Hardy-Sobolev inequalities in the half space

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Abstract

Our aim is to establish Hardy-Sobolev inequalities for Sobolev functions in Herz-Morrey spaces, which extend the classical Hardy inequalities in the L^p Lebesgue space.

1 Introduction

In the half space $\mathbb{H} = \{x = (x_1, ..., x_{n-1}, x_n) : x_n > 0\}$, the Hardy inequality says that

$$\int_{\mathbb{H}} |u(x)|^p x_n^{\beta-p}\, dx \le \left(\frac{p}{p-\beta-1}\right)^p \int_{\mathbb{H}} |(\partial u/\partial x_n)(x)|^p x_n^{\beta} dx \tag{1}$$

for $\beta < p-1$ and all $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $\partial\mathbb{H}$. Further,

$$\int_{\mathbb{H}} |u(x)|^p x_n^{\beta-p}\, dx \le \left(\frac{p}{\beta-p+1}\right)^p \int_{\mathbb{H}} |(\partial u/\partial x_n)(x)|^p x_n^{\beta} dx \tag{2}$$

for $\beta > p-1$ and all $u \in C_0^1(\mathbb{R}^n)$; see e.g. [7], [9], [10], [11] and so on.

In connection with the inequality in the book by Maz'ya [11, Theorem 1, p. 214], we have

$$\begin{aligned} &\int_{\mathbb{H}} \left(\frac{|u(x)|}{\sqrt{x_{n-1}^2 + x_n^2}}\right)^p x_n^{\beta} dx \\ \le\ & \left(\frac{p}{p-\beta-2}\right)^p \int_{\mathbb{H}} \left(\sqrt{(\partial u/\partial x_{n-1})^2 + (\partial u/\partial x_n)^2}\right)^p x_n^{\beta} dx \end{aligned} \tag{3}$$

for $\beta < p-2$ and all $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $\partial\mathbb{H}$, and

$$\begin{aligned} &\int_{\mathbb{H}} \left(\frac{|u(x)|}{\sqrt{x_{n-1}^2 + x_n^2}}\right)^p x_n^{\beta} dx \\ \le\ & \left(\frac{p}{\beta-p+2}\right)^p \int_{\mathbb{H}} \left(\sqrt{(\partial u/\partial x_{n-1})^2 + (\partial u/\partial x_n)^2}\right)^p x_n^{\beta} dx \end{aligned} \tag{4}$$

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for $\beta > p-2$ and all $u \in C_0^1(\mathbb{R}^n)$.

The proofs of those results are given by an elementary calculus with the aid of Minkowski's inequality and Hölder's inequality.

Our aim in this note is to extend those Hardy type inequalities in L^p to the Herz-Morrey space $M^{p,q,\beta}(\mathbb{R}^n)$ (see §3.1). It is well known that Morrey spaces and Herz spaces play an important role in harmonic analysis and PDE. For fundamental properties of Herz-Morrey spaces, we refer the reader to [2], [3], [4], [5], [6], [8], [13], [14], [15], [17], etc.

For $1 \le m \le n$, write

$$x = (x_1, ..., x_{n-m}, x_{n-m+1}, ..., x_n) = (x', x'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m.$$

As an extension of (1), we show the norm inequality:

$$\| |x''|^{-1} u\|_{M^{p,q,\beta}(\mathbb{R}^n)} \le C \| |\nabla'' u| \|_{M^{p,q,\beta}(\mathbb{R}^n)} \tag{5}$$

for $\beta < (p-m)/p$ and $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $\{x = (x', x'') : |x''| < 1\}$, where $\nabla'' u = (\partial u/\partial x_{n-m+1}, ..., \partial u/\partial x_n)$, which is an extension of (3). If $u \in C_0^1(\mathbb{R}^n)$, then (5) holds for $\beta > (p-m)/p$, which is an extension of (2) and (4).

The borderline case of (1) and (2), that is, the case $\beta = p-1 > 0$, is treated in the following:

$$\int_{\mathbb{H}} |u(x)|^p x_n^{-1} |\log(e/x_n)|^{-p}\, dx \quad \le \quad \left(\frac{p}{p-1}\right)^p \int_{\mathbb{H}} |\partial u/\partial x_n|^p x_n^{p-1}\, dx \tag{6}$$

for all $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $\{x = (x', x_n) : x_n > 1\}$ and

$$\int_{\mathbb{H}} |u(x)|^p x_n^{-1} |\log(e x_n)|^{-p}\, dx \quad \le \quad \left(\frac{p}{p-1}\right)^p \int_{\mathbb{H}} |\partial u/\partial x_n|^p x_n^{p-1}\, dx \tag{7}$$

for all $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $\{x = (x', x_n) : x_n < 1\}$. We extend (6) and (7) to the Herz-Morrey settings.

Our final goal is to establish Hardy-Sobolev inequalities for Sobolev functions on $\mathbb{H}$ in the Herz-Morrey settings.

2 The classical Hardy inequalities in the half space

For $1 \le m \le n$, write

$$x = (x_1, ..., x_{n-m}, x_{n-m+1}, ..., x_n) = (x', x'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m.$$

We show Hardy type inequality similar to Maz'ya [11], as an extension of (3).

THEOREM **2.1.** *Let $p \ge 1$ and $\beta < p - m$. Then there exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{H}} \left(\frac{|u(x)|}{|x''|} \right)^p x_n^\beta\, dx \right)^{1/p} \le \frac{p}{p-m-\beta} \left(\int_{\mathbb{H}} |\nabla'' u(x)|^p x_n^\beta\, dx \right)^{1/p}$$

for $u \in C^1(\mathbb{R}^N)$ such that $u = 0$ on $\partial\mathbb{H}$.

Proof. Let u be a function in $C^1(\mathbb{R}^n)$ such that $u=0$ on $\partial\mathbb{H}$. Then, for $x\in\mathbb{H}$ note that

$$u(x) \;=\; \int_0^1 \frac{d}{dr}u(x',rx'')\,dr,$$

so that

$$|x''|^{-1}|u(x)| \le \int_0^1 |\nabla'' u(x',rx'')|\,dr. \tag{8}$$

By Minkowski's inequality gives

$$\begin{aligned}
\left(\int_{\mathbb{H}} \left(|x''|^{-1}|u(x)|\right)^p x_n^{\beta} dx\right)^{1/p} &\le \left(\int_0^1 \left(\int_{\mathbb{H}} \left(|\nabla'' u(x',rx'')| x_n^{\beta/p}\right)^p dx\right)^{1/p}\right) dr \\
&\le \left(\int_{\mathbb{H}} \left(|\nabla'' u(y)| x_n^{\beta/p}\right)^p dy\right)^{1/p} \int_0^1 r^{-\beta/p-m/p} dr \\
&\le \frac{p}{p-m-\beta}\left(\int_{\mathbb{R}^n} \left(|v(y)||y''|^{\beta/p}\right)^p dy\right)^{1/p},
\end{aligned}$$

as required. □

Set

$$H(r)=\{x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R} : |x_n|\le r\}.$$

The borderline case of Theorem 2.1 is treated as follows.

THEOREM **2.2.** *Let $p>1$. If u is a function in $C^1(\mathbb{R}^n)$ such that $u=0$ on $\mathbb{R}^n\setminus H(1)$, then*

$$\left(\int_{\mathbb{H}} \left(\frac{|u(x)|}{|x''|}\right)^p x_n^{p-m} |\log(e/x_n)|^{-p}\,dx\right)^{1/p} \le \frac{p}{p-1}\left(\int_{\mathbb{H}} |\nabla'' u(x)|^p x_n^{p-m}\,dx\right)^{1/p}.$$

Proof. Let u be a function in $C^1(\mathbb{R}^n)$ such that $u=0$ on $\mathbb{R}^n\setminus H(1)$. Then, for $x\in\mathbb{H}$ note that

$$u(x) \;=\; -\int_1^\infty \frac{d}{dr}u(x',rx'')\,dr,$$

so that

$$|x''|^{-1}|u(x)| \le \int_1^\infty |\nabla'' u(x',rx'')|\,dr. \tag{9}$$

By Hölder's equality, we have for $x=(x',x'')\in\mathbb{H}$

$$\begin{aligned}
|x''|^{-1}|u(x)| &\le \left(\int_1^\infty \left(r^{1/p'}(\log(e/rx_n))^a|\nabla'' u(x',rx'')|\right)^p dr\right)^{1/p} \\
&\quad\times\left(\int_1^\infty (\log(e/rx_n))^{-ap'} r^{-1}\,dr\right)^{1/p'} \\
&\le \left(\int_1^{1/x_n} \left(r^{1/p'}(\log(e/rx_n))^a|\nabla'' u(x',rx'')|\right)^p dr\right)^{1/p} \\
&\quad\times\left(\frac{1}{1-ap'}\right)^{1/p'} (\log(e/x_n))^{-a+1/p'}
\end{aligned}$$

when $1-ap'>0$. Consequently, Minkowski's inequality gives

$$\begin{aligned}
&\int_{\mathbb{H}} ||x''|^{-1}u(x)|^p x_n^{p-m}|\log(e/x_n)|^{-p}dx \\
\leq\ &\left(\frac{1}{1-ap'}\right)^{p/p'}\int_1^\infty r^{p/p'}\left(\int_{\mathbb{H}}|\nabla'' u(x',rx'')|^p x_n^{p-m}|\log(e/rx_n)|^{ap}|\log(e/x_n)|^{-ap-1}dx\right)dr \\
\leq\ &\left(\frac{1}{1-ap'}\right)^{p/p'}\int_1^\infty r^{p/p'-p}\left(\int_{\mathbb{H}}|\nabla'' u(y)|^p y_n^{p-m}|\log(e/y_n)|^{ap}|\log(er/y_n)|^{-ap-1}dy\right)dr \\
\leq\ &\left(\frac{1}{1-ap'}\right)^{p/p'}\int_{\mathbb{H}}|\nabla'' u(y)|^p y_n^{p-m}|\log(e/y_n)|^{ap}\left(\int_1^{1/y_n}|\log(er/y_n)|^{-ap-1}r^{-1}\,dr\right)dy \\
\leq\ &\left(\frac{1}{1-ap'}\right)^{p/p'}\frac{1}{ap}\int_{\mathbb{H}}|\nabla'' u(y)|^p y_n^{p-m}dy.
\end{aligned}$$

Here we see that $\left(\frac{1}{1-ap'}\right)^{p/p'}\frac{1}{ap}$ has the minimum c_p at $a=1/(p+p')$. □

Next we give extensions of (4) and (7).

Theorem **2.3.** *Let $p\geq 1$ and $\beta>p-m$. Then there exists a constant $C>0$ such that*

$$\left(\int_{\mathbb{H}}|u(x)|^p\left(\frac{x_n^{\beta/p}}{|x''|}\right)^p dx\right)^{1/p}\leq\frac{p}{\beta-p+m}\left(\int_{\mathbb{H}}|\nabla'' u(x)|^p x_n^\beta\,dx\right)^{1/p}$$

for $u\in C_0^1(\mathbb{R}^N)$.

Proof. By (9) Minkowski's inequality gives

$$\begin{aligned}
\left(\int_{\mathbb{H}}\left(|x''|^{-1}|u(x)|\right)^p x_n^\beta dx\right)^{1/p} &\leq \left(\int_1^\infty\left(\int_{\mathbb{H}}\left(|\nabla'' u(x',rx'')|x_n^{\beta/p}\right)^p dx\right)^{1/p}\right)dr \\
&\leq \left(\int_{\mathbb{H}}\left(|\nabla'' u(y)|x_n^{\beta/p}\right)^p dy\right)^{1/p}\int_1^\infty r^{-\beta/p-m/p}dr \\
&\leq \frac{p}{\beta-p+m}\left(\int_{\mathbb{R}^n}\left(|v(y)||y''|^{\beta/p}\right)^p dy\right)^{1/p},
\end{aligned}$$

as required. □

Theorem **2.4.** *Let $p>1$. If u is a function in $C^1(\mathbb{R}^n)$ such that $u=0$ on $H(1)$, then*

$$\left(\int_{\mathbb{H}}\left(\frac{|u(x)|}{|x''|}\right)^p x_n^{p-m}|\log(ex_n)|^{-p}\,dx\right)^{1/p}\leq\frac{p}{p-1}\left(\int_{\mathbb{H}}|\nabla'' u(x)|^p x_n^{p-m}\,dx\right)^{1/p}.$$

Proof. By (8) and Hölder's equality, we have for $x = (x', x'')$

$$\begin{aligned}
|x''|^{-1}|u(x)| &\le \left(\int_0^1 \left(r^{1/p'}(\log(erx_n))^a|\nabla'' u(x', rx'')|\right)^p dr\right)^{1/p} \\
&\quad \times \left(\int_{1/x_n}^1 (\log(erx_n))^{-ap'} r^{-1}\, dr\right)^{1/p'} \\
&\le \left(\int_0^1 \left(r^{1/p'}(\log(erx_n))^a|\nabla'' u(x', rx'')|\right)^p dr\right)^{1/p} \\
&\quad \times \left(\frac{1}{1-ap'}\right)^{1/p'} (\log(ex_n))^{-a+1/p'}
\end{aligned}$$

when $1 - ap' > 0$. Consequently, Minkowski's inequality gives

$$\begin{aligned}
&\int_{\mathbb{H}} ||x''|^{-1}u(x)|^p x_n^{p-m}|\log(ex_n)|^{-p}dx \\
\le\ & \left(\frac{1}{1-ap'}\right)^{p/p'} \int_0^1 r^{p/p'} \left(\int_{\mathbb{H}} |\nabla'' u(x', rx'')|^p x_n^{p-m}|\log(erx_n)|^{ap}|\log(ex_n)|^{-ap-1}dx\right) dr \\
\le\ & \left(\frac{1}{1-ap'}\right)^{p/p'} \int_0^1 r^{p/p'-p} \left(\int_{\mathbb{H}} |\nabla'' u(y)|^p y_n^{p-m}|\log(ey_n)|^{ap}|\log(ey_n/r)|^{-ap-1}dy\right) dr \\
\le\ & \left(\frac{1}{1-ap'}\right)^{p/p'} \int_{\mathbb{H}} |\nabla'' u(y)|^p y_n^{p-m}|\log(ey_n)|^{ap} \left(\int_0^1 |\log(ey_n/r)|^{-ap-1}r^{-1}\, dr\right) dy \\
\le\ & \left(\frac{1}{1-ap'}\right)^{p/p'} \frac{1}{ap} \int_{\mathbb{H}} |\nabla'' u(y)|^p y_n^{p-m} dy.
\end{aligned}$$

Here we see that $\left(\frac{1}{1-ap'}\right)^{p/p'} \frac{1}{ap}$ has the minimum c_p at $a = 1/(p+p')$. □

3 Herz-Morrey space

3.1 Herz-Morrey space

For $1 \le m \le n$, write

$$x = (x_1, ..., x_{n-m}, x_{n-m+1}, ..., x_n) = (x', x'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m$$

and set

$$H_m(r) = \{x = (x', x'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m : |x''| \le r\}$$

and $A_m(r) = H_m(2r) \setminus H_m(r)$. To extend the Hardy inequality in the half space, for $p \ge 1$, $q > 0$, a weight ω and an open set $G \subset \mathbb{R}^n$, we consider the Herz-Morrey space

$$M_m^{p,q,\omega}(G) = \{f \in L^1_{\mathbf{loc}}(G) : \|f\|_{M_m^{p,q,\omega}(G)} < \infty\},$$

where

$$\|f\|_{M_m^{p,q,\omega}(G)} = \left(\int_0^\infty \left(\omega(r)\|f\|_{L^p(G\cap A_m(r))}\right)^q \frac{dr}{r}\right)^{1/q}$$

in case $q < \infty$; in case $q = \infty$, set

$$\|f\|_{M_m^{p,\infty,\omega}(G)} = \sup_{r>0} \omega(r)\|f\|_{L^p(G\cap A_m(r))}.$$

When $\omega(r) = r^\beta$, we write $M_m^{p,q,\beta}(G)$ for $M_m^{p,q,\omega}(G)$. For fundamental properties of Herz-Morrey spaces, we refer the reader to [2], [3], [4], [5], [6], [8], [13], [14], [15], [17], etc.

3.2 Hardy inequalities for Herz-Morrey space

Let us begin with the Hardy inequality in the setting of Herz-Morrey space in the half space.

THEOREM **3.1.** *Let $p \geq 1$ and $\beta < (p-m)/p$. Then for $1 \leq k \leq m$ there exists a constant $C > 0$ such that*

$$\||x''|^{-1}u\|_{M_k^{p,q,\beta}(\mathbb{R}^n)} \leq C\||\nabla''u|\|_{M_k^{p,q,\beta}(\mathbb{R}^n)}$$

for $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $H_k(1)$, where $\nabla''u = (\partial u/\partial x_{n-m+1}, ..., \partial u/\partial x_n)$.

THEOREM **3.2.** *Let $p \geq 1$ and $\beta > (p-m)/p$. Then for $1 \leq k \leq m$ there exists a constant $C > 0$ such that*

$$\||x''|^{-1}u\|_{M_k^{p,q,\beta}(\mathbb{R}^n)} \leq C\||\nabla''u|\|_{M_k^{p,q,\beta}(\mathbb{R}^n)}$$

for $u \in C_0^1(\mathbb{R}^n)$.

3.3 Corollaries to Theorem 3.1

Set

$$\mathbb{H} = \{x = (x', x_n) : x_n > 0\}.$$

As easy consequences of Theorem 3.1, we give the following.

COROLLARY **3.3.** *Let $p \geq 1$ and $\beta < (p-1)/p$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{M_1^{p,q,-1+\beta}(\mathbb{H})} \leq C\|\partial u/\partial x_n\|_{M_1^{p,q,\beta}(\mathbb{H})}$$

for $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $\partial\mathbb{H}$.

COROLLARY **3.4.** *Let $p \geq 1$ and $\beta < (p-m)/p$. Then there exists a constant $C > 0$ such that*

$$\||x''|^{-1}u\|_{M_1^{p,q,\beta}(\mathbb{H})} \leq C\||\nabla''u|\|_{M_1^{p,q,\beta}(\mathbb{H})}$$

for $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $H_1(1)$.

3.4 Corollaries to Theorem 3.2

COROLLARY **3.5.** *Let $p \geq 1$ and $\beta > (p-1)/p$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{M_1^{p,q,-1+\beta}(\mathbb{H})} \leq C \|\partial u / \partial x_n\|_{M_1^{p,q,\beta}(\mathbb{H})}$$

for $u \in C_0^1(\mathbb{R}^n)$.

COROLLARY **3.6.** *Let $p \geq 1$ and $\beta > (p-m)/p$. Then there exists a constant $C > 0$ such that*

$$\||x''|^{-1} u\|_{M_1^{p,q,\beta}(\mathbb{H})} \leq C \||\nabla'' u|\|_{M_1^{p,q,\beta}(\mathbb{H})}$$

for $u \in C_0^1(\mathbb{R}^n)$.

3.5 The borderline case of Theorem 3.2

THEOREM **3.7.** *Let $p > m \geq 1$ and set $\omega(r) = r^{(p-m)/p} |\log(e/r)|^{-\gamma}$, where*

$$\begin{cases} \gamma = 1 & \text{when } q \geq p; \\ \gamma = 1 - 1/p + 1/q & \text{when } 0 < q < p. \end{cases}$$

Then for $1 \leq k \leq m$ there exists a constant $C > 0$ such that

$$\||x''|^{-1} u\|_{M_k^{p,q,\omega}(\mathbb{R}^n)} \leq C \||\nabla'' u|\|_{M_k^{p,q,(p-m)/p}(\mathbb{R}^n)}$$

for $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $\mathbb{R}^n \setminus H_k(1)$.

3.6 The borderline case of Theorem 3.1

In the same manner we treat the borderline case of Theorem 3.1, as an extension of (5).

THEOREM **3.8.** *Let $p > m \geq 1$ and set $\omega(r) = r^{(p-m)/p} |\log(er)|^{-\gamma}$, where γ is the constant appearing in Theorem 3.7. Then for $1 \leq k \leq m$ there exists a constant $C > 0$ such that*

$$\||x''|^{-1} u\|_{M_k^{p,q,\omega}(\mathbb{R}^n)} \leq C \||\nabla'' u|\|_{M_k^{p,q,(p-m)/p}(\mathbb{R}^n)}$$

for $u \in C^1(\mathbb{R}^n)$ such that $u = 0$ on $H_k(1)$.

4 Hardy-Sobolev inequalities for Sobolev functions

4.1 Hardy-Sobolev inequality for Sobolev functions

Our aim in this section is to give Hardy-Sobolev inequalities for Sobolev functions on $\mathbb{H}$. We write $M_k^{p,q,\omega}(\mathbb{H})$ for $M_k^{p,q,\omega}(\mathbb{R}^n)$ whose functions are considered on $\mathbb{H}$. We also write M and H for M_1 and H_1, respectively.

THEOREM **4.1.** *Let $p \geq 1$ and $\beta \neq (p-1)/p$. For $0 \leq \lambda \leq 1$, set $1/p_\lambda = 1/p - \lambda/n > 0$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{M^{p_\lambda,q,\lambda-1+\beta}(\mathbb{H})} \leq C \||\nabla u|\|_{M^{p,q,\beta}(\mathbb{H})}$$

for $u \in C_0^1(\mathbb{H})$.

When $q = p$, we have the weighted inequality for Sobolev functions.

Corollary **4.2.** *Let $p \geq 1$, $\beta \neq (p-1)/p$, $0 \leq \lambda \leq 1$ and $1/p_\lambda = 1/p - \lambda/n$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{H}} |u(x)|^{p_\lambda} x_n^{(\lambda-1+\beta)p_\lambda}\, dx \leq C \tag{10}$$

for all $u \in C_0^1(\mathbb{H})$ such that $\int_{\mathbb{H}} |\nabla u(x)|^p x_n^{\beta p}\, dx \leq 1$.

This gives Sobolev's inequality when $\lambda = 1$, and Hardy's inequality (no account of the best constant) when $\lambda = 0$.

To show Theorem 4.1, we prepare the following two lemmas.

Lemma **4.3** (Sobolev's inequality (see e.g. [1], [11], [12]))**.** *Let $p \geq 1$ and $1/p^* = 1/p - 1/n > 0$. Then there is a constant $C > 0$ such that*

$$\|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C \| |\nabla v| \|_{L^p(\mathbb{R}^n)}$$

for $v \in C_0^1(\mathbb{R}^n)$.

Let $\tilde{A}(r) = H(4r) \setminus H(r/2)$.

Lemma **4.4.** *Let $p \geq 1$ and $1/p^* = 1/p - 1/n > 0$. Then there is a constant $C > 0$ such that*

$$\|u\|_{L^{p^*}(A(r))} \leq C\{r^{-1}\|u\|_{L^p(\tilde{A}(r))} + \| |\nabla u| \|_{L^p(\tilde{A}(r))}\}$$

for $u \in C_0^1(\mathbb{R}^n)$ and $r > 0$.

For this, take $\varphi \in C_0^1(\tilde{A}(1))$ such that $\varphi = 1$ on $A(1)$ and $|\nabla \varphi| \leq 2$, and apply Lemma 4.3 with $v(x) = \varphi(x_n/r)u(x)$.

4.2 The borderline case of Theorem 4.1

In the borderline case $\beta = (p-1)/p$, we establish the following results by using Theorems 3.7 and 3.8.

Theorem **4.5.** *Let $p > 1$. For $0 \leq \lambda \leq 1$, set $1/p_\lambda = 1/p - \lambda/n$ and $\omega(r) = r^{\lambda-1+(p-1)/p}|\log(e/r)|^{-\gamma}$, where γ is the constant appearing in Theorem 3.7. Then there exists a constant $C > 0$ such that*

$$\|u\|_{M^{p_\lambda,q,\omega}(\mathbb{H})} \leq C \| |\nabla u| \|_{M^{p,q,(p-1)/p}(\mathbb{H})}$$

for all $u \in C_0^1(\mathbb{H})$ such that $u = 0$ on $\mathbb{H} \setminus H(1)$.

Theorem **4.6.** *Let $p > 1$ and set $\omega(r) = r^{\lambda-1+(p-1)/p}|\log(er)|^{-\gamma}$, where γ is the constant appearing in Theorem 3.7. Then there exists a constant $C > 0$ such that*

$$\|u\|_{M^{p_\lambda,q,\omega}(\mathbb{H})} \leq C \| |\nabla u| \|_{M^{p,q,(p-1)/p}(\mathbb{H})}$$

for $u \in C_0^1(\mathbb{H})$ such that $u = 0$ on $H(1)$.

4.3 Corollaries

When $q = p$, we have the weighted inequalities for Sobolev functions.

Corollary **4.7.** *Let* $p > 1$, $0 \leq \lambda \leq 1$ *and* $1/p_\lambda = 1/p - \lambda/n$. *Then*

$$\int_{\mathbb{H}} |u(x)|^{p_\lambda} \left(x_n^{\lambda-1+(p-1)/p} |\log(e/x_n)|^{-1}\right)^{p_\lambda} dx \leq C \tag{11}$$

for all $u \in C_0^1(\mathbb{H})$ *such that* $u = 0$ *on* $\mathbb{H} \setminus H(1)$ *and* $\||\nabla u|\|_{H^{p,p,(p-1)/p}(\mathbb{H})} \leq 1$.

Corollary **4.8.** *Let* $p > 1$, $0 \leq \lambda \leq 1$ *and* $1/p_\lambda = 1/p - \lambda/n$. *Then*

$$\int_{\mathbb{H}} |u(x)|^{p_\lambda} \left(x_n^{\lambda-1+(p-1)/p} |\log(ex_n)|^{-1}\right)^{p_\lambda} dx \leq C \tag{12}$$

for all $u \in C_0^1(\mathbb{H})$ *such that* $u = 0$ *on* $H(1)$ *and* $\||\nabla u|\|_{H^{p,p,(p-1)/p}(\mathbb{H})} \leq 1$.

Our results here give (weighted) Sobolev's inequality when $\lambda = 1$, and Hardy's inequality (no account of the best constant) when $\lambda = 0$.

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