On the Structure of Hrushovski's Pseudoplanes Associated to Irrational Numbers

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Abstract

Let α be an irrational number, and a/b a reduced fraction. Suppose $2/3 < \alpha < a/b < 3/4$ and b is sufficiently large. Let B be a canonical twig for a/b and A the set of all leaves in B. Let $p \in B$ be a good vertex of B over A. Let M be the generic structure for $(\mathbf{K}_f, <)$ where f is the Hrushovski's log-like function associated to α. Assume that B is a closed subset of M. Let D be the orbit of p over A in M. Then $M = \mathrm{cl}(D)$. Actually, we can prove this only assuming $0 < \alpha < a/b < 1$.

1 Introduction

We show that Hrushovski's pseudoplanes associated irrational numbers introduced in his 1988 preprint [6] is a closure of an orbit of some point p over some finite set A. The "rank" of the type of p over A can be arbitrarily small positive real number. This statement is a weaker version of the monodimensionality introduced by D. Evans, Z. Ghadernezhad, and K. Tent [4].

In this paper, we assume that the irrational number α satisfies $2/3 < \alpha < 3/4$ instead of $1/2 < \alpha < 2/3$ assumed in Hrushovski's preprint [6]. With little modification, we can prove the same statement assuing $1/2 < \alpha < 2/3$, or even $0 < \alpha < 1$. We essentially use notation and terminology from Baldwin-Shi [2] and Wagner [15]. We also use some terminology from graph theory [3].

For a set X, $[X]^n$ denotes the set of all subsets of X of size n, and $|X|$ the cardinality of X.

We recall some of the basic notions in graph theory we use in this paper. These appear in [3]. Let G be a graph. $V(G)$ denotes the set of vertices of G. Vertices will be also called *points*. $E(G)$ is the set of edges of G. $E(G)$ is a subset of $[V(G)]^2$. $|G|$ denotes $|V(G)|$ and $e(G)$ denotes $|E(G)|$. The *degree* of a vertex v is the number of edges at v. A vertex of degree 1 is a *leaf*. G is a *path* $x_0x_1\dots x_k$ if $V(G) = \{x_0, x_1, \dots, x_k\}$ and $E(G) = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$ where the x_i are all distinct. x_0 and x_k are *ends* of G. The number of edges of a path is its *length*. A path of length 0 is a single vertex. G is a *cycle* $x_0x_1\dots x_{k-1}x_0$ if $k \geq 3$, $V(G) = \{x_0, x_1, \dots, x_{k-1}\}$ and $E(G) = \{x_0x_1, x_1x_2, \dots, x_{k-2}x_{k-1}, x_{k-1}x_0\}$ where the x_i are all distinct. The number of edges of a cycle is its *length*. A non-empty graph G is *connected* if any two of its vertices are linked by a path in G. A *connected component* of a graph G is a maximal connected subgraph of G. A *forest* is a graph not containing any cycles. A *tree* is a connected forest.

To see a graph G as a structure in the model theoretic sense, it is a structure in language $\{E\}$ where E is a binary relation symbol. $V(G)$ will be the universe, and $E(G)$ will be the interpretation of E. The language $\{E\}$ will be called *the graph language*.

Suppose A is a graph. If $X \subseteq V(A)$, $A|X$ denotes the substructure B of A such that $V(B) = X$. If there is no ambiguity, X denotes $A|X$. We usually follow this convention. $B \subseteq A$ means that B is a substructure of A. A substructure of a graph is an induced subgraph in graph theory. $A|X$ is the same as $A[X]$ in Diestel's book [3].

Let A, B, C be graphs such that $A \subseteq C$ and $B \subseteq C$. AB denotes $C|(V(A) \cup V(B))$, $A \cap B$ denotes $C|(V(A) \cap V(B))$, and $A - B$ denotes $C|(V(A) - V(B))$. If $A \cap B = \emptyset$, $E(A,B)$ denotes the set of edges xy such that $x \in A$ and $y \in B$. We put $e(A,B) = |E(A,B)|$. $E(A,B)$ and $e(A,B)$ depend on the graph in which we are working.

Let D be a graph and A, B, and C substructures of D. We write $D = B \otimes_A C$ if $D = BC$, $B \cap C = A$, and $E(D) = E(B) \cup E(C)$. $E(D) = E(B) \cup E(C)$ means that there are no edges between $B - A$ and $C - A$. D is called a *free amalgam of B and C over A*. If A is empty, we write $D = B \otimes C$, and D is also called a *free amalgam of B and C*.

Definition 1.1. Let α be a real number such that $0 < \alpha < 1$.

(1) For a finite graph A, we define a predimension function δ by $\delta(A) = |A| - e(A)\alpha$.

(2) Let A and B be substructures of a common graph. Put $\delta(A/B) = \delta(AB) - \delta(B)$.

Definition 1.2. Let A and B be graphs with $A \subseteq B$, and suppose A is finite.

$A < B$ if whenever $A \subsetneq X \subseteq B$ with X finite then $\delta(A) < \delta(X)$.

We say that A is *closed* in B if $A < B$. We also say that B is a *strong* extension of A.

We say that A is *almost closed* in B, written $A <^{-} B$, if whenever $A \subsetneq X \subsetneq B$ with X finite then $\delta(A) < \delta(X)$.

Let $\mathbf{K}_\alpha$ be the class of all finite graphs A such that $\emptyset < A$.

Some facts about $<$ appear in [2, 15, 16]. Some proofs are given in [11].

Fact 1.3. *Let A and B be disjoint substructures of a common graph. Then $\delta(A/B) = \delta(A) + e(A,B)$.*

Fact 1.4. *If $A < B \subseteq D$ and $C \subseteq D$ then $A \cap C < B \cap C$.*

Fact 1.5. *Let $D = B \otimes_A C$.*

(1) $\delta(D/A) = \delta(B/A) + \delta(C/A)$.

(2) *If $A < C$ then $B < D$.*

(3) *If $A < B$ and $A < C$ then $A < D$.*

Let B, C be graphs and $g : B \to C$ a graph embedding. g is a *closed embedding* of B into C if $g(B) < C$. Let A be a graph with $A \subseteq B$ and $A \subseteq C$. g is a *closed embedding over A* if g is a closed embedding and $g(x) = x$ for any $x \in A$.

In the rest of the paper, $\mathbf{K}$ denotes a class of finite graphs closed under isomorphisms.

Definition 1.6. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *amalgamation property* if for any finite graphs $A, B, C \in \mathbf{K}$, whenever $g_1 : A \to B$ and $g_2 : A \to C$ are closed embeddings then there is a graph $D \in \mathbf{K}$ and closed embeddings $h_1 : B \to D$ and $g_2 : C \to D$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs A, B, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

A countable graph M is a *generic structure* of $(\mathbf{K}, <)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and A is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B < M$.

(2) If $A \subseteq M$ then $A \in \mathbf{K}$.

(3) For any A, $B \in \mathbf{K}$, if $A < M$ and $A < B$ then there is a closed embedding of B into M over A.

Let A be a finite structure of M. There is a smallest B satisfying $A \subseteq B < M$, written $\mathrm{cl}(A)$. The set $\mathrm{cl}(A)$ is called the *closure* of A in M.

Fact 1.7 ([2, 15, 16])**.** *Let $(\mathbf{K}, <)$ be an amalgamation class. Then there is a generic structure of $(\mathbf{K}, <)$. Let M be a generic structure of $(\mathbf{K}, <)$. Then any isomorphism between finite closed substructures of M can be extended to an automorphism of M.*

Definition 1.8. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *free amalgamation property* if whenever $D = B \otimes_A C$ with $B, C \in \mathbf{K}$, $A < B$ and $A < C$ then $D \in \mathbf{K}$.

By Fact 1.5 (2), we have the following.

Fact 1.9. *Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. If $(\mathbf{K}, <)$ has the free amalgamation property then it has the amalgamation property.*

Definition 1.10. Let $\mathbb{R}^+$ be the set of non-negative real numbers. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing concave (convex upward) unbounded function. Assume that $f(0) = 0$, and $f(1) \leq 1$. We assume that f is piecewise smooth. $f'_+(x)$ denotes the right-hand derivative at x. We have $f(x+h) \leq f(x) + f'_+(x)h$ for $h > 0$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_\alpha \mid B \subseteq A \Rightarrow \delta(B) \geq f(|B|)\}.$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic structure of $(\mathbf{K}_f, <)$ has a countably categorical theory [16].

A graph X is *normal to* f if $\delta(X) \geq f(|X|)$. A graph A belongs to $\mathbf{K}_f$ if and only if U is normal to f for any substructure U of A.

2 Hrushovski's Log-like Functions

Definition 2.1. Let α be a positive real number. x is called a *best approximation of α strictly from above with a denominator at most n* if x is a smallest rational number r such that $r = k/d > \alpha$ with $d \leq n$ where k and d are positive integers.

Definition 2.2 ([6])**.** Let α be a positive real number. We define x_n, e_n, k_n, d_n for integers $n \geq 1$ by induction as follows: Put $x_1 = 2$ and $e_1 = 1$. Assume that x_n and e_n are defined. Let r_n be the best approximation of α strictly from above with a denominator at most e_n. Let k_n/d_n be the reduced fraction satisfying $k_n/d_n = r_n$. Finally, let $x_{n+1} = x_n + k_n$, and $e_{n+1} = e_n + d_n$.

Let $a_0 = (0,0)$, and $a_n = (x_n, x_n - e_n\alpha)$ for $n \geq 1$. Let f_α be a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ whose graph on interval $[x_n, x_{n+1}]$ with $n \geq 0$ is a line segment connecting a_n and a_{n+1}. We call f_α a *Hrushovski's log-like function associated to* α.

Fact 2.3 ([6])**.** *Let f_α be a Hrushovski's log-like function and $\{x_i\}$, $\{e_i\}$, $\{k_i\}$, $\{d_i\}$ sequences in the definition of f_α.*

Suppose C is an extension of B by x points and z edges, $|B| \geq x_n$ and $x/z \geq k_n/d_n$ for some n, and B is normal to f_α. Then C is normal to f_α.

Fact 2.4 ([6])**.** *Let $D = B \otimes_A C$. If $\delta(A) < \delta(B)$, $\delta(A) < \delta(C)$, and A, B, C are normal to f_α then D is normal to f_α.*

Fact 2.5 ([6])**.** *Let α be a real number with $0 < \alpha < 1$. Then f_α is strictly increasing and concave, and $(\mathbf{K}_{f_\alpha}, <)$ has the free amalgamation property. Therefore, there is a generic structure of $(\mathbf{K}_{f_\alpha}, <)$. Any one point structure is closed in any structure in $\mathbf{K}_{f_\alpha}$. If α is rational then f_α is unbounded.*

The following is easy.

Lemma 2.6. *Let $C = A \otimes_p B$ where p is a single vertex and $A, B \in \mathbf{K}_f$. Then $C \in \mathbf{K}_f$. Any finite forests belong to $\mathbf{K}_f$.*

Lemma 2.7. *Suppose* $2/3 < \alpha < 3/4$.

(1) *The first several terms of the sequences defining f_α are given by the following chart with (k_5, d_5) being either $(3,4)$ or $(5,7)$:*

x_i	2	3	4	5	8	$\cdots$
e_i	1	2	3	4	8	$\cdots$
k_i	1	1	1	3	k_5	$\cdots$
d_i	1	1	1	4	d_5	$\cdots$

(2) *Suppose C is an extension of B by x points and z edges, $5 \leq |B|$, $3/4 \leq x/z$, and B is normal to f_α. Then C is normal to f_α.*

(3) *Suppose C is an extension of B by x points and z edges, $5 \leq |B|$, $z \leq (4/7)|B|$, $\alpha < x/z$, and B is normal to f_α. Then C is normal to f_α.*

Proof. (1) is straightforward. (2) holds by Fact 2.3 and (1).
(3) Choose i satisfying $x_i \leq |B| < x_{i+1}$. Since $x_4 = 5 \leq |B|$, we have $4 \leq x_i$. Then $x_i - 1 \leq e_i$ and $k_i/d_i \leq 3/4$. Also, we have $d_i \leq e_i$. So, $|B| < x_{i+1} = x_i + k_i = x_i + (k_i/d_i)d_i \leq (e_i + 1) + (3/4)e_i = (7/4)e_i + 1$. Hence, $|B| \leq (7/4)e_i$ and thus $z \leq (4/7)|B| \leq e_i$. By the choice of k_i/d_i, we have $k_i/d_i \leq x/z$. Since $x_i \leq |B|$, C is normal to f_α by Fact 2.3. □

3 Special Structures

Definition 3.1. Let h/k and h'/k' be reduced fractions of non-negative integers. $(h+h')/(k+k')$ is called a *mediant* of h/k and h'/k'. We say that $(h/k, h'/k')$ is a *Farey pair* if $h'k - hk' = 1$. Note that $0 \leq h/k < h'/k'$.

The following lemma is well-known.

Lemma 3.2. *Let $(h/k, h'/k')$ be a Farey pair and u, v positive integers.*

(1) *If $h/k < u/v < h'/k'$ then $k + k' \leq v$.*

(2) *Let h''/k'' be the mediant of h/k and h'/k'. Then $(h/k, h''/k'')$ and $(h''/k'', h'/k')$ are Farey pairs.*

Definition 3.3. Let u/v be a reduced fraction of positive integers. A graph W is called a *general twig* for u/v if the number of edges of W is v, the number of non-leaf vertices of W is u, and the set of all leaves of W is almost closed in W with respect to $\delta_{u/v}$. A general twig W for u/v is called a *twig* for u/v if there is a path $P = p_0 \cdots p_k$ in W such that p_0 is a leaf of W, p_k is a non-leaf vertex of W, and the paths from leaves of W other than p_0 to P are independent paths. The path P is called the *main path* of the twig W, p_0 the *left end* of the main path of W, and p_k the *right end* of the main path of W. Note that the left end of the main path of a twig is a leaf of the twig, and the right end of the main path is a non-leaf vertex of the twig. A twig is a twig for some reduced fraction.

Lemma 3.4. *Let $(h/k, h'/k')$ be a Farey pair, A a general twig for h'/k' and B a general twig for h/k. Suppose $D = A \otimes_c B$ where c is a non-leaf vertex of A as well as a leaf of B. Then D is a general twig for $(h+h')/(k+k')$.*

Proof. First of all, it is clear that the number of all edges in D is $k+k'$. Since vertex c is a leaf in B as well as a non-leaf vertex in A, the number of all non-leaf vertices in D is $h+h'$.

Let F be the set of all leaves of D, X a proper substructure of D with $F \subsetneq X$. Put $X_A = X \cap A$ and $X_B = X \cap B$. Then $X = X_A \otimes X_B$ if $c \notin X$ and $X = X_A \otimes_c X_B$ if $c \in X$. Let u be the number of all non-leaf vertices of A in X, v the number of all edges of A in X, u' the number of all non-leaf vertices of B in X, v' the number of all edges of B in X. Since c is a non-leaf vertex in A as well as a leaf in B, the number of non-leaf vertices of D in X is $u+u'$ and the number of edges of D in X is $v+v'$. So, $\delta(X/F) = (u+u') - (v+v')\alpha$ where $\alpha = (h+h')/(k+k')$. We have $h/k < h'/k' \leq u/v$ because A is a general twig for h'/k', and We also have $h/k \leq u'/v'$ becuase B is a general twig for h/k. Hence, $h/k < (u+u')/(v+v')$. Since the number of all edges in D is $k+k'$, X is a proper substructure of D, and D is connected, we have $v+v' < k+k'$. Note that h/k and $(h+h')/(k+k')$ form a Farey pair by Lemma 3.2 (2). Hence, we have $(h+h')/(k+k') \leq (u+u')/(v+v')$ by Lemma 3.2 (1). Since $v+v' < k+k'$, we cannot have $(u+u')/(v+v') = (h+h')/(k+k')$. □

Lemma 3.5. (1) *A path of length* 4 *is a general twig for* $3/4$. *It can be considered as a twig for* $3/4$ *having a main path of length* 2 *and a uniform height* 2. *This twig will be called a* 2-twig *for* $3/4$.

(2) *A path of length* 3 *is a general twig for* $2/3$. *It can be considered as a twig for* $2/3$ *having a main path of length* 1 *and a uniform height* 2. *This twig will be called a* 1-twig *for* $2/3$.

Definition 3.6. Two twigs are said to be *isomorphic* twigs if there is a graph isomorphism between them which preserves the main paths. A graph W is called a *concatenation* of two twigs W_1 and W_2 if $W = W_1' \otimes_c W_2'$ where W_1' is a twig isomorphic to W_1, W_2' is a twig isomorphic to W_2, and c is the left end of the main path of W_1' as well as the right end of the main path of W_2'. A graph $W = W_1 \otimes_{p_1} W_2 \otimes_{p_2} \cdots \otimes_{p_{k-1}} W_k$ is called a *chain of twigs* if each W_i is a twig and each p_i is a right end of the main path of W_i as well as the right end of the main path of W_{i+1} for $i = 1, \ldots, k-1$. $W_1 \otimes_{p_1} W_2 \otimes_{p_2} \cdots \otimes_{p_{j-1}} W_j$ with $j \leq k$ will be called a *left prefix* of W. W is said to be a chain of twigs satisfying certain property if each W_i has the property. For example, W is a chain of twigs for $2/3$ if each W_i is a twig for $2/3$. Let p_0 be the right end of the main path of W_1 and p_k the left end of the main path of W_k. The path from p_0 to p_k in W is called the main path of W, p_0 the left end of the main path of W, p_k the right end of the main path of W. Note that the paths from leaves of W other than p_0 to P are independent paths. We say that a chain of twigs has a *uniform height n* if the distance from any leaves other than the left end of the main path is n.

Lemma 3.7. *Let $(h/k, h'/k')$ be a Farey pair, W a twig for h/k, and W' a twig for h'/k'. Let u/v be a reduced fraction with $h/k < u/v < h'/k'$. Then there is a twig for u/v which is also a chain of twigs isomorphic to W or W'.*

Proof. We prove the lemma by induction on $v-(k+k')$. Let W'' be a concatenation of W and W'. Let h''/k'' be the mediant of h/k and h'/k'.

Suppose $u/v = h''/k''$. Then W'' is a twig for u/v by Lemma 3.4. We have the lemma in this case.

Suppose $u/v \neq h''/k''$. Then $h/k < u/v < h''/k''$ or $h''/k'' < u/v < h'/k'$.

Case $h/k < u/v < h''/k''$. Since $k'' = k+k' > k'$, we have $v-(k+k'') < v-(k+k')$. By induction hypothesis, there is a twig W''' for u/v which is also a chain of twigs isomorphic to W or W''. Since W'' is a concatenation of W and W', W''' is also a chain of twigs isomorphic to W or W'.

Case $h''/k'' < u/v < h'/k'$. The proof for this case is similar to the proof for the previous case. □

Definition 3.8. Let a/b be a reduced fraction with $2/3 < a/b < 3/4$. A twig for a/b is called a *canonical* twig if it is a chain of twigs isomorphic to a 2-twig for 4/3 or a 1-twig for 2/3. Canonical twigs exist for any such a/b.

4 Almost Monodimensionality

In this section, there are many cases that we want to show some structures are normal to f. Note that any trees are normal to f and any single vertex is closed in structures normal to f. Also, the free amalgamation property holds for the class of structures normal to f. So, if a structure is normal to f then any extension by a tree over a single vertex is also normal to f.

Definition 4.1. Let B be a graph and A a substructure of B. A substructure X of B is said to be *smooth* over A if any leaves of X belong to A.

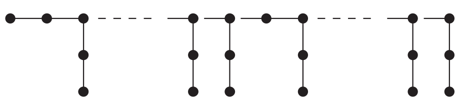
Definition 4.2. Let B be a graph and A a substructure of B, and $p \in B$. $d^c_B(p/A)$ denotes the smallest value of $\delta_\alpha(X/A)$ where $A \subseteq X \subseteq B$ and there is a path from p to A in X.

Definition 4.3. Let B be a graph, A a substructure of B, and β a real number. B is called a $3/4$-*extension* of A if $x = |B| - |A|$ and $z = e(B) - e(A)$ then $x/z \geq 3/4$.

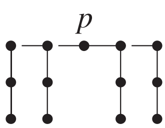
Definition 4.4. Suppose $A < B$. $p \in B$ is called a *good* vertex of B over A if $p \in B - A$ and whenever $p \in X \subset B$ with $X \cap A \neq \emptyset$ then either $7 \leq |X - A|$ or $X \otimes_p pp_1p_2p_3$ is a $3/4$-extension of $X'p_3$ for some $X' \subseteq X$ with $X \cap A \subseteq X'$. Here, $pp_1p_2p_3$ is a path of length 3 with ends p and p_3.

Proposition 4.5. *Let α be an irrational number, and a/b a reduced fraction. Suppose $2/3 < \alpha < a/b < 3/4$ and b is sufficiently large. Let B be a canonical twig for a/b and A the set of all leaves in B. Then there is a good vertex of B over A whose distance from A is* 3.

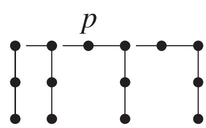
Proof. Note that for any reduced fractions a'/b' with $2/3 < a'/b' < 3/4$, the canonical twig for a'/b' begins from the left end with a twig for $3/4$ and ends with a twig for $2/3$ at the right end. Since b is sufficiently large, the canonical twigs B for a/b look like the following:



Hence, there is a substructure of B which is isomorphic to one of the following pictures:



(1)



(2)

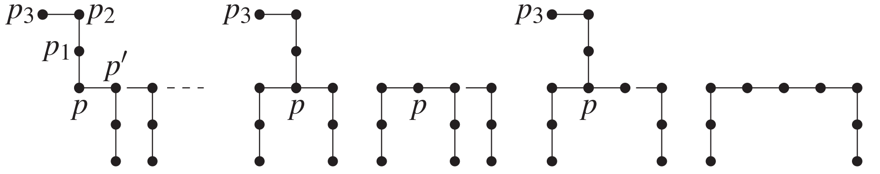
Let us assume that there is a substructure of B isomorphic to (1) above. Choose a vertex p as indicated in the figure. We show that p is a good vertex of B over A.

Let X be a smooth and connected substructure of B over pA with $p \in X$ and $X \cap A \neq \emptyset$. Suppose that X does not contain a vertex in B adjacent to p. Then X contains the other vertex in B adjacent to p, say p'. Then $X \otimes_p pp_1p_2p_3 = (X - p) \otimes_{p'} p'pp_1p_2p_3$. Therefore, it is a $3/4$-extension of $(X - p)p_3$. See (3) in the figure below.

Now, suppose that X contains both vertices adjacent to p. If X contains at least 5 vertices from the main path of B, then X contains at least 2 more paths from the main path of B to A. Each such path has length 2 and thus contains an inner vertex. Hence $X-A$ contais at least 7 vertices. See (7) in the figure below.

If X contains exactly 3 vertices from the main path of B, then $X \otimes_p pp_1p_2p_3$ looks like (4) in the figure below. It is an extension of $(X \cap A)p_3$ by 7 vertices and 9 edges. Since $7/9 > 3/4$, it is a 3/4-extension of $(X \cap A)p_3$.

If X contains exactly 4 vertices from the main path of B, (a) X is isomorphic to (5) or (b) $X \otimes_p pp_1p_2p_3$ is isomorphic to (6) in the figure below. In the case (a), $X-A$ contains 7 vertices. In the case (b), $X \otimes_p pp_1p_2p_3$ is an extension of $(X \cap A)p_3$ by 8 vertices and 10 edges. Since $8/10 = 4/5 > 3/4$, it is a 3/4-extension of $(X \cap A)p_3$.



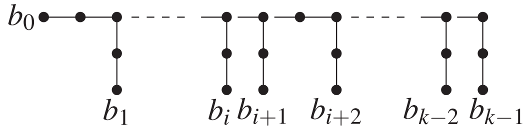
(3) (4) (5) (6) (7)

We have shown that vertex p is a good vertex of B over A when we choose p as in (1). When we choose p as in (2), we can show that p is a good vertex of B over A similarly. □

Lemma 4.6. *Let α be an irrational number with $2/3 < \alpha < 3/4$, u/v a reduced fraction with $u/v < \alpha$ such that whenever $u/v < u'/v' < \alpha$ then $v < v'$. Let $f = f_\alpha$ be the Hrushovski's log-like function associated to α. Assume that $B \in \mathbf{K}_f$ with $A < B$ and there is a good vertex b of B over A, W is a canonical twig for u/v, C the set of all leaves of W, and $k = |C|$. Let $D = (B_0 \otimes_A B_1 \otimes_A B_2 \otimes_A \ldots \otimes_A B_{k-1}) \otimes_C W$ where $C = \{b_0, b_1, \ldots, b_{k-1}\}$, B_i is isomorphic to B over A and $b_i \in B_i$ is the isomorphic image of b for each $i = 0, \ldots, k-1$. Then for sufficiently large v, D belongs to $\mathbf{K}_f$, and there is a good vertex p of D over A such that $d^c_D(p/A) > d^c_B(b/A) + \min\{d^c_B(b/A), 3(1-\alpha)\}$.*

Proof. We show that D belongs to $\mathbf{K}_f$ by choosing v sufficiently large. It is straightforward to prove other statements.

The b_i are the leaves of W. We can assume that b_0 is the left end of the main path of W, and $b_1, b_2, \ldots, b_{k-1}$ are ordered from left to right respecting the order of vertices in the main path of W connected to b_i by a path of length 2 in W.



For j with $1 \leq j \leq k$, let $D_j = (B_0 \otimes_A B_1 \otimes_A B_2 \otimes_A \ldots \otimes_A B_j) \otimes_{C_j} W_j$ where $C_j = \{b_0, b_1, \ldots, b_j\}$, and W_j is the left prefix of W with the right most leaf b_j. Note that $D = D_{k-1}$.

Now, let X be a substructure of D. Our aim is to show that X is normal to f. By Fact 2.4 (the free amalgamation property for the structures normal to f), we can assume that $X \cap A \neq \emptyset$, X is smooth over A, and $X \cap W$ is connected.

Put $Y_j = (X \cap B_0) \otimes_{X \cap A} \cdots \otimes_{X \cap A} (X \cap B_j)$. Then $Y_j \in \mathbf{K}_f$ for any j. In particular, $|Y_{k'}| > 7k'$. Also, the number of all edges in $W_{k'}$ is at most $4k'$ and $C_{k'} < W_{k'}$. By Lemma 2.7 (3), $X \cap D_{k'} = Y_{k'} \otimes_{C_{k'}} W_{k'}$ is normal to f.

Now, consider $X \cap D_{k'+1}$. There are two cases for $W_{k'+1}$: $W_{k'+1} = W_{k'} \otimes_p P_{k'+1}$ where $P_{k'+1}$ is a path of length 4 or a path of length 3 with ends $p \in W_{k'}$ and $b_{k'+1}$. We have $D_{k'+1} = (D_{k'} \otimes_A B_{k'+1}) \otimes_{p, b_{k'+1}} P$.

If the length is 4, then $X \cap D_{k'+1}$ is a $3/4$-extension of $(X \cap D_{k'}) \otimes_{X \cap A} (X \cap B_{k'+1})$, which is normal to f. Hence, $X \cap D_{k'+1}$ is also normal to f by Lemma 2.7 (2). If the length is 3, then $X \cap D_{k'+1}$ is a $3/4$-extension of $(X \cap D_{k'}) \otimes_{X \cap A} X'$ for some X' with $X \cap A \subseteq X' \subsetneq X \cap B_{k'+1}$ because $b_{k'+1}$ is a good vertex of $B_{k'+1}$ over A. $X \cap D_{k'} \otimes_{X \cap A} X'$ is normal to f by Fact 2.4, so is $X \cap D_{k'+1}$ by Lemma 2.7 (2). Repeating the similar arguments, we see that $X \cap D_{k-1}$ is normal to f.

The essential remaining case is the case where $W \subseteq X$ and $|X \cap B_j| \geq 7$ for all j. Since v is sufficiently large, We can assume $0 > \delta_\alpha(W/C) > -\delta_\alpha(B/A)$. We can also assume that k is very large. Then $X \cap D$ is normal to f. □

Now, we prove the main theorem.

Theorem 4.7. *Let α be an irrational number, and a/b a reduced fraction. Suppose $2/3 < \alpha < a/b < 3/4$ and b is sufficiently large. Let B be a canonical twig for a/b and A the set of all leaves in B. Let $p \in B$ be a good vertex of B over A. Let M be the generic structure for $(\mathbf{K}_f, <)$ where f is the Hrushovski's log-like function associated to α. Assume that B is a closed subset of M. Let D be the orbit of p over A in M. Then $M = \mathrm{cl}(D)$.*

Proof. We first claim that any points in M independent from A over the empty set belong to $\mathrm{cl}(D)$.

Note that a good vertex of B over A exists by Proposition 4.5. Let $B_1 < M$ be the embedded image of D obtained by By Lemma 4.6 from B. Then $B_1 \subseteq \mathrm{cl}(D, A)$,

$A < B_1$, there is a good vertex p_1 of B_1 over A. Repeating this process, we get $A < B_1 < B_2 < \ldots < B_j < M$ for any natural number j, and a good vertex p_i of B_i over A for each $i \leq j$. Each p_{i+1} for i belongs to $\mathrm{cl}(\mathrm{Orb}(p_i/A))$. Therefore, each p_{i+1} for i belongs to $\mathrm{cl}(\mathrm{Orb}(p/A))$.

Let $\varepsilon = \min\{d^c_B(p/A), 3(1-\alpha)\}$. By the structures of B_i, $d^c_{B_1}(p_1/A) > 2\varepsilon$, $d^c_{B_2}(p_2/A) > 3\varepsilon$, and so on. We have $d^c_{B_j}(p_j/A) > (j+1)\varepsilon$. For sufficiently large j, we have $d^c_{B_j}(p_j/A) > 1$. Therefore, there is j such that $d(p_j/A) = 1 = d(p_j)$ and $p_j \in \mathrm{cl}(D)$. Suppose x is not adjacent to vertices in A and $xA < M$. Since $p_jA < M$ and xA is isomorphic to p_jA, there is an automorphism of M which sends x to p_j and fixes A pointwise. Hence, x belong to $\mathrm{cl}(D)$ also because D is invariant under the automorphisms fixing A pointwise. We have shown the first claim.

Choose a reduced fraction u/v with $u/v < \alpha$ which is a good approximation of α from below. Using twigs for u/v, make a big tree W such that there is a root x of W such that for all the leaves y of W, yx is not an edge of W, and $yx < W$.

Now, let $x \in M$. Consider $\mathrm{cl}(xA)$. Consider $W \otimes_x \mathrm{cl}(xA) > \mathrm{cl}(xA)$. We can embed $W \otimes_x \mathrm{cl}(xA)$ into M over $\mathrm{cl}(xA)$ as a closed structure. Let y be a leaf of W. Suppose $yA \subseteq X \subseteq W \otimes_x \mathrm{cl}(xA)$. If $x \notin X$, then $X = (X \cap W) \otimes (X \cap \mathrm{cl}(xA))$. In this case, $y < (X \cap W)$ and $A < X \cap \mathrm{cl}(xA)$. Hence, $\delta(yA) < \delta(X)$ unless $yA = X$. Suppose $x \in X$. $X = (X \cap W) \otimes_x (X \cap \mathrm{cl}(xA))$. We have $\delta(yx) < \delta(X \cap W)$ unless $X \cap W = yx$. Also, we have $\delta(A) < \delta(X \cap \mathrm{cl}(xA))$ since $A < M$ and $A \subsetneq X \cap \mathrm{cl}(xA)$.

Suppose $yx \subsetneq X \cap W$. We have

$$\delta(X) = \delta(X \cap W) + \delta(X \cap \mathrm{cl}(xA)) - 1 > \delta(yx) - 1 + \delta(A) = 1 + \delta(A).$$

Therefore, yA is closed in $W \otimes_x \mathrm{cl}(xA)$, and thus $yA < M$. This shows that all the leaves of W belong to $\mathrm{cl}(D)$. So, x belongs to $\mathrm{cl}(D)$. □

Acknowledgments

The work is supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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