On locally o-minimal structures

前園久智 (Hisatomo Maesono)
早稲田大学グローバルエデュケーションセンター
(Global Education Center, Waseda University)

概要

abstract Locally o-minimal structures are some local adaptation from o-minimal structures. They were investigated, e.g. in [1], [2]. We characterize types of definably complete locally o-minimal structures. In particular, we argue about the Rudin-Keisler order of them.

1. Introduction

At first we recall some definitions and fundamental facts.

Definition 1. Let M be a densely linearly ordered structure without endpoints.

M is *o-minimal* if every definable subset of M^1 is a finite union of points and intervals.

M is *locally o-minimal* if for any element $a \in M$ and any definable subset $X \subset M^1$, there is an open interval $I \subset M$ such that $I \ni a$ and $I \cap X$ is a finite union of points and intervals.

M is *definably complete* if any definable subset X of M^1 has the supremum and infimum in $M \cup \{\pm\infty\}$.

Example 2. [1], [2]

$(\mathbf{R}, +, <, \mathbf{Z})$ where $\mathbf{Z}$ is the interpretation of a unary predicate, and $(\mathbf{R}, +, <, \sin)$ are definably complete locally o-minimal structures.

Fact 3. [1] *Definably complete local o-minimality is preserved under elementary equivalence.*

Thus we argue in a sufficiently large saturated model $\mathcal{M}$ and we assume that the theory T is countable in this note.

We characterize locally o-minimal structures by means of behavior of 1-variable types. They consider two kinds of 1-types by the way to cut linear orders of parameter sets, e.g. in [6]. Here we consider nonisolated types only.

Definition 4. Let M be a densely linearly ordered structure and $A \subset M$. And let $p(x) \in$

$S_1(A)$.

We say that $p(x)$ is *cut over* A if for any $a \in A$, if $a < x \in p(x)$, then there is $b \in A$ such that $a < b < x \in p(x)$, and similarly if $x < a \in p(x)$, then there is $c \in A$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1(A)$ is *noncut over* A if $q(x)$ is not a cut type.

Remark 5. *Let M be a densely linearly ordered structure and $A \subset M$. And let $p(x) \in S_1(A)$ be noncut.*

There are four kinds of noncut types ;

$p(x) = \{b < x < a : b < a \in A\}$ for some fixed a, or $\{a < x < b : a < b \in A\}$ for some fixed a.

Here we call these types bounded noncut types.

And $p(x) = \{b < x : b \in A\}$ or $\{x < b : b \in A\}$.

We call these types unbounded noncut types.

For $p(x) \in S_1(A)$, if $p(x) \restriction < \vdash p(x)$, then we say that $p(x)$ is order-complete or $<$-complete, and otherwise, if $p(x) \restriction < \nvdash p(x)$, then we say that $p(x)$ is order-incomplete or $<$-incomplete. ($p(x) \restriction <$ means the partial type restricted to the order relation.)

Fact 6. [4] *Let M be a definably complete locally o-minimal structure and $p(x) \in S_1(M)$.*

(Here we consider types over structures only.)

If $p(x)$ is a bounded noncut type, then $p(x)$ is $<$-complete and definable, and

If $p(x)$ is an unbounded noncut type, then $p(x)$ may be both $<$-complete and $<$-incomplete, if $p(x)$ is $<$-complete, then it is definable.

And if $p(x)$ is a cut type, then $p(x)$ may be both $<$-complete and $<$-incomplete, and $p(x)$ is not definable in general.

Apology : There is incorrect description in my proceeding [4] about fact above. It is pointed out by M.Fujita. I apologize and correct it here.

We recall the next proposition which is used frequently in the argument. We call the fact strong local monotonicity property.

Proposition 7. [3] *Let M be a definably complete locally o-minimal structure.*

And let I be an interval and $f : I \longrightarrow M$ be a definable function.

Then there is a definable partition of $I = X_d \cup X_c \cup X_+ \cup X_-$ satisfying the following conditions :

(1) *X_d is discrete and closed,*

(2) *X_c is open and f is locally constant on X_c,*

(3) *X_+ is open and f is locally strictly increasing and continuous on X_+,*

(4) X_- *is open and* f *is locally strictly decreasing and continuous on* X_-.

2. Prime models of definably complete locally o-minimal theory

I referred to the next fact at RIMS meeting 2021 and wrote it in the proceeding [4].

Lemma 8. *Let* M *be a definably complete locally o-minimal structure and* $A \subset M$ *with* $dcl(A) \neq \emptyset$.

Then the isolated types of $Th(M, a)_{a \in A}$ *are dense.*

After that I confirmed the next fact.

Proposition 9. *Let* M *be a definably complete locally o-minimal structure.*

Then for any $A \subset \mathcal{M}$ *with* $dcl(A) \neq \emptyset$*, there is a unique prime model over* A *up to isomorphism over* A.

We recall the next lemma which I use in the following. This lemma proved by A.Tsuboi and W.Komine first. By means of the fact, the proposition above is proved similarly in [5].

Lemma 10. *Let* M *be a definably complete locally o-minimal structure and* $dcl(\emptyset) \neq \emptyset$.

Then for any $a, b \in \mathcal{M}$ *and* $A \subset \mathcal{M}$*, if* $b \in dcl(aA) \setminus dcl(A)$*, then* $a \in dcl(bA)$.

Sketch of proof ;

Let $b \in dcl(aA) \setminus dcl(A)$*. We may assume that* A *is finite. Thus there is a definable function* f *over* A *such that* $b = f(a)$*. As there is a prime model over* aA *which has some definable element, and by the strong monotonicity property and* $b \notin dcl(A)$*, there is an interval* $I = (c, d)$ *such that* c*,* $d \in dcl(A)$ *and* $a \in I$*, and* f *is monotone and continuous on* I*. Then* $a \in dcl(bA)$.

∎

3. Rudin-Keisler order of types in definably complete locally o-minimal structures

We recall some definitions at first.

Definition 11. Let $p(\bar{x})$, $q(\bar{x}) \in S_n(M)$ for some model M of a complete theory T.

We say that $p(\bar{x})$ is greater than or equal to $q(\bar{x})$ for the $Rudin-Keisler\ ordering$, and we write $q \leq_{RK} p$, if $q(\bar{x})$ is realized in $M(p)$ where $M(p)$ is a prime model over $M \cup \{\bar{a}\}$ for some realization $\bar{a}$ of $p(\bar{x})$.

In general, for types p, q, the lengths of variables may be different. We consider the RK-ordering for 1-types of definably complete locally o-minimal structures.

In the following, M is a definably complete locally o-minimal structure and we assume that $dcl(\emptyset) \neq \emptyset$ for the sake of simplicity.

The next proposition is proved similarly in [6]. But we must use the strong monotonicity property instead of the ordinary monotonicity theorem in o-minimal structures.

We consider three kinds of types ; bounded noncut, unbounded noncut and cut $\cdots\cdots(*)$

Proposition 12. *Let $p(x),\ q(x) \in S_1(M)$ and these be different kinds of types in $(*)$.*
Then $p(x) \not\leq_{RK} q(x)$ and $q(x) \not\leq_{RK} p(x)$.

And we can prove the next fact.

Proposition 13. *Let $p(x),\ q(x) \in S_1(M)$ and let $p(x)$ be $<$-complete and $q(x)$ be $<$-incomplete.*
Then $p(x) \not\leq_{RK} q(x)$ and $q(x) \not\leq_{RK} p(x)$.

Lemma 14. *Let $p(x),\ q(x) \in S_1(M)$ and both types be $<$-complete.*
Then $p(x) \leq_{RK} q(x)$ is an equivalence relation.

Sketch of proof ;
Let $a \models p(x)$ and $b \models q(x)$. If $b \in M(a)$ where $M(a)$ is a prime model over Ma, then there is a realization c of $q(x)$ such that $c \in dcl(Ma)$. By the exchange of acl, $a \in dcl(Mc)$. ∎

Lemma 15. *Let $p(x),\ q(x) \in S_1(M)$ and both types be $<$-incomplete.*
Suppose that $q(x) \leq_{RK} p(x)$ by some $a \models p(x)$ and $b \models q(x)$.
Then if $b \notin dcl(aM)$, then there is a definable function f over M such that ;
for intervals $I = (a,c)$ or (c,a) and $J = (d,e)$ with $b \in (d,e)$, and $c,d,e \in dcl(Ma)$,
$f : I \longrightarrow J$ is monotone and continuous, and I and J generate complete types over Ma.

We try to characterize the RK-order of types along the argument in ω-stable case.

Lemma 16. *Let $p(x),\ q(x) \in S_1(M)$ be either bounded noncut types or $<$-complete unbounded noncut types.*
For any M' with $M \prec M'$, let $p'(x), q'(x) \in S_1(M')$ be their heirs over M'.
If $p(x) \leq_{RM} q(x)$, then $p'(x) \leq_{RM} q'(x)$.

In the next proposition, M is not a locally o-minimal structure. This fact is well known.

Proposition 17. *Let T be ω-stable and let $p(\bar{x}),\ q(\bar{x}) \in S_n(M)$ be RK-minimal.*

Then the following conditions are equivalent ;
(1) $p(\bar{x})$ *is orthogonal to* $q(\bar{x})$.
(2) $p(\bar{x})$ *is almost orthogonal to* $q(\bar{x})$.
(3) $p(\bar{x})$ *and* $q(\bar{x})$ *are not RK-equivalent.*

For the parallel argument in local o-minimality context, we must modify some definitions in the proposition above.

Definition 18. Let $p(\bar{x})$, $q(\bar{y}) \in S(\mathcal{M})$ be invariant types.
We define the type $p(\bar{x}) \otimes q(\bar{y}) \in S_{\bar{x}\bar{y}}(\mathcal{M})$ as $\mathrm{tp}(\bar{a}\bar{b}/\mathcal{M})$ where $\bar{b} \models q$ and $\bar{a} \models p|\mathcal{M}\bar{b}$.
Two invariant types $p(\bar{x})$ and $q(\bar{y})$ *commute* if $p(\bar{x}) \otimes q(\bar{y}) = q(\bar{y}) \otimes p(\bar{x})$.
Let $p(\bar{x})$, $q(\bar{y}) \in S(A)$.
$p(\bar{x})$ and $q(\bar{y})$ are *weakly orthogonal* if $p(\bar{x}) \cup q(\bar{y})$ implies a complete type over A.
$p(\bar{x})$ and $q(\bar{y})$ are *orthogonal* if they are weakly orthogonal as global types (when $A = \mathcal{M}$).

Lemma 19. *Let* $p(x)$, $q(y) \in S_1(M)$, *and both* $p(x)$ *and* $q(y)$ *be definable types.*
Assume that $q(y) \not\leq_{RK} p(x)$.
Then for the definable extensions $p'(x)$, $q'(y) \in S_1(\mathcal{M})$ *of* $p(x)$ *and* $q(y)$ *are commute over* M, *that is,* $p'(x) \otimes q'(y)\,|_M = q'(y) \otimes p'(x)\,|_M$.

Sketch of proof ;
Let $q(y) = \{\, b < y < m \,:\, b < m \in M \,\}$ *for some fixed* $b \in M$. *(Other cases are proved similarly.) And let* $q'(y) \otimes p'(x)\,|_M \vdash \phi(x, y, \bar{m})$ *where* $\bar{m} \in M$. *For* $\phi(x, y, \bar{m})$, *there is the definition* $d_p\phi(y, \bar{m}')$ *over* M *of* $p'(x)$. *Let* $a \models p'(x)$. *Thus* $q'(x)\,|\,\mathcal{M}a \vdash \phi(a, y, \bar{m}) \wedge d_p\phi(y, \bar{m}')^\eta$ *for* $\eta = 0, 1$ *(where according to* η, *it means affirmation or negation). Let* $\theta(y) = \phi(a, y, \bar{m}) \wedge d_p\phi(y, \bar{m}')^\eta$. *So there is* $b_\theta \in M(a)$ *such that for any* c *with* $b < c < b_\theta$, $\models \theta(c)$. *As* $q(y) \not\leq_{RK} p(x)$, *there is* $m'' \in M$ *such that* $b < m'' < b_\theta$. *Thus* $\models \phi(a, m'', \bar{m}) \wedge d_p\phi(m'', \bar{m}')^\eta$. *That is,* $\phi(x, m'', \bar{m}) \in p(x)$ *and* $d_p\phi(y, \bar{m}') \in q(y)$. *Then for any* $b' \models q'(y)$, $\phi(x, b', \bar{m}) \in p'(x)\,|\,\mathcal{M}b'$. ∎

4. Further problems

In ω-stable context, the RK-order is characterized by strongly regular types. They have the minimal Morley rank and degree = 1. Can we have analogous argument by means of dp-rank and so on ?

And the RK-order is extended to the domination order of types over $\aleph_\epsilon$-saturated models. Can we generalize the argument in the same way (to a certain extent) ?

References

[1] C.Toffalori and K.Vozoris, *Note on local o-minimality*, Math.Log.Quart., 55, pp 617–632, 2009.

[2] T.Kawakami, K.Takeuchi, H.Tanaka and A.Tsuboi, *Locally o-minimal structures*, J. Math. Soc. Japan, vol.64, no.3, pp 783–797, 2012.

[3] M.Fujita, *Locally o-minimal structures with tame topological properties*, J. Symb. Logic, to appear.

[4] H.Maesono, *Some remarks on locally o-minimal structures*, *RIMS Kôkyûroku*, no.2218, pp 119–126, 2022.

[5] A.Pillay and C.Steinhorn, *Definable sets in ordered structures. I*, Trans of A.M.S., vol.295, no.2, pp 565–592, 1986.

[6] D.Marker, *Omitting types in o-minimal theories*, J. Symb. Logic, vol.51, no.1, pp 63–74, 1986.

[7] D.Marker and C.Steinhorn, *Definable types in o-minimal theories*, J. Symb. Logic, vol.59, no.1, pp 185–198, 1994.

[8] H.Schoutens, *O-minimalism*, J. Symb. Logic, vol.79, no.2, pp 355–409, 2014.

[9] S.Shelah, *Strongly dependent theories*, Israel J. Math., vol.204, pp 1–83, 2014.

[10] E.Hrushovski and A.Pillay, *On NIP and invariant measures*, J. Eur. Math. Soc., vol.13, pp 1005–1061, 2011.

[11] P.Simon, *On dp-minimal ordered structures*, J. Symb. Logic, vol.76, no.2, pp 448–460, 2011.

[12] P.Simon, *Dp-minimality : invariant types and dp-rank*, J. Symb. Logic, vol.79, no.4, pp 1025–1045, 2014.

[13] A.Chernikov, *Theories without the tree property of the second kind*, Ann. Pure. and Appl. Logic, vol.165, pp 695–723, 2014.

[14] P.A.Estevan and I.Kaplan, *Non-forking and preservation of NIP and dp-rank*, Annals.P.A.Logic, 172, pp 1–30, 2021.

[15] M.Makkai, *A survey of basic stability theory with emphasis on orthgonality and regularity*, Israel J. Math., vol.49, pp 181–238, 1984.

[16] D.Lascar, *Stability in model theory*, Pitman monographs and surveys in pure and applied mathematics, vol.36, Longman scientific and technical, 1987.

[17] L.van den Dries, *Tame topology and o-minimal structures*, London Math. Soc. Lecture Note Ser, 248, Cambridge University Press, 1998.