# Attractive points, acute points and convergence theorems for families of nonlinear mappings

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**Abstract**

In this paper, we study the asymptotic behavior of orbits of nonexpansive semigroups in Banach spaces.

## 1 Introduction

Let $E$ be a real Banach space, let $C$ be a nonempty subset of $E$. For a mapping $T : C \to E$, we denote by $F(T)$ the set of *fixed points* of $T$ and by $A(T)$ the set of *attractive points* [18] of $T$, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;

(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

The behavior of the sequence of Picard iterates of $T$ is one of the important problems in metric fixed point theory because this allows us to approximate a fixed point in the simplest way. Moreau [12] proved that if $C$ is a closed subset of a Hilbert space and if $F(T)$ has nonempty interior, then for each $x \in C$, the sequence $\{T^n x\}$ converge strongly to a point in $F(T)$. Kirk and Sims [13] generalized this result to Banach spaces. Grzesik, Kaczor, Kuczumow and Reich [8] proved convergence of iterates of nonexpansive mappings: Let $C$ be a bounded closed

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and convex subset of a uniformly convex Banach space $E$. Assume that $C$ has nonempty interior and that it is locally uniformly rotund. Let $T$ be a nonexpansive mapping of $C$ into itself and let $x \in C$. If $T$ has no fixed point in the interior of $C$, then there exists a unique point $z_0$ on the boundary $\partial C$ of $C$ such that each sequence $\{T^n x : n = 1,2,3,\dots\}$ converges strongly to $z_0$. They [8] also proved the convergence of orbits of one-parameter nonexpansive semigroups.

In this paper, we study the asymptotic behavior of orbits of nonexpansive semigroups with no common fixed points in the interior of their domains (see [1]). Motivated by [8], we give convergence theorems for abstract semigroups. We also give some convergence theorems for nonexpansive mappings and nonexpansive semigroups.

# 2 Preliminaries and notations

Throughout this paper, we assume that $E$ is a real Banach space with norm $\|\cdot\|$. We denote by $E^*$ the topological dual space of $E$. We denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in $E$ converges strongly to $x$. We also write $x_n \rightharpoonup x$ (or it w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in $E$ converges weakly to $x$. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset $A$ of $E$, co$A$ and $\overline{\text{co}}A$ mean the convex hull of $A$ and the closure of convex hull of $A$, respectively.

Let $S$ be a semitopological semigroup, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from $S$ to $S$ are continuous. In the case when $S$ is commutative, we denote $st$ by $s+t$. Let $B(S)$ be the Banach space of all bounded real-valued functions defined on $S$ with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on $S$. For each $s \in S$ and $g \in B(S)$, we can define an element $\ell_s g \in B(S)$ by $(\ell_s g)(t) = g(st)$ for all $t \in S$. We also denote by $\ell_s^*$ the conjugate operator of $\ell_s$. Let $C(S)^*$ be the dual space of $C(S)$. A linear functional $\mu$ on $C(S)$ is called a mean on $C(S)$ if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(g(t))$ or $\int g(t) d\mu(t)$ instead of $\mu(g)$ for $\mu \in C(S)^*$ and $g \in C(S)$. A mean $\mu$ on $C(S)$ is called invariant if $\mu(\ell_s g) = \mu(g)$ for all $s \in S$ and $g \in C(S)$. For $s \in S$, we can define a point evaluation $\delta_s$ by $\delta_s(g) = g(s)$ for every $g \in B(S)$. A convex combination of point evaluations is called a finite mean on $S$. A finite mean $\mu$ on $S$ is also a mean on $C(S)$ containing constants.

The following definition which was introduced by Takahashi [16] is crucial in the nonlinear ergodic theory for abstract semigroups (see also [9]). Let $h$ be a continuous function of $S$ into $E$ such that the weak closure of $\{h(t) : t \in S\}$ is weakly compact. Then, for any $\mu \in C(S)^*$ there exists a unique element $h_\mu \in E$

such that

$$\langle h_\mu, x^* \rangle = (\mu)_t \langle h(t), x^* \rangle = \int \langle h(t), x^* \rangle \, d\mu(t)$$

for all $x^* \in E^*$. If $\mu$ is a mean on $C(S)$, then $h_\mu$ is contained in $\overline{\mathrm{co}}\{h(t) : t \in S\}$ (for example, see [16, 17]). Sometimes, $h_\mu$ will be denoted by $\int h(t) d\mu(t)$.

Throughout this paper, $S$ is a commutative semitopological semigroup with identity. Let $C$ be a closed convex subset of a Banach space $E$. Then, a family $\mathscr{S} = \{T(s) : s \in S\}$ of mappings of $C$ into itself is called a nonexpansive semigroup on $C$ if it satisfies the following conditions:

(a) $T(s+t) = T(s)T(t)$ for all $s, t \in S$;

(b) $s \mapsto T(s)x$ is continuous;

(c) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(\mathscr{S})$ the set of common fixed points of $T(t), t \in S$. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$. Assume that for each $x \in C$ and $x^* \in E^*$, the weak closure of $\{T(t)x : t \in S\}$ is weakly compact. Let $\mu$ be a mean on $C(S)$. Following [14], we also write $T_\mu x$ instead of $\int T(t)x \, d\mu(t)$ for $x \in C$. We remark that $T_\mu$ is nonexpansive on C and $T_\mu x = x$ for each $x \in F(\mathscr{S})$. If $\mu$ is a finite mean, i.e.,

$$\mu = \sum_{i=1}^{n} a_i \delta_{t_i} \ (t_i \in S, a_i \geq 0, \ \sum_{i=1}^{n} a_i = 1),$$

then

$$T_\mu x = \sum_{i=1}^{n} a_i T(t_i) x.$$

A Banach space $E$ is said to be strictly convex if $\dfrac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every $\varepsilon$ with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of $E$ by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon \right\}.$$

A Banach space $E$ is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If $E$ is uniformly convex, then for $r, \varepsilon$ with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\| \frac{x+y}{2} \right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Let $S_E = \{x \in$

$E : \|x\| = 1\}$ be unit sphere in a Banach space $E$. A Banach space $E$ is said to be *locally uniformly rotund* if for each $x \in S_E$ and for each $\varepsilon \in (0,2]$,
there exists $\delta(x,\varepsilon) > 0$ such that
for each $y \in S_E$ with $\|x-y\| \geq \varepsilon$, we have

$$1 - \left\| \frac{x+y}{2} \right\| \geq \delta(x,\varepsilon)$$

For more details, see [11].

Let $E$ be a Banach space, let $C$ be a nonempty bounded closed and convex subset of $E$. Assume that $C$ have nonempty interior, that is, $\mathrm{int}(C) \neq \emptyset$. We say that $C$ is *locally uniformly rotund* if for each $x \in \partial C$ and for each $\varepsilon \in (0, d_x)$, where $d_x = \sup\{\|x-y\| : y \in C\}$, there exists $\delta(x,\varepsilon) > 0$ such that for each $y \in C$ with $\|x-y\| \geq \varepsilon$, we have

$$\mathrm{dist}\left(\frac{x+y}{2}, \partial C\right) := \inf\left\{ \left\| \frac{x+y}{2} - x' \right\| : x' \in \partial C \right\} \geq \delta(x,\varepsilon).$$

Let $C$ be a nonempty bounded closed and convex subset of a Banach space $E$. Assume that $C$ have nonempty interior, that is, $\mathrm{int}(C) \neq \emptyset$. We say that $C$ is uniformly convex if for each $\varepsilon \in (0, \mathrm{diam(C)})$, there exists $\eta_C(\varepsilon) > 0$ such that for each $x, y \in C$ with $\|x-y\| \geq \varepsilon$, we have

$$\mathrm{dist}\left(\frac{x+y}{2}, \partial C\right) := \inf\left\{ \left\| \frac{x+y}{2} - x' \right\| : x' \in \partial C \right\} \geq \eta_C(\varepsilon).$$

Now, we present a simple example of a bounded closed and convex subset of a Hilbert space, which is locally uniformly rotund but not uniformly convex (see [11]).

**Example 1** ( [11]). Let $H = \ell^2$. Let

$$C = \left\{ x = \{x^i\} \in H = \ell^2 : \sum_{k=2}^{\infty} \left(|x^{2k-1}|^k + |x^{2k}|^k\right)^{\frac{2}{k}} \leq 1 \right\}$$

Then, $C$ is bounded, closed, convex and has nonempty interior. Moreover, $C$ is locally uniformly rotund, but not uniformly convex.

Let $C$ be a subset of a Banach space $E$ and let $T$ be mapping of $C$ into $E$. The mapping $T$ is said to be *demiclosed* if for any sequence $\{x_n\} \subset C$ the following implication hold:

$$\text{w-}\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \|Tx_n - y\| = 0$$

imply that

$$Tx = y$$

(see [6]).

**Theorem 2.1** ([6]). *Let $C$ be a nonempty closed and convex subset of a uniformly convex Banach space $E$. Let $T$ be nonexpansive mapping of $C$ into itself and let $I$ be the identity mapping. Then, $I-T$ is demiclosed at* 0*, that is,*

$$\text{w-}\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

*imply that*

$$Tx = x.$$

Throughout the rest of this paper, $S$ is a commutative semitopological semigroup with identity. The following theorem has been essentially established in [7] (see also [3, 4, 10, 17]).

**Theorem 2.2** ([7]). *Let $C$ be a nonempty bounded closed and convex subset of a uniformly convex Banach space $E$. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$. Then, $F(\mathscr{S})$ is nonempty.*

The following theorem has been essentially established in [3] (see also [4, 7, 17]).

**Theorem 2.3** ([3]). *Let $C$ be a closed and convex subset of a strictly convex Banach space $E$. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$ such that $F(\mathscr{S}) \neq \emptyset$. Then, the set $F(\mathscr{S})$ is closed and convex.*

# 3 Convergence theorems for nonexpansive semigroups

In this section, we give the convergence theorems for nonexpansive semigroups with no common fixed points in the interior of their domains. Throughout the rest of this paper, $S$ is a commutative semitopological semigroup with identity.

For $h \in (0,\infty)$, we denote by $C_h$ the set $C \cap \{x \in E : \|x - z_0\| \geq h\}$. Let $z_0 \in C$ and let $x^* \in E^*$ with $\|x^*\| = 1$. We denote by $V_{k,z_0}$ the hyperplane

$$\{x \in E : x^*(x) = k\}$$

which supports $C$ at the point $z_0$, where $k \in (0,\infty)$, $x^*(z_0) = k$.

The following was proved in [8].

**Lemma 3.1** ([8]). *Let $E$ be a Banach space and let $C$ be a bounded, closed and convex subset of $E$. Assume that* $\text{int}(C)$ *is nonempty,* $0 \in \text{int}(C)$ *and that $C$ is locally uniformly rotund. Let $z_0 \in \partial C$, let $x^* \in E^*$ with $\|x^*\| = 1$ and let the hyperplane*

$$V_{k,z_0} = \{x \in E : x^*(x) = k\}$$

*which supports $C$ at the point $z_0$ be given, where $k \in (0,\infty)$. If $r \in (0,\infty)$ and the set*

$$C_r = C \cap \{x \in E : \|x - z_0\| \geq r\}$$

*is nonempty, then there exists $k_1 \in \mathbb{R}$ such that $0 < k_1 < k$ and*

$$C_r \subset \{x \in E : x^*(x) \leq k_1\}.$$

The following lemma plays an important role in our main results (see [2, 9, 15]).

**Lemma 3.2** ([2]). *Let $C$ be a nonempty bounded, closed convex subset of a uniformly convex Banach space $E$. Let $S$ be a commutative semitopological semigroup with identity. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that $\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$ for each $s \in S$. Then, for each $t \in S$,*

$$\lim_{n\to\infty} \sup_{y\in C} \left\| T_{\mu_n} y - T(t) T_{\mu_n} y \right\| = 0.$$

A sequence $\{x_n\}$ in $C$ is said to be an *approximating sequence* of a nonexpansive mapping $T : C \to C$ if

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

(for example, see [8]). A sequence $\{x_n\}$ in $C$ is said to be an *approximating sequence* of a nonexpansive semigroup $\mathscr{S} = \{T(t) : t \in S\}$ if

$$\lim_{n\to\infty} \|x_n - T(t) x_n\| = 0$$

for each $t \in S$ (for example, see [8]). We study the behavior of approximating sequences of nonexpansive semigroups (see [1]).

**Theorem 3.3** ([1]). *Let $E$ be a reflexive Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $\mathscr{S} = \{T(t){:}t \in S\}$ be a nonexpansive semigroup on $C$. Assume that $I - T(t)$ is demiclosed at $0$ for each $t \in S$. If $\mathscr{S} = \{T(t){:}t \in S\}$ has a unique common fixed point $z_0$ and $z_0$ lies on the boundary $\partial C$ of $C$, then every approximating sequence $\{x_n\}$ of $\mathscr{S}$ converges strongly to $z_0$.*

We get convergence of orbits of nonexpansive semigroups with no common fixed points in the interior of their domains (see [1] ).

**Theorem 3.4** ([1]). *Let $E$ be a uniformly convex Banach space and let $C$ be a bounded closed and convex subset of $E$. Assume that $C$ has nonempty interior and that it is locally uniformly rotund. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$. If $\mathscr{S} = \{T(t) : t \in S\}$ has no common fixed point in the interior of $C$, then there exists a unique point $z_0$ on the boundary $\partial C$ of $C$ such that each orbit $\{T(t)x : t \in S\}$ converges strongly to $z_0$.*

The following example shows that the assumption that $C$ is locally uniformly rotund is crucial (see[8]).

**Example 2.** Let $H = \mathbb{R}^2$ be endowed with the standard Euclidean norm and let $C = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$. If $T(x,y) = (1,-y)$ for $(x,y) \in C$, then $T$ is nonexpansive and $(1,0) \in \partial C$ is its unique fixed point, but $\{T^n(1,1), n = 1,2,\ldots\}$, do not converge to $(1,0)$.

# 4 Deduced theorems

Using theorems 3.3 and 3.4, we get some convergence theorems (see [8]).

Let $C$ be a closed convex subset of a Banach space $E$. Then, a family $\mathscr{S} = \{T(s) : s \in \mathbb{R}^+\}$ of mappings of $C$ into itself is called a one-parameter nonexpansive semigroup on $C$ if it satisfies the following conditions:

(a) $T(s+t) = T(s)T(t)$ for all $s,t \in \mathbb{R}^+$;

(b) $T(0)x = x$ for each $x \in C$;

(c) $s \mapsto T(s)x$ is continuous;

(d) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x,y \in C$ and $s \in \mathbb{R}^+$

Using Theorem 3.3 and Lemma 3.2, we obtain the following convergence theorem. (see also [1, 5]).

**Theorem 4.1.** *Let $E$ be a uniformly convex Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$. Assume that $\mathscr{S} = \{T(t) : t \in S\}$ has a unique common fixed point $z_0$ and that $z_0$ lies on the boundary $\partial C$ of $C$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that*

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$$

*for each $s \in S$. Let $x \in C$ and let $\{z_n\}$ be the sequence defined by*

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) T_{\mu_n} z_n \quad \text{for each} \quad n \in \mathbb{N}.$$

*Then, $\{z_n\}$ converges strongly to $z_0$.*

Using Theorem 3.3 and Lemma 3.2, we also obtain the following convergence theorem. (see also [1, 19]).

**Theorem 4.2.** *Let $E$ be a uniformly convex Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $\mathscr{S} = \{T(t):t \in S\}$ be a nonexpansive semigroup on $C$. Assume that $\mathscr{S} = \{T(t):t \in S\}$ has a unique common fixed point $z_0$ and that $z_0$ lies on the boundary $\partial C$ of $C$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that*

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$$

*for each $s \in S$. Let $u_0 = x \in C$ and let $\{u_n\}$ be the sequence defined by*

$$u_n = \frac{1}{n}u_{n-1} + \left(1 - \frac{1}{n}\right) T_{\mu_n} u_n \quad \text{for each} \quad n \in \mathbb{N}.$$

*Then, $\{u_n\}$ converges strongly to $z_0$.*

Using Theorem 3.3, we obtain the following convergence theorems (see [1, 8]).

**Theorem 4.3.** *Let $E$ be a reflexive Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $T$ be a nonexpansive mapping of $C$ into itself. Assume that $I - T$ is demiclosed at $0$. If $T$ has a unique fixed point $z_0$ and $z_0$ lies on the boundary $\partial C$ of $C$, then every approximating sequence $\{x_n\}$ of $T$ converges strongly to $z_0$.*

**Theorem 4.4.** *Let $E$ be a reflexive Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $\mathscr{S} = \{T(t):t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on $C$. Assume that $I - T(t)$ is demiclosed at $0$ for each $t \in S$. If $\mathscr{S} = \{T(t):t \in \mathbb{R}^+\}$ has a unique common fixed point $z_0$ and $z_0$ lies on the boundary $\partial C$ of $C$, then every approximating sequence $\{x_n\}$ of $\mathscr{S}$ converges strongly to $z_0$.*

**Theorem 4.5.** *Let $E$ be a reflexive Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $T$ be a nonexpansive mapping of $C$ into itself. Assume that $I - T$ is demiclosed at $0$ and that $T$ has a unique fixed point $z_0$ which lies on the boundary $\partial C$ of $C$. Let $x \in C$ and let $\{z_n\}$ be the sequence defined by*

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) T z_n \quad \text{for each} \quad n \in \mathbb{N}.$$

*Then, $\{z_n\}$ converges strongly to $z_0$.*

**Theorem 4.6.** *Let $E$ be a reflexive Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $T$ be a nonexpansive mapping of $C$ into itself. Assume that $I-T$ is demiclosed at 0 and that $T$ has a unique fixed point $z_0$ which lies on the boundary $\partial C$ of $C$. Let $u_0 = x \in C$ and let $\{u_n\}$ be the sequence defined by*

$$u_n = \frac{1}{n}u_{n-1} + \left(1 - \frac{1}{n}\right) Tu_n \quad \textit{for each} \quad n \in \mathbb{N}.$$

*Then, $\{z_n\}$ converges strongly to $z_0$.*

By Theorem 4.1, we get the following convergence theorem (see also [1, 17]).

**Theorem 4.7.** *Let $E$ be a uniformly convex Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on $C$. Assume that $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ has a unique common fixed point $z_0$ and that $z_0$ lies on the boundary $\partial C$ of $C$. Let $\{t_n\}$ be a sequence in $(0,\infty)$ with $t_n \to \infty$. Let $x \in C$ and let $\{z_n\}$ be the sequence defined by*

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) \frac{1}{t_n} \int_0^{t_n} T(t) z_n dt \quad \textit{for each} \quad n \in \mathbb{N}.$$

*Then, $\{z_n\}$ converges strongly to $z_0$.*

By Theorem 4.2, we get the following theorem.

**Theorem 4.8.** *Let $E$ be a uniformly convex Banach space and let $C$ be a bounded, closed and convex subset of $E$ with nonempty interior. Assume further that $C$ is locally uniformly rotund. Let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on $C$. Assume that $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ has a unique common fixed point $z_0$ and that $z_0$ lies on the boundary $\partial C$ of $C$. Let $\{t_n\}$ be a sequence in $(0,\infty)$ with $t_n \to \infty$. Let $u_0 = x \in C$ and let $\{u_n\}$ be the sequence defined by*

$$u_n = \frac{1}{n}u_{n-1} + \left(1 - \frac{1}{n}\right) \frac{1}{t_n} \int_0^{t_n} T(t) u_n dt \quad \textit{for each} \quad n \in \mathbb{N}.$$

*Then, $\{z_n\}$ converges strongly to $z_0$.*

Using theorem 3.4, we get the following theorems (see [1, 8]).

**Theorem 4.9.** *Let $E$ be a uniformly convex Banach space and let $C$ be a bounded closed and convex subset of $E$. Assume that $C$ has nonempty interior and that it is locally uniformly rotund. Let $T$ be a nonexpansive mapping of $C$ into itself. If $T$ has no fixed point in the interior of $C$, then there exists a unique point $z_0$ on the boundary $\partial C$ of $C$ such that each sequence $\{T^n x : n = 1,2,3,\ldots\}$ converges strongly to $z_0$.*

**Theorem 4.10.** *Let $E$ be a uniformly convex Banach space and let $C$ be a bounded closed and convex subset of $E$. Assume that $C$ has nonempty interior and that it is locally uniformly rotund. Let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on $C$. If $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ has no common fixed point in the interior of $C$, then there exists a unique point $z_0$ on the boundary $\partial C$ of $C$ such that each orbit $\{T(t)x : t \in \mathbb{R}^+\}$ converges strongly to $z_0$.*


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