(\mathbb{M}, P) \models T_P$ (which is not necessarily a complete theory). Let $\kappa$ and $\kappa'$ be cardinals such that $2^{|T_P|} < \kappa < \kappa'$ and $cf(\kappa) = \kappa$. Suppose for a contradiction that $\mathrm{Th}(\mathbb{M}, P)$ has ATP witnessed by an $\mathcal{L}_P$-formula $\varphi(x, y)$ with a strongly indiscernible tree $(a_\eta)_{\eta \in 2^{<\kappa'}}$ (such a tree of this form exists, similarly as Remark 1.6). By [AKL21, Theorem 3.17], we may assume that $|x| = 1$.

Let $(\mathrm{acl}(a_\eta))_{\eta \in 2^{<\kappa'}}$ be a tree of tuples where each enumeration of $\mathrm{acl}(a_\eta)$ starts with $a_\eta$. Then $(\mathrm{acl}(a_\eta))_{\eta \in 2^{<\kappa'}}$ itself might not be strongly indiscernible, but by Fact 1.3 and compactness, there is a strongly indiscernible $(\mathrm{acl}(a^*_\eta))_{\eta \in 2^{<\kappa'}}$ which is strongly based on $(\mathrm{acl}(a_\eta))_{\eta \in 2^{<\kappa'}}$. Then with dummy variables, an $\mathcal{L}_P$-formula $\varphi(x, y') \equiv \varphi(x, y)$ with a strongly indiscernible tree $(\mathrm{acl}(a^*_\eta))_{\eta \in 2^{<\kappa'}}$ witnesses ATP of $T_P$. Thus we may replace each $a_\eta$ by $a^*_\eta$ and say that $(\mathrm{acl}(a_\eta))_{\eta \in 2^{<\kappa'}}$ is strongly indiscernible; whenever an enumeration of $\mathrm{acl}(a_\eta)$ is concerned in the rest of this proof, we refer to the enumeration fixed here. Note that $(\mathrm{acl}(a_\eta))_{\eta \in 2^{<\kappa'}}$ is strongly indiscernible over $D := \mathrm{acl}(a_\emptyset) \cap \mathrm{acl}(a_0) (= \mathrm{acl}(a_\eta) \cap \mathrm{acl}(a_\nu)$ for any $\eta, \nu \in 2^{<\kappa'})$.

Put $A = \{a_\eta : \eta \in 2^{<\kappa'}\}$. Recall that by Remark 1.10, $\{\varphi(x, a_\eta) : \eta \in 2^\kappa\}$ has infinitely many realizations. Thus (by Ramsey's Theorem and compactness) we can find a non-constant $A$-indiscernible sequence $(b_i)_{i<\omega}$ not in $\operatorname{acl}(A)$ such that each $b_i$ realizes $\{\varphi(x, a_\eta) : \eta \in 2^\kappa\}$. For each $i$, put $\bar{b}_i$ some fixed enumeration of $\operatorname{acl}(b_i D)$ starting with $b_i$ such that $\bar{b}_i \equiv_A \bar{b}_0$.

Let $B = \bigcup_{i<\omega} \operatorname{acl}(b_i D)$, $C = \{\operatorname{tp}(\operatorname{acl}(a_\eta)/B)) : \eta \in 2^\kappa\}$, $f : 2^\kappa \to C$ be a function such that $f(\eta) = \operatorname{tp}(\operatorname{acl}(a_\eta)/B)$ and $c : C \to S_{y'}(B)$ be an inclusion map where $|y'| = |\operatorname{acl}(a_\eta)|$. Then letting $\lambda = 2^{|T_P|}(= |S_{y'}(B)|)$, we can find a subset $S \subseteq 2^\kappa$ given in the Fact 1.7 so that

- for any $\eta, \nu \in S$, $\operatorname{tp}(\operatorname{acl}(a_\eta)/B) = \operatorname{tp}(\operatorname{acl}(a_\nu)/B))$;
- for any $k < \omega$, there exists some tuple in $S$ strongly isomorphic to the lexicographic enumeration of $2^k$.

Now, choose an element $\rho \in S$ arbitrary, put $p(\bar{x}, \operatorname{acl}(a_\rho)) = \operatorname{tp}_T(\bar{b}_0/\operatorname{acl}(a_\rho))$. By Remark 1.11, $p(\bar{x}, \operatorname{acl}(a_\rho)) \cup p(\bar{x}, \operatorname{acl}(a_{\rho^\frown 0}))$ has infinitely many realizations, whose first coordinates are all distinct. Then by compactness, we can find $\bar{b}$ such that $\bar{b} \models p(\bar{x}, \operatorname{acl}(a_\rho)) \cup p(\bar{x}, \operatorname{acl}(a_{\rho^\frown 0}))$ and the first element, say $b$, of $\bar{b}$ is not in $\operatorname{acl}(A)$ $\cdots(*)$. Via an elementary map, $\bar{b}$ is an enumeration of $\operatorname{acl}(bD)$.

By Remark 1.14, 2.2(2) and that $b_0 \notin \operatorname{acl}(A)$,following relations between algebraic closures hold (†):

(1) $\operatorname{acl}(b_0 D) \cap \operatorname{acl}(a_\rho a_{\rho^\frown 0}) = \operatorname{acl}(D)$;
(2) $\operatorname{acl}(b_0 a_\rho) \cap \operatorname{acl}(a_\rho a_{\rho^\frown 0}) = \operatorname{acl}(a_\rho)$;
(3) $\operatorname{acl}(b_0 a_{\rho^\frown 0}) \cap \operatorname{acl}(a_\rho a_{\rho^\frown 0}) = \operatorname{acl}(a_{\rho^\frown 0})$;
(4) $\operatorname{acl}(b_0 a_\rho) \cap \operatorname{acl}(b_0 a_{\rho^\frown 0}) = \operatorname{acl}(b_0 D)(= \bar{b}_0)$.

Let $\bar{c}_0$ be an enumeration of $\operatorname{acl}(b_0 a_\rho) \setminus (\operatorname{acl}(b_0 D) \cup \operatorname{acl}(a_\rho))$ and denote $\bar{b}_0 \setminus \operatorname{acl}(D)$ a subsequence of $\bar{b}_0$, formed by deleting coordinates of $\bar{b}_0$ in $\operatorname{acl}(D)$. By (†) and Remark 2.2(2), the sets of all coordinates of $\bar{b}_0 \setminus \operatorname{acl}(D)$, $\bar{c}_0$ and $\operatorname{acl}(a_\rho a_{\rho^\frown 0})$ are pairwise disjoint.

Recall $p(\bar{x}, \operatorname{acl}(a_\rho)) = \operatorname{tp}_T(\bar{b}_0/\operatorname{acl}(a_\rho))$ and $\bar{b} \models p(\bar{x}, \operatorname{acl}(a_\rho)) \cup p(\bar{x}, \operatorname{acl}(a_{\rho^\frown 0}))$. Using $\mathcal{L}$-elementary maps $f_0, g_0$ where $f_0$ fixes $\operatorname{acl}(a_\rho)$, $f_0((\bar{b}_0 \setminus \operatorname{acl}(D)) \operatorname{acl}(a_\rho)) = (\bar{b} \setminus \operatorname{acl}(D)) \operatorname{acl}(a_\rho)$ and $g_0((\bar{b}_0 \setminus \operatorname{acl}(D) \operatorname{acl}(a_\rho)) = (\bar{b} \setminus \operatorname{acl}(D) \operatorname{acl}(a_{\rho^\frown 0})$, we obtain the following enumerations which are automorphic images of $\bar{c}_0$:

- $\bar{c}_\rho$ of $\operatorname{acl}(b a_\rho) \setminus (\operatorname{acl}(bD) \cup \operatorname{acl}(a_\rho))$ and
- $\bar{c}_{\rho^\frown 0}$ of $\operatorname{acl}(b a_{\rho^\frown 0}) \setminus (\operatorname{acl}(bD) \cup \operatorname{acl}(a_{\rho^\frown 0}))$.

Note that

- $(\bar{b} \setminus \operatorname{acl}(D))\bar{c}_\rho \operatorname{acl}(a_\rho)$ is an enumeration of $\operatorname{acl}(b a_\rho)$ and
- $(\bar{b} \setminus \operatorname{acl}(D))\bar{c}_{\rho^\frown 0} \operatorname{acl}(a_{\rho^\frown 0})$ is of $\operatorname{acl}(b a_{\rho^\frown 0})$.

Now let

$$q(\bar{x}\bar{w}, \operatorname{acl}(a_\rho)) = \operatorname{tp}_T((\bar{b}_0 \setminus \operatorname{acl}(D))\bar{c}_0/\operatorname{acl}(a_\rho)).$$

Then for any formula $\psi \in q(\bar{x}\bar{w}, \operatorname{acl}(a_\rho)) \cup q(\bar{x}\bar{w}_*, \operatorname{acl}(a_{\rho^\frown 0}))$ where $\bar{w} \cap \bar{w}_* = \emptyset$, say $\psi(\bar{x}'\bar{w}'\bar{w}'_*, \bar{a}\bar{a}_*) \equiv \psi_\rho(\bar{x}'\bar{w}', \bar{a}) \wedge \psi_{\rho^\frown 0}(\bar{x}'\bar{w}'_*, \bar{a}_*)$ where

(1) $q(\bar{x}\bar{w}, \operatorname{acl}(a_\rho)) \vdash \psi_\rho(\bar{x}'\bar{w}', \bar{a})$,
(2) $q(\bar{x}\bar{w}_*, \operatorname{acl}(a_{\rho^\frown 0})) \vdash \psi_{\rho^\frown 0}(\bar{x}'\bar{w}'_*, \bar{a}_*)$,
(3) $\bar{x}' \subseteq \bar{x}$, $\bar{w}' \subseteq \bar{w}$, $\bar{w}'_* \subseteq \bar{w}_*$ are finite tuples of variables,

the following formula(Fact 2.1(1))

$$\psi(\bar{x}'\bar{w}'\bar{w}'_*, \bar{a}\bar{a}_*) \wedge (\bar{x}'\bar{w}'\bar{w}'_* \cap \operatorname{acl}(\bar{a}\bar{a}_*) = \emptyset) \wedge \bigwedge_{v_i, v_j \in \bar{x}'\bar{w}'\bar{w}'_*, i \neq j} (v_i \neq v_j)$$

has a realization in $(\bar{b}\setminus \operatorname{acl}(D))\bar{c}_\rho\bar{c}_{\rho^\frown 0}$ (by $(*)$, $b\notin \operatorname{acl}(A)$. Then it can be checked that $\operatorname{acl}(ba_\rho)\cap\operatorname{acl}(A)=\operatorname{acl}(a_\rho)$ by exchange property). In particular, $b\notin\operatorname{acl}(a_\rho a_{\rho^\frown 0})\ \cdots(**)$.

Then by Fact 2.1(1) and compactness, there is $(\bar{b}'\setminus\operatorname{acl}(D))\bar{c}'_\rho\bar{c}'_{\rho^\frown 0}\models q(\bar{x}\bar{w},\operatorname{acl}(a_\rho))\cup q(\bar{x}\bar{w}_*,\operatorname{acl}(a_{\rho^\frown 0}))$ where $\bar{b}'$ is an enumeration of $\operatorname{acl}(b'D)$ starting with $b'$ and we can arbitrarily choose which of the elements of $(\bar{b}'\setminus\operatorname{acl}(D))\bar{c}'_\rho\bar{c}'_{\rho^\frown 0}$ are in $P$.

To summarize, now we have the following:

- There are $\mathcal{L}$-automorphisms $f_0,f_1$ of $\mathbb{M}$ fixing $\operatorname{acl}(a_\rho)$ such that
$$f_0((\bar{b}_0\setminus\operatorname{acl}(D))\bar{c}_0\operatorname{acl}(a_\rho))=(\bar{b}\setminus\operatorname{acl}(D))\bar{c}_\rho\operatorname{acl}(a_\rho);$$
$$f_1((\bar{b}\setminus\operatorname{acl}(D))\bar{c}_\rho\operatorname{acl}(a_\rho))=(\bar{b}'\setminus\operatorname{acl}(D))\bar{c}'_\rho\operatorname{acl}(a_\rho).$$
where $(\bar{b}'\setminus\operatorname{acl}(D))\bar{c}'_\rho\operatorname{acl}(a_\rho)$ is an enumeration of $\operatorname{acl}(b'a_\rho)$.
- There are $\mathcal{L}$-automorphisms $g_0,g_1$ of $\mathbb{M}$ such that
$$g_0((\bar{b}_0\setminus\operatorname{acl}(D))\bar{c}_0\operatorname{acl}(a_\rho))=(\bar{b}\setminus\operatorname{acl}(D))\bar{c}_{\rho^\frown 0}\operatorname{acl}(a_{\rho^\frown 0}),$$
$$g_1((\bar{b}\setminus\operatorname{acl}(D))\bar{c}_{\rho^\frown 0}\operatorname{acl}(a_{\rho^\frown 0}))=(\bar{b}'\setminus\operatorname{acl}(D))\bar{c}'_{\rho^\frown 0}\operatorname{acl}(a_{\rho^\frown 0})$$
where $(\bar{b}'\setminus\operatorname{acl}(D))\bar{c}'_{\rho^\frown 0}\operatorname{acl}(a_{\rho^\frown 0})$ is an enumeration of $\operatorname{acl}(b'a_{\rho^\frown 0})$.

Since we can freely choose $P\cap((\bar{b}'\setminus\operatorname{acl}(D))\bar{c}'_\rho\bar{c}'_{\rho^\frown 0})$,
$$f_1f_0:(\operatorname{acl}(b_0a_\rho),P\cap\operatorname{acl}(b_0a_\rho))\to(\operatorname{acl}(b'a_\rho),P\cap\operatorname{acl}(b'a_\rho))\text{ and}$$
$$g_1g_0:(\operatorname{acl}(b_0a_\rho),P\cap\operatorname{acl}(b_0a_\rho))\to(\operatorname{acl}(b'a_{\rho^\frown 0}),P\cap\operatorname{acl}(b'a_{\rho^\frown 0}))$$
can be simultaneously regarded as $\mathcal{L}_P$-isomorphisms between $\mathcal{L}_P$-substructures since

- $\operatorname{acl}(b'a_\rho)\cap\operatorname{acl}(b'a_{\rho^\frown 0})=\operatorname{acl}(b'D)$ by $(**)$ and Remark 1.14(4);
- $f_1f_0(\bar{b}_0)=g_1g_0(\bar{b}_0)=\bar{b}'$;
- $f_1f_0$ fixes $\operatorname{acl}(a_\rho)$ pointwise;
- $g_1g_0(\operatorname{acl}(a_\rho))=\operatorname{acl}(a_{\rho^\frown 0})$ and preserves $P$-coloring by the ($\mathcal{L}_P$-)strong indiscernibility of $(\operatorname{acl}(a_\eta))_{\eta\in 2^{<\kappa'}}$.

Therefore by Fact 2.1(2), $\operatorname{tp}_{T_P}(b_0a_\rho)=\operatorname{tp}_{T_P}(b'a_\rho)=\operatorname{tp}_{T_P}(b'a_{\rho^\frown 0})$. Since $\models\varphi(b_0,a_\rho)$, we have $\models\varphi(b',a_\rho)\wedge\varphi(b',a_{\rho^\frown 0})$, which contradicts that $\varphi$ witnesses ATP with $(a_\eta)_{\eta\in 2^{<\kappa'}}$. □

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