EXISTENTIALLY CLOSED FIELDS WITH DIFFERENCE/DIFFERENTIAL OPERATORS

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Abstract. We survey separably differentially closed fields and separable differential closure in comparison with differentially closed fields and differential closure. Also we observe several topics around the theory of separably differentially closed fields.

Contents

1. Differentially closed fields and differential closure

In this section, we describe classical results for differentially closed fields and differential closure in characteristic zero. We fix a differential field (K, δ) is of characteristic zero where a derivation δ on K is an additive homomorphism satisfying Leibniz rule: $\delta(ab) = a\delta(b) + b\delta(a)$. We denote a differential polynomial ring over (K, δ) in a differential indeterminate x by $K\{x\}$.

A. Robinson [40], in 1959, introduced the notion of differentially closed fields as analogous to algebraically closed fields of characteristic zero; every finite system of algebraic differential equations and inequations over K with a solution in an differential extension of K already has solution in K. One can show that every differential fields of characteristic zero has a differentially algebraic extension that is differentially closed. He axiomatized them and showed that the theory of differentially closed fields is complete, model complete and decidable. Through the notion of existential closedness, We may also define a differential field (K, δ) being *differentially closed* if it is existentially closed in every differential field extension. (Recall that a field K is algebraically closed if it is existentially closed in every field extension.)

L. Blum [1] simplified Robinson's axiom. One can observe the shape of the prime differential ideals for an ingredient for this; Let P is a nonzero prime differential

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ideal of $K\{x\}$. For any irreducible f of minimal rank in P, we have $P = [f] : s_f^\infty$ where a *separant* of f, s_f, is obtained by taking the partial derivative of f with respect to its highest order variable, and $[f]$ denote the differential ideal generated by f, and for an ideal I of $K\{x\}$ and $s \in K\{x\}$, the saturated ideal of I over s is defined as $I : s^\infty = \{h \in K\{x\} : s^m h \in I \text{ for some } m \geq 0\}$. Conversely, if $f \in K\{x\}$ is irreducible, then $[f] : s_f^\infty$ is a prime differential ideal and f is of minimal rank in it. In conclusion, Blum [1, §III] showed that a differential field (K, δ) is differentially closed if and only if for any nonzero $f, g \in K\{x\}$ and $\operatorname{ord} g < \operatorname{ord} f$, there exists $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$ where this scheme of conditions is expressed in a first-order fashion and denoted by DCF_0.

E.R. Kolchin [25] introduced the notion called constrained elements and constrained extension since every differential closed fields has a nontrivial differentially algebraic extension unlike the property of algebraically closed fields; let $\alpha = (\alpha_i)_{i \in I}$ be a family of elements from a differential field extension of K, and then α is said to be *constrained* over K if there exists $g \in K\{\bar{x}\}$ with $g(\bar{\alpha}) \neq 0$ such that for every differential specialisation β of α over K if $g(\beta) \neq 0$ then β is a generic differential specialisation. Kolchin [25] defined a differential field extension of K to be a constrained extension if every finite family of elements from it is constrained over K, and K having no proper constrained extension is called *constrainedly closed*. (Remark that this Kolchin's definition is for Characteristic zero is slightly different from the case for positive characteristic. See next Section). Note that every constrained extension is differentially algebraic, but the converse is not generally true. In conclusion, he showed that K is differentially closed if and only if K is constrainedly closed.

D. Pierce and A. Pillay [37] reworked the axiom of differentially closed fields in a geometric setting in terms of prolongations of affine algebraic varieties; namely, given an algebraic variety V over K there exists an algebraic bundle $\pi : \tau V \to V$ over K called that prolongation of V that has the following characteristic property: for any differential field extension (L, δ) of (K, δ) if $a \in V(L)$ then $(a, \delta a) \in \tau V(L)$. In the case when V is affine, say $K[V] = K[x_1, \ldots, x_n]/I$ with I a radical ideal, the following equations define the first prolongation of V, τV,

$$f(\bar{x}) = 0 \quad \text{and} \quad \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{x}) \cdot y_i + f^\delta(\bar{x}) = 0,$$

as f varies in generators of I, the defining ideal of V.

Their geometric axiom has been adapted to other settings; for instance, fields of characteristic zero with several commuting derivations [30], fields of positive characteristic with derivation of Frobenius [15, 26], difference-differential fields of characteristic zero [4], or with several commuting derivations [29, 16].

In the end, we give a differential version of the Basis Theorem in terms of radical differential ideals; namely, every radical differential ideal in $K\{\bar{x}\}$ is finitely generated. To sum up these works by A. Robinson, L. Blum, E.R. Kolchin, D. Pierce and A. Pillay, we have the following:

Fact 1.1. *Let (K, δ) be a differential field of characteristic zero. Then, the following are equivalent:*

(1) *(K, δ) is differentially closed.*

(2) (K,δ) *is existentially closed in every differentially algebraic extension.*
(3) (K,δ) *is constrainedly closed.*
(4) *for each* n *and every prime differential ideal* P *of* $K\{x_1,\ldots,x_n\}$*, if* $g \in K\{x_1,\ldots,x_n\} \setminus P$ *then there is* $\bar{a} \in K^n$ *such that* $f(\bar{a}) = 0$ *for all* $f \in P$ *and* $g(\bar{a}) \neq 0$*.*
(5) *for any nonzero* $f, g \in K\{x\}$ *and* $\operatorname{ord} g < \operatorname{ord} f$*, there exists* $a \in K$ *such that* $f(a) = 0$ *and* $g(a) \neq 0$*.*
(6) *let* V *and* W *be* K*-irreducible affine varieties over* K *with* $W \subseteq \tau V$ *and* $\pi|_W : W \to V$ *dominant. If* O_V *and* O_W *are nonempty Zariski-open subsets over* K *of* V *and* W*, respectively, then there is a* K*-rational point* $a \in O_V$ *such that* $(a, \delta a) \in O_W$*.*

Differential closure is defined as analogous to algebraic closure; $\bar{K} \supseteq K$ is a *differential closure* of K if $\bar{K}$ is differentially closed and for any differentially closed $L \supseteq K$, there exists a differential K-embedding $f : \bar{K} \to L$. L. Blum observed a prime model extension in DCF_0 is exactly differential closure, and she also showed that the theory DCF_0 is ω-stable. In general, in ω-stable theory, a prime model extension exists [35] and it is unique [43]. However, a differential closure does not have exactly the same properties as an algebraic closure. S. Shelah [44], E. R. Kolchin [25] and M. Rosenlicht [41] independently showed that a differential closure is not minimal in general, that is, there exists a differential closed field L such that $K \subseteq L \subset \bar{K}$). For instance, the differential closure of $\mathbb{Q}$ is not minimal.

Strong minimality of the solution set of algebraic differential equations has been studied with many successful applications to functional transcendence results. A differential equation over (K,δ), $f(x) = 0$ with $\operatorname{ord} f = n$, is strongly minimal if f is absolutely irreducible and for all element a with $f(a) = 0$ and any differential field extension (L,δ) of (K,δ), we have

$$tr.deg(L\langle a\rangle/L) = 0 \text{ or } n.$$

For instance, the differential equation satisfied by the j-function [14], Painlevé equations with generic coefficients [36], equations satisfied by Γ-automorphic functions on upper half-plain for Γ a Fuchsian group of first kind [5], to name a few (see also [12]), are all strongly minimal. Also Zilber's Trichotomy in DCF_0 for strongly minimal sets is one of the reason for strong minimality to have deep applications; for instance, differential Galois theory [38], the proof of Modell-Lang conjecture for function fields [18], (Manin-Manford conjecture [19] in the context of difference field) among others.

2. Separably differentially closed fields and separable differential closure

In this section, we describe separably differentially closed fields and separable differential closure. We fix a differential field (K,δ) is of arbitrary characteristic p.

As differentially closed fields of characteristic zero and their model-theoretic properties provide us a suitable universal domain to study for the solution sets of algebraic differential equations, differential Galois theory, differential algebraic group, differential algebraic geometry and its application to number theory, we expect that separably differentially closed fields have the right universal domain for the case of those in positive characteristic. (Remark that C. Wood established the foundation of the theory of differentially closed fields of positive characteristic, and the differential closures over differential perfect fields where we recall that K is differentially perfect if either its characteristic is zero or the p-th power of K coincides with the constants of K, that is, $[C_K : K^p] = 1$).

Separably differentially closed fields were introduced by the author and O. León Sánchez in [20, 21] as a differential analogous to separably closed fields; (K, δ) is said to be *separably differentially closed* if it is existentially closed in every differential field extension that is separable as a field. In other words, if (L, δ) is a differential extension of (K, δ) and L/K is separable, then (K, δ) is existentially closed in (L, δ).

In the spirit of L.Blum, we characterised them by analysing the shape of separable prime differential ideals; if P is a nonzero separable prime differential ideal of $K\{x\}$, then $P = [f] : s_f^\infty$ for any irreducible f of minimal rank in P. Conversely, if $f \in K\{x\}$ is irreducible with $s_f \neq 0$, then $[f] : s_f^\infty$ is a separable prime differential ideal and f is of minimal rank in it. In conclusion, we showed that (K, δ) is separably differentially closed if and only if for any nonzero $f, g \in K\{x\}$ with $s_f \neq 0$ and $\operatorname{ord} g < \operatorname{ord} f$, there exists $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$, where this scheme of conditions is expressed in a first-order fashion and denoted by SDCF.

Kolchin's constrained elements [24] (in arbitrary characteristic) are given as follows; let $\bar{\alpha}$ be an n-tuple from a differential field extension of K. We say that $\bar{\alpha}$ is *constrained* over K if $K\langle\bar{\alpha}\rangle/K$ is separable and there exists $g \in K\{\bar{x}\}$ with $g(\bar{\alpha}) \neq 0$ such that for every differential specialisation $\bar{\beta}$ of $\bar{\alpha}$ over K, with $K\langle\bar{\beta}\rangle/K$ separable, if $g(\bar{\beta}) \neq 0$ then $\bar{\beta}$ is a generic differential specialisation. Then our definition [20, 21] of being *constrained closed* (in arbitrary characteristic) is that for every finite tuple $\bar{a}$ from a differential field extension, if $\bar{\alpha}$ is constrained over K then each entry of $\bar{\alpha}$ is in K. We take this definition which is different from the case of characteristic zero because $K\langle\bar{\alpha}\rangle/K$ is not generally constrained over K for every $\bar{\alpha}$ constrained over K (4. [21]). But this definition is reduced to the Kolchin's definition [25] if K is of characteristic zero.

In the manner of D. Pierce and A. Pillay, we characterised SDCF in a geometric setting by adding separability conditions; namely, given K-irreducible varieties V and W, we say that V is separable if the function field $K(V)$ is separable over K. We say that a morphism $\phi : W \to V$ over K is separable if it is dominant and the function field $K(W)$ is separable over the function field $K(V)$.

Before summarizing our results, we make some comments for Differential Basis Theorem in positive characteristic. Y.Ershov [13] showed that the completions of the theory of separably closed fields are determined by specifying the degree of imperfection e; that is, the degree of K over K^p, $[K : K^p] = p^e$. In my Thesis [20], all the work has been done with the restriction of what we call *differential degree of imperfection* being finite, $[C_K : K^p] < \infty$. Thus the following Basis theorem are used through the Thesis [24, §III]; namely, every radical differential ideal in $K\{\bar{x}\}$ is finitely generated if and only if $[C_K : K^p]$ is finite. However, in the paper [21], we extended the work including infinite differential degree of imperfection, $[C_K : K^p] = \infty$, then we have the following Basis Theorem [24, §III]; every separable prime differential ideal of $K\{\bar{x}\}$ is finitely generated as a radical differential ideal.

To sum up, separably differentially closed fields are characterised as the following [20, 21].

Theorem 2.1. *Let (K, δ) be a differential field of arbitrary characteristic. Then, the following are equivalent:*

(1) *(K, δ) is separably differentially closed.*
(2) *for every differentially algebraic extension (L, δ) of (K, δ), if L/K is separable (as fields) then (K, δ) is existentially closed in (L, δ).*
(3) *(K, δ) is constrainedly closed.*
(4) *for each n and every separable prime differential ideal P of $K\{x_1, \ldots, x_n\}$, if $g \in K\{x_1, \ldots, x_n\} \setminus P$ then there is $\bar{a} \in K^n$ such that $f(\bar{a}) = 0$ for all $f \in P$ and $g(\bar{a}) \neq 0$.*
(5) *For any nonzero $f, g \in K\{x\}$ with $s_f \neq 0$ and $\operatorname{ord} g < \operatorname{ord} f$, there exists $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.*
(6) *Let V and W be K-irreducible affine varieties over K with $W \subseteq \tau V$, W separable, and $\pi|_W : W \to V$ separable. If O_V and O_W are nonempty Zariski-open subsets over K of V and W, respectively, then there is a K-rational point $a \in O_V$ such that $(a, \delta a) \in O_W$.*

Remark that the theorem above is reduced to the case of characteristic zero by the following points; namely, if K is characteristic zero, then

(i) Separably differentially closed fields are differentially closed fields.
(ii) Separable prime differential ideals are prime differential ideals
(iii) Constrainedly closed differential fields and constrained elements (by definition in this section) are constrainedly closed differential fields and constrained elements (by the definition in previous section)

The completions of the theory of separably differentially closed fields are determined by specifying the differential degree of imperfection ϵ and characteristic p. Adding differential λ-functions as differential analogous to its algebraic one, one can show that the theory is stable.

Given (K, δ) a differential field of positive characteristic p with a differential degree of imperfection ϵ as analogous to the separable closure, its suitable closure should preserve same differential degree of imperfection. Then $\bar{K} \supseteq K$ is said to be a *separable differential closure* of K if $\bar{K}$ is separably differentially closed with differential degree of imperfection ϵ and for any separably differentially closed extension L of K with the differential degree of imperfection ϵ, there exists a differential K-embedding $f : \bar{K} \hookrightarrow L$.

The existence of the separable differential closure is guaranteed by the constrained construction [21, chapter 5]. We showed that the existence and uniqueness of prime model extension over each differential fields of positive characteristic p with a differential degree of imperfection ϵ [20, 21]. Since by quantifier elimination in $\mathrm{SDCF}^{\ell}_{p,\epsilon}$, any embeddings of the model is elementary. Thus, the existence of a prime model over K is deduced by the existence of a separable differential closure. The results that the theory is stable and countable implies the uniqueness of the prime model extensions [21, chapter 6].

One can naturally ask if a separable differential closure is minimal in general (for instance, the minimality of a (separable) differential closure of $\mathbb{F}_p$). Also we have a question for a version of Zilber's Dichotomy in SDCF. (Remark that if (K, δ)

is separably differentially closed fields, then K and C_K are separably closed fields with infinite differential degree of imperfection [21], but we only know that Zilber's Dichotomy holds for the case of finite degree of imperfection [18].)

3. Differential large fields with stable theory

Infinite ω-stable fields and infinite superstable fields are algebraically closed by A. Macintyre [32] and G. Cherlin and S. Shelah [10] respectively. C. Wood [47] showed that separably closed fields are stable, and conversely the famous conjecture that "Infinite stable fields are separably closed" has remained open for a long time. (Remark that T. Scalon showed that infinite stable fields are Artin-Schreier closed [23]). Several special cases of the conjecture have been shown; for instance, W. Jonson, C. Tran, E. Walsberg and J. Ye [22] showed that a large stable field is separably closed where we recall that a field K is large if every irreducible variety over K with smooth K-rational point has Zariski dense set of K-rational points.

O. León Sánchez and M. Tressl [31] introduced the notion of differentially largeness in characteristic zero as an analogue to largeness of fields; a differential field (K, δ) of characteristic zero is differentially large if it is large and (K, δ) is existentially closed in every differential extension in which K is existentially closed as a field. The author and W. Mikolaj conjectured that every differential large stable fields is separably differentially closed. O. León Sánchez's (unpublished) work suggested that a differential field (K, δ) of arbitrary characteristic is differentially large if and only if it is large and (K, δ) is existentially closed in every differential extension in which K is existentially closed as a field. He also axiomatised them as follows; (K, δ) is differentially large if and only if for every pair of nonzero (f, g) in $K\{x\}$ with $S_f \neq 0$ and $\operatorname{ord} f > \operatorname{ord} g$, if $f = 0$ and $gS_f \neq 0$ has a algebraic solution in K, then $f = 0$ and $g \neq 0$ has a differential solution in K. Then one can easily show that every differentially large stable field is separably differentially closed; assuming that (K, δ) is differentially large and stable. Since an infinite stable fields is separably closed, K is separably closed, and (K, δ) is differentially large, we have (K, δ) being existentially closed in every differential field extension which is separable, that is, (K, δ) is separably differentially closed.

Corollary 3.1. *Every differentially large stable field is separably differentially closed.*

4. Difference-differential fields

R.Cohn [6], in 1970, studied difference-differential algebra, and then Hardouin and Singer [17] studied linear difference-differential Galois theory. E. Hrushovski showed the existence of model companion of the theory of difference-differential fields in characteristic zero and R. Bustamante [3, 4] studied their algebraic and model-thoeretic properties. These studies are extended to the partial differential case by O. León Sánchez [29] (see also [16]). The theory for both ordinary and partial cases is supersimple and satisfy Zilber's Dichotomy which is shown by using proper jet spaces [39] (see also [34]).

Let (K, δ) be a differential field of characteristic zero. A difference-differential field (K, δ, σ) is a differential field (K, δ) with an automorphism σ which commutes with the derivation δ. A version of Basis Theorem for difference-differential fields is the following; every perfect (δ,σ)-ideal is finitely generated where a perfect (δ,σ)-ideal I is a differential ideal and $f^m\sigma(f)^n \in I$ implies $f \in I$.

We now introduce two types of the form of model companion of the theory of difference-differential fields in the spirit of R. Bustamante and O. León Sánchez with restriction to the ordinal differential case.

Fact 4.1. *Let (K,δ,σ) be a difference-differential field of characteristic zero. Then, the following are equivalent:*

(1) *(K,δ,σ) is existentially closed.*
(2) *Suppose that (K,δ) is differentially closed. Let V and W be K-irreducible δ-varieties over K with $W \subseteq V \times V^\sigma$ such that W projects generically both onto V and V^σ. If O_V and O_W are nonempty δ-open subsets over K of V and W, respectively, then there is a K-rational point $a \in O_V$ such that $(a,\delta a) \in O_W$.*
(3) *Suppose that (K,δ) is differentially closed. Let U, V and W be K-irreducible affine varieties over K with $U \subseteq V \times V^\sigma$ and $W \subseteq \tau V$ such that U generically both onto V and V^σ and W projects generically onto U. If $\pi_1(W)^\sigma = \pi_2(W)$ and δ-generic point of W projects onto a δ-generic point of $\pi_1(W)^\sigma$ and onto δ-generic point of $\pi_2(W)$, then there is a K-rational point $a \in V$ such that $(a,\sigma a) \in U$ and $(a,\delta a,\sigma a,\sigma\delta a) \in W$ where we let $\pi_1 : \tau(V) \times \tau(V)^\sigma \to \tau(V)$ and $\pi_2 : \tau(V) \times \tau(V)^\sigma \to \tau(V)^\sigma$, with which identify $\tau(V \times V^\sigma)$.*

Note that in the notations of (3) in the fact above, we say that a point a in an extension of K is a δ-generic of W over K if $(a,\delta a)$ is a generic of W over K, and $tr.deg(\delta^i a/K(a,\ldots,\delta^{i-1}a)) = tr.deg(\delta a/K(a))$ for all $i > 1$. We denote by V^σ the δ-variety obtained by applying σ to the coefficients of the differential polynomials defining V. Remark that the statement of (2) above is described by differential variety, but that of (3) is by algebraic variety.

One can naturally ask if the theory of separably differentially closed fields with a generic differential automorphism has model companion. We note that Z. Chatzidakis [8] showed that in the case of separably closed fields, such model companion exists.

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