ON CHARACTERIZATIONS OF $VMO_{\Delta_N}(\mathbb{R}^n)$ SPACE

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Abstract. In this note, we shall give a resumé of a joint work with Mingming Cao [5]. We state several different characterizations of the vanishing mean oscillation space associated with Neumann Laplacian Δ_N, written $VMO_{\Delta_N}(\mathbb{R}^n)$. We first describe it with the classical $VMO(\mathbb{R}^n)$ and certain VMO on the half-spaces. Then we comment that $VMO_{\Delta_N}(\mathbb{R}^n)$ is actually $BMO_{\Delta_N}(\mathbb{R}^n)$-closure of the space of the smooth functions with compact supports. Beyond that, it can be characterized in terms of the compact commutators of Riesz transforms and fractional integral operators associated to the Neumann Laplacian. Additionally, we by means of the functional analysis obtain the duality between certain VMO and the corresponding Hardy spaces on the half-spaces. Finally, we present an useful approximation for BMO functions on the space of homogeneous type, which can be applied to our argument and otherwhere.

1. Introduction

A locally integrable function f on $\mathbb{R}^n$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_{Q\subseteq\mathbb{R}^n} \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx < \infty,$$

where f_Q denotes the average value of f on the cube Q.@ (F. John and L. Nirenberg, 1961.)

Let $VMO(\mathbb{R}^n)$ denote the closure of $C_c^\infty(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$. Additionally, the space $VMO(\mathbb{R}^n)$ is endowed with the norm of $BMO(\mathbb{R}^n)$. (R.R. Coifman and G. Weiss, 1977.)

• $H^1(\mathbb{R}^n) = VMO(\mathbb{R}^n)'$.

• Let $1 < p < \infty$ and R_j be the j-th Riesz transform on $\mathbb{R}^n$. Then Uchiyama 1978 showed that for $b \in \cup_{q>1} L^q_{\rm loc}(\mathbb{R}^n)$

$$b \in BMO(\mathbb{R}^n) \quad \text{if and only if} \quad [b, R_j] \in \mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n)), \tag{1.1}$$

and

$$b \in VMO(\mathbb{R}^n) \quad \text{if and only if} \quad [b, R_j] \in \mathcal{K}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n)), \tag{1.2}$$

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where $\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ is the set of all bounded linear operators from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, and $\mathcal{K}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ is the set of all compact operators from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Prototypes were considered by Coifmam, Rochberg and Weiss 1976, and P. Hartman 1958 and D. Sarason 1975, respectively.

Moreover

Proposition 1.1 ([34]). *Let $f \in BMO(\mathbb{R}^n)$. Then $f \in VMO(\mathbb{R}^n)$ if and only if f satisfies the following three conditions:*

(a) $\gamma_1(f) := \lim\limits_{r\to 0} \sup\limits_{Q:\ell(Q)\le r} \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^2 dx\right)^{1/2} = 0,$

(b) $\gamma_2(f) := \lim\limits_{r\to \infty} \sup\limits_{Q:\ell(Q)\ge r} \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^2 dx\right)^{1/2} = 0,$

(c) $\gamma_3(f) := \lim\limits_{r\to \infty} \sup\limits_{Q\subset Q(0,r)^c} \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^2 dx\right)^{1/2} = 0.$

Suppose that Ω is an open subset of $\mathbb{R}^n$. Define

$$\mathcal{M}(\Omega) := \left\{ f \in L^1_{loc}(\Omega) : \exists \epsilon > 0 \text{ s.t. } \int_\Omega \frac{|f(x)|^2}{1 + |x|^{n+\epsilon}} dx < \infty \right\}.$$

Definition 1.2. *(X. T. Duong and L. Yan 2005) We say that $f \in \mathcal{M}(\Omega)$ is of bounded mean oscillation associated with an operator L (abbreviated as $BMO_L(\Omega)$) if*

$$\|f\|_{BMO_L(\Omega)} := \sup_Q \frac{1}{|Q|} \int_Q \left| f(x) - e^{-\ell(Q)^2 L} f(x) \right| dx < \infty,$$

where the supremum is taken over all cubes Q in Ω.

Definition 1.3. *(D. G. Deng, X. T. Duong, L. Song, C. Tan and L. Yan, 2008) We say that a function $f \in BMO_L(\Omega)$ belongs to $VMO_L(\Omega)$, the space of functions of vanishing mean oscillation associated with the semigroup $\{e^{-tL}\}_{t>0}$, if it satisfies the limiting conditions*

$$\gamma_1(f;L) := \lim_{r\to 0} \sup_{Q\subseteq\Omega:\ell(Q)\le r} \left(\frac{1}{|Q|} \int_Q |f(x) - e^{-\ell(Q)^2 L} f(x)|^2 dx \right)^{1/2} = 0,$$

$$\gamma_2(f;L) := \lim_{r\to \infty} \sup_{Q\subseteq\Omega:\ell(Q)\ge r} \left(\frac{1}{|Q|} \int_Q |f(x) - e^{-\ell(Q)^2 L} f(x)|^2 dx \right)^{1/2} = 0,$$

$$\gamma_3(f;L) := \lim_{r\to \infty} \sup_{Q\subseteq\Omega\cap Q(0,r)^c} \left(\frac{1}{|Q|} \int_Q |f(x) - e^{-\ell(Q)^2 L} f(x)|^2 dx \right)^{1/2} = 0.$$

We endow $VMO_L(\Omega)$ with the norm of $BMO_L(\Omega)$.

2. Preliminaries

2.1. **The Neumann Laplacian.** The Neumann problem on the half line $(0, \infty)$ is given by the following:

$$\begin{cases} u_t - u_{xx} = 0, & x, t \in (0, \infty), \\ u(x, 0) = \phi(x), \\ u_x(0, t) = 0. \end{cases} \tag{2.1}$$

Let Δ_{1,N_+} be the Laplacian corresponding to (2.1). According to [33, Section 3.1], we see that

$$u(x, t) = e^{-t\Delta_{1,N_+}}(\phi)(x).$$

For $n > 1$, write $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$. And we define the Neumann Laplacian on $\mathbb{R}^n_+$ by

$$\Delta_{N_+} := \Delta_{n,N_+} = \Delta_{n-1} + \Delta_{1,N_+},$$

where Δ_{n-1} is the Laplacian on $\mathbb{R}^{n-1}$. Similarly, we can define Neumann Laplacian $\Delta_{N_-} := \Delta_{n,N_-}$ on $\mathbb{R}^n_-$.

The Laplacian Δ and Neumann Laplacian $\Delta_{N_\pm}$ are positive definite self-adjoint operators. By the spectral theorem one can define the semigroups generated by these operators $\{e^{-t\Delta}\}_{t\geq 0}$ and $\{e^{-t\Delta_{N_\pm}}\}_{t\geq 0}$. Set $p_t(x, y)$ and $p_{t,\Delta_{N_\pm}}(x, y)$ to be the heat kernels corresponding to the semigroups generated by Δ and $\Delta_{N_\pm}$, respectively. Then there holds

$$p_t(x, y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}.$$

It follows from the reflection method [33, p. 60] that

$$p_{t,\Delta_{N_+}}(x, y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x'-y'|^2}{4t}} \left(e^{-\frac{|x_n-y_n|^2}{4t}} + e^{-\frac{|x_n+y_n|^2}{4t}}\right), \quad x, y \in \mathbb{R}^n_+;$$

$$p_{t,\Delta_{N_-}}(x, y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x'-y'|^2}{4t}} \left(e^{-\frac{|x_n-y_n|^2}{4t}} + e^{-\frac{|x_n+y_n|^2}{4t}}\right), \quad x, y \in \mathbb{R}^n_-.$$

Let us introduce some notation. For any subset $A \subset \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{C}$, denote by $f|_A$ the restriction of f to A. For any function f on $\mathbb{R}^n$, we set

$$f_+ = f|_{\mathbb{R}^n_+} \text{ and } f_- = f|_{\mathbb{R}^n_-}.$$

For any $x = (x', x_n) \in \mathbb{R}^n$ we set $\widetilde{x} = (x', -x_n)$. If f is a function defined on $\mathbb{R}^n_+$, its even extension and zero extension defined on $\mathbb{R}^n$ are respectively given by

$$f_e(x) := \begin{cases} f(x), & \text{if } x \in \mathbb{R}^n_+, \\ f(\widetilde{x}), & \text{if } x \in \mathbb{R}^n_-, \end{cases} \qquad f_z(x) := \begin{cases} f(x), & \text{if } x \in \mathbb{R}^n_+, \\ 0, & \text{if } x \in \mathbb{R}^n_-. \end{cases}$$

Now let Δ_N be the uniquely determined unbounded operator acting on $L^2(\mathbb{R}^n)$ such that

$$(\Delta_N f)_+ = \Delta_{N_+} f_+ \text{ and } (\Delta_N f)_- = \Delta_{N_-} f_- \tag{2.2}$$

for all $f : \mathbb{R}^n \to \mathbb{R}$ such that $f_+ \in W^{1,2}(\mathbb{R}^n_+)$ and $f_- \in W^{1,2}(\mathbb{R}^n_-)$. Then Δ_N is a positive self-adjoint operator and

$$\left(e^{-t\Delta_N} f\right)_+ = e^{-t\Delta_{N+}} f_+ \quad \text{and} \quad \left(e^{-t\Delta_N} f\right)_- = e^{-t\Delta_{N-}} f_-. \tag{2.3}$$

The heat kernel of $e^{-t\Delta_N}$, denoted by $p_{t,\Delta_N}(x,y)$, is given by

$$p_{t,\Delta_N}(x,y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x'-y'|^2}{4t}} \left(e^{-\frac{|x_n-y_n|^2}{4t}} + e^{-\frac{|x_n+y_n|^2}{4t}}\right) H(x_n y_n),$$

where $H : \mathbb{R} \to \{0,1\}$ is the Heaviside function given by

$$H(t) = 1, \text{ if } t \geq 0; \quad H(t) = 0, \text{ if } t < 0.$$

Note that

- The operators Δ, $\Delta_{N_\pm}$ and Δ_N are self-adjoint and they generate bounded analytic positive semigroups acting on all $L^p(\mathbb{R}^n)$ spaces for $1 \leq p \leq \infty$;
- Let $p_{t,L}(x,y)$ be the kernel corresponding to the semigroup generated by one of the operators L listed above. Then $p_{t,L}(x,y)$ satisfies Gaussian bounds:

$$|p_{t,L}(x,y)| \lesssim t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{t}},$$

for all $x, y \in \Omega$, where $\Omega = \mathbb{R}^n$ for Δ and Δ_N; $\Omega = \mathbb{R}^n_+$ for Δ_{N_+} and $\Omega = \mathbb{R}^n_-$ for Δ_{N_-}.

3. $BMO_{\Delta_N}(\mathbb{R}^n)$ and $VMO_{\Delta_N}(\mathbb{R}^n)$ spaces

Definition 3.1. *Let f be a function on $\mathbb{R}^n_+$.*

(1) *f is said to be in $BMO_r(\mathbb{R}^n_+)$ if there exists $F \in BMO(\mathbb{R}^n)$ such that $F|_{\mathbb{R}^n_+} = f$. If $f \in BMO_r(\mathbb{R}^n_+)$, we set $\|f\|_{BMO_r(\mathbb{R}^n_+)} := \inf\left\{\|F\|_{BMO(\mathbb{R}^n)} : F|_{\mathbb{R}^n_+} = f\right\}$.*

(2) *f is said to be in $BMO_z(\mathbb{R}^n_+)$ if its zero extension f_z belongs to $BMO(\mathbb{R}^n)$. If $f \in BMO_z(\mathbb{R}^n_+)$, we set $\|f\|_{BMO_z(\mathbb{R}^n_+)} := \|f_z\|_{BMO(\mathbb{R}^n)}$.*

(3) *f is said to be $BMO_e(\mathbb{R}^n_+)$ if $f_e \in BMO(\mathbb{R}^n)$. Moreover, $BMO_e(\mathbb{R}^n_+)$ is endowed with the norm $\|f\|_{BMO_e(\mathbb{R}^n_+)} := \|f_e\|_{BMO(\mathbb{R}^n)}$.*

Similarly one can define the spaces $BMO_r(\mathbb{R}^n_-)$, $BMO_z(\mathbb{R}^n_-)$ and $BMO_e(\mathbb{R}^n_-)$.

The different type BMO spaces enjoy the following properties.

Proposition 3.2 ([15]). *(D. G. Deng, X. T. Duong, A. Sikora and L. X. Yan, 2008)*

There hold that

$$\|f\|_{BMO_{\Delta_{N_+}}(\mathbb{R}^n_+)} \simeq \|f\|_{BMO_e(\mathbb{R}^n_+)} \simeq \|f\|_{BMO_r(\mathbb{R}^n_+)},$$
$$\|f\|_{BMO_{\Delta_{N_-}}(\mathbb{R}^n_-)} \simeq \|f\|_{BMO_e(\mathbb{R}^n_-)} \simeq \|f\|_{BMO_r(\mathbb{R}^n_-)},$$
$$\|f\|_{BMO_{\Delta_N}(\mathbb{R}^n)} \simeq \|f_{+,e}\|_{BMO(\mathbb{R}^n)} + \|f_{-,e}\|_{BMO(\mathbb{R}^n)}.$$

Additionally, P. Auscher and E. Russ 2003 in [1] and P. Auscher, E. Russ and P. Tchamitchian 2005 in [2] further investigated the $BMO_r(\Omega)$, $BMO_z(\Omega)$ and corresponding Hardy spaces if Ω is a Lipschitz domain. The local case can be found in D. C. Chang [6] 1994.

As one has seen, the theory of the classical BMO and VMO is closely connected to the Laplacian Δ. On the other hand, the generalization of the operator L brings the new challenges to study the VMO_L space. As far as we know, there is almost no literature to explore its other properties except for the duality. Thus, three basic questions arising from (1.2) motivate our work:

- Question 1: Does (1.2) hold for Riesz transforms $\nabla L^{-1/2}$ associated with the operator L other than the Laplacian?
- Question 2: What type of VMO_L spaces is suitable to (1.2) for Riesz transforms $\nabla L^{-\frac{1}{2}}$?
- Question 3: Are there other new properties for VMO_L?

Before addressing these questions, let us get a glimpse of the possibility. If L is the Dirichlet Laplacian Δ_{D_+} on $\mathbb{R}^n_+$, then the $BMO_{\Delta_{D_+}}(\mathbb{R}^n_+)$ space cannot be characterized by the boundedness of $[b, \nabla\Delta_{D_+}^{-1/2}]$ (see X. T. Duong, I. Holmes, J. Li, B. D. Wick and D. Yang 2019 [17, Theorem 1.4]). This indicates that the equation (1.2) does not hold for $\nabla L^{-\frac{1}{2}}$ in a very general framework. On the other hand, (1.2) holds for certain special operator, for example the Bessel operator Δ_λ in [18]. Furthermore, as we know, the boundedness is prior condition for the compactness. Taking into consideration some research on the Neumann Laplacian Δ_N [15] and the boundedness of commutators of $\nabla\Delta_N^{-1/2}$ in [28], we will pay our attention to the Neumann Laplacian Δ_N. We postpone all the definitions and notation in Section 2.

We begin with giving the answers to Question 1.

Theorem 3.3. *Let $1 < p < \infty$ and $j = 1, \ldots, n$. Then $b \in VMO_{\Delta_N}(\mathbb{R}^n)$ if and only if $[b, R_{N,j}]$ is a compact operator on $L^p(\mathbb{R}^n)$.*

Our next main result is to indicate that the equation (1.2) also holds for the fractional integrals associated with the Neumann Laplacian Δ_N.

Theorem 3.4. *Let* $0 < \alpha < n$, $1 < p < q < \infty$ *with* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then* $b \in VMO_{\Delta_N}(\mathbb{R}^n)$ *if and only if* $[b, \Delta_N^{-\alpha/2}]$ *is a compact operator from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

Theorems 3.3 and 3.4 also provide positive answers to Question 2. Additionally, VMO_L space is suitable to (1.2) for Riesz transforms $\nabla L^{-\frac{1}{2}}$ when L is the Neumann Laplacian Δ_{N_+} (Δ_{N_-}) on the upper (lower) half-space. Actually, we have established the desired properties for the corresponding VMO spaces on the half-space in Section 3. The approach in Section 5 is easily modified to the setting of half-spaces. The details are left to the readers.

Considering Question 3, we first build a bridge between the $VMO_{\Delta_N}(\mathbb{R}^n)$ and the classical VMO space. As we will see, it is quite valuable to further study the $VMO_{\Delta_N}(\mathbb{R}^n)$ space.

Theorem 3.5. *The* $VMO_{\Delta_N}(\mathbb{R}^n)$ *space can be characterized in the following way:*

$$VMO_{\Delta_N}(\mathbb{R}^n) = \{f \in \mathcal{M}(\mathbb{R}^n) : f_{+,e} \in VMO(\mathbb{R}^n) \text{ and } f_{-,e} \in VMO(\mathbb{R}^n)\}.$$

Moreover, we have that

$$\|f\|_{VMO_{\Delta_N}(\mathbb{R}^n)} \simeq \|f_{+,e}\|_{VMO(\mathbb{R}^n)} + \|f_{-,e}\|_{VMO(\mathbb{R}^n)}.$$

Beyond that, we can understand the $VMO_{\Delta_N}(\mathbb{R}^n)$ space in the following way.

Theorem 3.6. *The* $VMO_{\Delta_N}(\mathbb{R}^n)$ *space is the* $BMO_{\Delta_N}(\mathbb{R}^n)$*-closure of* $C_c^\infty(\mathbb{R}^n)$.

We also amalyze the other properties, including characterizations, duality and weak*-convergence, of $VMO_{\Delta_N}(\mathbb{R}^n)$ and associated spaces.

Now let us discuss the strategy of the proof. Generally, the proof of (1.2), as well as other known results about the compactness of commutators, makes use of a characterization of precompactness in Lebesgue spaces, which is the called Fréchet-Kolmogorov theorem. Such theorem has been adapted for various spaces for examples, [8], [10], [11] and [18]. Even so, it seems to be invalid for the Neumann Laplacian Δ_N. One main reason is that the smooth properties on $\mathbb{R}^n$ are not enough although the Riesz transforms $\nabla\Delta_N$ are Calderón-Zygmund operators on both $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$. In order to circumvent this obstacle, we reduce our question to that in $L^p_e(\mathbb{R}^n)$, which is a closed subspace of $L^p(\mathbb{R}^n)$ and contains all even functions with respect to the last variable. Theorem 3.5 is based on the reflection argument on $\mathbb{R}^n$. Thus it allows us to focus on the analysis on half-spaces. The proof of Theorem 3.6 is constructive but different from Uchiyama's. We mainly apply some BMO estimates for smooth functions with compact support. In view of Theorem 3.5, it needs to connect the functions on the upper and lower spaces by continuity and smoothness. As we mentioned above, the $VMO_{\Delta_N}(\mathbb{R}^n)$ space is closely related to those on half-spaces, such as $VMO_e(\mathbb{R}^n_+)$, $VMO_r(\mathbb{R}^n_+)$

and $VMO_z(\mathbb{R}^n_+)$. Hence, we also investigate their duality to understand $VMO_{\Delta_N}(\mathbb{R}^n)$ well. Our method is motivated by [13] and [6]. Some results from functional analysis is quite effective on our conclusion. Not only that, we utilize an approximation for BMO functions by the continuous functions with bounded support. The general case will be presented in Section 6.

This article is organized as follows. In Section 2, we recall the definitions of the Neumann Laplacian Δ_{N_+} and the reflection Neumann Laplacian Δ_N. We also collect some known results related to various types of BMO spaces. In Section 3, we introduce the vanishing mean oscillation space $VMO_{\Delta_N}(\mathbb{R}^n)$ associated with Δ_N, and provide its characterizations by means of the classical $VMO(\mathbb{R}^n)$ space, the VMO on the half-spaces, and smooth functions with compact supports. Section 4 is devoted to the duality between certain VMO spaces and the corresponding Hardy spaces. After that, in Section 5, we establish other characterizations of $VMO_{\Delta_N}(\mathbb{R}^n)$ using the compact commutators of Riesz transforms and fractional integral operators associated with Δ_N. Finally, in Section 6, an approximation is presented for BMO functions on the space of homogeneous type in the sense of Coifman-Weiss.

Let us introduce several types of VMO spaces on the half-spaces.

Definition 3.7. *Let f be a function on $\mathbb{R}^n_+$.*

(1) *f is said to be in $VMO_r(\mathbb{R}^n_+)$ if there exists $F \in VMO(\mathbb{R}^n)$ such that $F|_{\mathbb{R}^n_+} = f$. If $f \in VMO_r(\mathbb{R}^n_+)$, we set $\|f\|_{VMO_r(\mathbb{R}^n_+)} := \inf\left\{\|F\|_{VMO(\mathbb{R}^n)} : F|_{\mathbb{R}^n_+} = f\right\}$.*

(2) *f is said to be in $VMO_z(\mathbb{R}^n_+)$ if the function f_z belongs to $VMO(\mathbb{R}^n)$. If $f \in VMO_z(\mathbb{R}^n_+)$, we set $\|f\|_{VMO_z(\mathbb{R}^n_+)} := \|f_z\|_{VMO(\mathbb{R}^n)}$.*

(3) *f is said to be $VMO_e(\mathbb{R}^n_+)$ if $f_e \in VMO(\mathbb{R}^n)$. Moreover, $VMO_e(\mathbb{R}^n_+)$ is endowed with the norm $\|f\|_{VMO_e(\mathbb{R}^n_+)} := \|f_e\|_{VMO(\mathbb{R}^n)}$.*

Similarly one can define the spaces $VMO_r(\mathbb{R}^n_-)$, $VMO_z(\mathbb{R}^n_-)$ and $VMO_e(\mathbb{R}^n_-)$.

Theorem 3.8. *The spaces $VMO_{\Delta_{N_+}}(\mathbb{R}^n_+)$, $VMO_e(\mathbb{R}^n_+)$ and $VMO_r(\mathbb{R}^n_+)$ coincide, with equivalent norms*

$$\|f\|_{VMO_{\Delta_{N_+}}(\mathbb{R}^n_+)} \simeq \|f\|_{VMO_e(\mathbb{R}^n_+)} \simeq \|f\|_{VMO_r(\mathbb{R}^n_+)}.$$

Similar results hold for $VMO_{\Delta_{N_-}}(\mathbb{R}^n_-)$, $VMO_e(\mathbb{R}^n_-)$ and $VMO_r(\mathbb{R}^n_-)$.

To understand the $VMO_{\Delta_N}(\mathbb{R}^n)$ space well, let us describe it in terms of VMO spaces on the upper/lower half-spaces.

Theorem 3.9. *The $VMO_{\Delta_N}(\mathbb{R}^n)$ space can be described as*

$$VMO_{\Delta_N}(\mathbb{R}^n) = \{f \in \mathcal{M}(\mathbb{R}^n) : f_+ \in VMO_{\Delta_{N_+}}(\mathbb{R}^n_+) \text{ and } f_- \in VMO_{\Delta_{N_-}}(\mathbb{R}^n_-)\}.$$

Moreover, we have that

$$\|f\|_{VMO_{\Delta_N}(\mathbb{R}^n)} \simeq \|f_+\|_{VMO_{\Delta_{N_+}}(\mathbb{R}^n_+)} + \|f_-\|_{VMO_{\Delta_{N_-}}(\mathbb{R}^n_-)}.$$

As a consequence, Theorem 3.5 immediately follows from Theorems 3.9 and 3.8.

We here give the comparison among the different spaces.

Theorem 3.10. *The following inclusions hold*

$$VMO_\Delta(\mathbb{R}^n) = VMO_{\sqrt{\Delta}}(\mathbb{R}^n) = VMO(\mathbb{R}^n) \subsetneqq VMO_{\Delta_N}(\mathbb{R}^n) \subsetneqq BMO_{\Delta_N}(\mathbb{R}^n).$$

Proof. The equivalence $VMO_\Delta(\mathbb{R}^n) = VMO_{\sqrt{\Delta}}(\mathbb{R}^n) = VMO(\mathbb{R}^n)$ was proved in [16, Proposition 3.6]. $VMO(\mathbb{R}^n) \subsetneqq VMO_{\Delta_N}(\mathbb{R}^n)$ and $VMO(\mathbb{R}^n) \subseteq VMO_{\Delta_N}(\mathbb{R}^n)$ can be easily checked. In order to certify the strict inclusion, we give examples. □

Let X and Y be Banach spaces. For the convenience of notation, we denote by $\overline{X}^Y$ the closure of X in Y. Now we characterize VMO spaces via smooth functions with compact supports.

Theorem 3.11. *We have*

$$VMO_{\Delta_N}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{BMO_{\Delta_N}(\mathbb{R}^n)}, \tag{3.1}$$

$$VMO_{\Delta_{N_+}}(\mathbb{R}^n_+) = \overline{C_c^\infty(\mathbb{R}^n_+)}^{BMO_{\Delta_{N_+}}(\mathbb{R}^n_+)}, \tag{3.2}$$

$$VMO_e(\mathbb{R}^n_+) = \overline{C_c^\infty(\mathbb{R}^n_+)}^{BMO_e(\mathbb{R}^n_+)}, \tag{3.3}$$

$$VMO_r(\mathbb{R}^n_+) = \overline{C_c^\infty(\mathbb{R}^n_+)}^{BMO_r(\mathbb{R}^n_+)}, \tag{3.4}$$

$$VMO_z(\mathbb{R}^n_+) = \overline{C_c^\infty(\mathbb{R}^n_+)}^{BMO_z(\mathbb{R}^n_+)}. \tag{3.5}$$

Moreover, the similar results hold for the lower half-space.

Proof. We only show (3.1). We will use the fact

$$VMO(\mathbb{R}^n) = \overline{C_c(\mathbb{R}^n)}^{BMO(\mathbb{R}^n)}. \tag{3.6}$$

We first prove $\overline{C_c^\infty(\mathbb{R}^n)}^{BMO_{\Delta_N}(\mathbb{R}^n)} \subseteq VMO_{\Delta_N}(\mathbb{R}^n)$. Assume that $f \in \overline{C_c^\infty(\mathbb{R}^n)}^{BMO_{\Delta_N}(\mathbb{R}^n)}$. Then for any $\epsilon > 0$, there exists $g \in C_c^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{BMO_{\Delta_N}(\mathbb{R}^n)} < \epsilon$. Observe that $g_{+,e}, g_{-,e} \in C_c(\mathbb{R}^n)$. Indeed, if $\mathrm{supp}(g) \subseteq \mathbb{R}^n_+$, then $g_{+,e} \in C_c(\mathbb{R}^n)$ and $g_{-,e} \equiv 0$. If $\mathrm{supp}(g) \subseteq \mathbb{R}^n_-$, then $g_{+,e} \equiv 0$ and $g_{-,e} \in C_c(\mathbb{R}^n)$. If $\mathrm{supp}(g) \cap \mathbb{R}^n_+ \neq \emptyset$ and $\mathrm{supp}(g) \cap \mathbb{R}^n_- \neq \emptyset$, then $g_{+,e} \in C_c(\mathbb{R}^n)$ and $g_{-,e} \in C_c(\mathbb{R}^n)$. Moreover, it follows from Proposition 3.2 that $\|f_{+,e} - g_{+,e}\|_{BMO(\mathbb{R}^n)} \lesssim \epsilon$ and $\|f_{-,e} - g_{-,e}\|_{BMO(\mathbb{R}^n)} \lesssim \epsilon$. By (3.6), we have $f_{+,e} \in VMO(\mathbb{R}^n)$ and $f_{-,e} \in VMO(\mathbb{R}^n)$, which together with Theorem 3.5 gives $f \in VMO_{\Delta_N}(\mathbb{R}^n)$.

Now we are in the position to show the converse. Assume that $f \in VMO_{\Delta_N}(\mathbb{R}^n)$, which by Theorem 3.5 gives that $f_{+,e}, f_{-,e} \in VMO(\mathbb{R}^n)$. Hence for any $\epsilon > 0$, there exists $\tilde{g}_1 \in C_c^\infty(\mathbb{R}^n)$ such that $\|f_{+,e} - \tilde{g}_1\|_{BMO(\mathbb{R}^n)} < \epsilon$. Set

$$g_1(x) = (\tilde{g}_1(x) + \tilde{g}_1(x', -x_n))/2, \quad x = (x', x_n) \in \mathbb{R}^n.$$

Then we see that $g_1(x) = g_1(x', -x_n) =: g_1(\widetilde{x})$ and

$$\begin{aligned}\|f_{+,e} - g_1\|_{BMO(\mathbb{R}^n)} &= \|f_{+,e} - (\tilde{g}_1(x) + \tilde{g}_1(x', -x_n))/2\|_{BMO(\mathbb{R}^n)} \\ &\leq \frac{1}{2}\left(\|f_{+,e} - \tilde{g}_1\|_{BMO(\mathbb{R}^n)} + \|f_{+,e} - \tilde{g}_1(x', -x_n)\|_{BMO(\mathbb{R}^n)}\right) \\ &= \|f_{+,e} - \tilde{g}_1\|_{BMO(\mathbb{R}^n)} < \epsilon.\end{aligned}$$

Similarly, there exist $g_2 \in C_c^\infty(\mathbb{R}^n)$ such that

$$g_2(x) = g_2(\widetilde{x}) \ \text{ and } \ \|f_{-,e} - g_2\|_{BMO(\mathbb{R}^n)} < \epsilon.$$

By Lemma 3.14 below there exist even functions $\psi_1,\ \psi_2 \in C_c^\infty(\mathbb{R})$ such that $\psi_1(0) = \psi_2(0) = 1$ and

$$\|g_1(x', 0)\psi_1(x_n)\|_{BMO(\mathbb{R}^n)} + \|g_2(x', 0)\psi_2(x_n)\|_{BMO(\mathbb{R}^n)} < \varepsilon.$$

Define

$$h(x) = \begin{cases} g_1(x) + g_2(x', 0)\psi_2(x_n), & x \in \mathbb{R}^n_+, \\ g_1(x', 0)\psi_1(x_n) + g_2(x), & x \in \mathbb{R}^n_-. \end{cases}$$

It immediately yields that

$$h_{+,e}(x) = g_1(x) + g_2(x', 0)\psi_2(x_n), \quad h_{-,e}(x) = g_1(x', 0)\psi_1(x_n) + g_2(x),$$

$h \in C_c(\mathbb{R}^n)$ and $h_{+,e}, h_{-,e} \in C_c^\infty(\mathbb{R}^n) \subseteq VMO(\mathbb{R}^n)$. Consequently, we deduce that

$$\begin{aligned}\|f_{+,e} - h_{+,e}\|_{BMO(\mathbb{R}^n)} &\leq \|f_{+,e} - g_1\|_{BMO(\mathbb{R}^n)} + \|g_1 - h_{+,e}\|_{BMO(\mathbb{R}^n)} \\ &\leq \epsilon + \|g_2(x', 0)\psi_2(x_n)\|_{BMO(\mathbb{R}^n)} < 2\epsilon, \\ \|f_{-,e} - h_{-,e}\|_{BMO(\mathbb{R}^n)} &\leq \|f_{-,e} - g_2\|_{BMO(\mathbb{R}^n)} + \|g_2 - h_{-,e}\|_{BMO(\mathbb{R}^n)} \\ &\leq \epsilon + \|g_1(x', 0)\psi_1(x_n)\|_{BMO(\mathbb{R}^n)} < 2\epsilon,\end{aligned}$$

and

$$\|f - h\|_{VMO_{\Delta_N}(\mathbb{R}^n)} \simeq \|f_{+,e} - h_{+,e}\|_{VMO(\mathbb{R}^n)} + \|f_{-,e} - h_{-,e}\|_{VMO(\mathbb{R}^n)} \lesssim \epsilon.$$

This implies that $C_c(\mathbb{R}^n)$ is dense in $VMO_{\Delta_N}(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n)$ under the $L^\infty(\mathbb{R}^n)$ norm, we see that $C_c^\infty(\mathbb{R}^n)$ is dense in $VMO_{\Delta_N}(\mathbb{R}^n)$.

□

The remainder of this section is devoted to showing Lemma 3.14. To this end, we first present the *BMO* estimates for smooth functions with compact supports.

Lemma 3.12. *Let $\varphi_j(x) \in C_c^\infty(\mathbb{R})$ be nonnegative and satisfy $\chi_{\{|x|\le 2^j-1\}} \le \varphi_j(x) \le \chi_{\{|x|\le 2^j\}}$, $j \in \mathbb{N}$, and*

$$\psi_\ell(x) = \frac{1}{\ell}\sum_{j=1}^{\ell}\varphi_j(x), \quad \ell = 3, 4, \ldots.$$

Then it holds that $\psi_\ell \in C_c^\infty(\mathbb{R})$, $0 \le \psi_\ell(x) \le 1$, $\psi_\ell(x) = 1$ for $|x| \le 1$, $\psi_\ell(x) = 0$ for $|x| \ge 2^\ell$ and $\|\psi_\ell\|_{BMO(\mathbb{R})} \le 16/\ell$,

Proof. We have only to prove $\|\ell\,\psi_\ell\|_{BMO(\mathbb{R})} \le 16$. Let $I = [a, b]$ be an interval. If $|I| < 2$, then we see easily that $(2^{j+1} - 1) - (2^j - 1) = 2^j \ge 2$ for any $j \ge 1$ and

$$\frac{1}{|I|}\int_I |\ell\psi_\ell(x) - (\ell\psi_\ell)_I|dx \le \sum_{j=1}^{\ell}\frac{1}{|I|}\int_I |\varphi_j(x) - (\varphi_j)_I|dx \le 2.$$

If $|I| \ge 2$, there exists $j \in \mathbb{N}_+$ such that $2^j \le |I| < 2^{j+1}$. We shall consider the following two cases: (a) $j \ge \ell - 1$ and (b) $1 \le j \le \ell - 2$.

Case (a): $j \ge \ell - 1$**.**

$$\begin{aligned}\frac{1}{|I|}\int_I |\ell\phi_\ell(x)dx &= \sum_{i=1}^{\ell-2}\frac{1}{|I|}\int_I \varphi_i(x)dx + \frac{1}{|I|}\int_I \varphi_{\ell-1}(x)dx \\ &\le \sum_{i=1}^{\ell-2}\frac{1}{2^i}\cdot 2\cdot 2^i + 2 \\ &= \frac{2^{\ell-1}}{2^j} + 2 < 3.\end{aligned}$$

Case (b): $1 \le j \le \ell - 2$**.** If $a \ge 2^j$, then $b - a < 2^{j+1}$ i.e. $b < a + 2^{j+1}$, and so I contains at most one point of the form 2^i $(j + 1 \le i \le \ell)$. Hence, it yields that

$$\frac{1}{|I|}\int_I |\ell\psi_\ell(x) - (\ell - i)|dx \le 1.$$

If $-2^j \le a < 2^j$, then $b < 2^{j+1} + 2^j < 2^{j+2} \le 2^\ell$, which implies that

$$\frac{1}{|I|}\int_I |\ell\psi_\ell(x) - (\ell - j - 1)|dx \le \frac{1}{2^j}\sum_{i=1}^{j+1} 2\cdot 2^i \le 2 \le \frac{1}{2^j}\cdot 2\cdot 2^{j+2} = 8.$$

.Thus, we have the desired estimate if $a \ge -2^j$.

Note that if $a < -2^j$, then it follows $b < 2^{j+1} - 2^j = 2^j$. So, similarly, the above estimates hold for $b < 2^j$, and hence if $a < -2^j$.

All together we get

$$\frac{1}{|I|}\int_I |\ell\psi_\ell(x) - (\ell\psi_\ell)_I| dx \le 16.$$

□

Lemma 3.13. *For any ϵ, $\eta > 0$, there exists $\psi \in C_c^\infty(\mathbb{R})$ with* $\operatorname{supp}\psi \subset [-\eta, \eta]$ *such that*

$$\|\psi\|_{BMO(\mathbb{R})} < \epsilon,\ 0 \le \psi(x) \le 1,\ \ \textit{and}\ \ \psi(0) = 1.$$

Proof. We use the notations in Lemma 3.12. First we take $\ell \in \mathbb{N}_+$ so that $\|\psi_\ell\|_{BMO(\mathbb{R})} \le 16/\ell < \epsilon$, and we set

$$\psi(x) = \psi_\ell(2^\ell x/\eta).$$

Then from the dilation invariance of BMO norm, we get $\|\psi\|_{BMO(\mathbb{R})} < \epsilon$. Since the $\operatorname{supp}\psi_\ell \subset [-2^\ell, 2^\ell]$, we see that $\operatorname{supp}\psi \subset [-\eta, \eta]$. This ψ also satisfies $\psi(0) = 1$ and $0 \le \psi(x) \le 1$ for $x \in \mathbb{R}$. □

Lemma 3.14. *For any $\epsilon > 0$ and $g \in C_c^\infty(\mathbb{R}^n)$, there exists $\psi \in C_c^\infty(\mathbb{R})$ such that*

$$0 \le \psi(x_n) \le 1,\ \psi(0) = 1 \ \textit{and}\ \|g(x', 0)\psi(x_n)\|_{BMO(\mathbb{R}^n)} < \epsilon,$$

where $x = (x', x_n) \in \mathbb{R}^n$.

Proof. In the case $n = 1$, for $\epsilon > 0$ take $\epsilon_1 > 0$ satisfying $|g(0)|\epsilon_1 < \epsilon$. Takeing ψ_ℓ in Lemma 3.12 so that $16/\ell < \epsilon_1$, we get $\|g(0)\psi_\ell(x_n)\|_{BMO(\mathbb{R})} < \epsilon$. So this ψ_ℓ is a desired function. In the case $n \ge 2$, we proceed as follows. Let $g \in C_c^\infty(\mathbb{R}^n)$ and $\epsilon > 0$. Then $g(x', 0) \in C_c^\infty(\mathbb{R}^{n-1}) \subset VMO(\mathbb{R}^{n-1})$. Let $Q = (I', I)$ be any cube in $\mathbb{R}^n$, where I' is a cube in $\mathbb{R}^{n-1}$ and I be an interval in $\mathbb{R}$. Since $g(x', 0)$ is also a $VMO(\mathbb{R}^{n-1})$ function, there exists $\delta > 0$ such that

$$\frac{1}{|I'|}\int_{I'} |g(x', 0) - g(\cdot, 0)_{I'}| dx' < \epsilon \ \text{ if } |I'| < \delta^{n-1}.$$

By Lemma 3.13 for $\eta = \epsilon\delta/2$, there exists $\psi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \subset [-\eta, \eta]$ such that

$$\|\psi\|_{BMO(\mathbb{R})} < \epsilon,\ \psi(0) = 1, \text{ and } 0 \le \psi(x) \le 1,\ x \in \mathbb{R}.$$

Now we deduce that

$$\begin{aligned} J &:= \frac{1}{|Q|}\int_{I'}\int_I |g(x', 0)\psi(x_n) - g(\cdot, 0)_{I'}\psi_I| dx_n dx' \\ &\le \frac{1}{|Q|}\int_{I'}\int_I |g(x', 0)\psi(x_n) - g(x', 0)\psi_I| dx_n dx' \\ &\quad + \frac{1}{|Q|}\int_{I'}\int_I |g(x', 0)\psi_I - g(\cdot, 0)_{I'}\psi_I| dx_n dx' \end{aligned}$$

$$\leq \frac{1}{|I'|}\int_{I'}|g(x',0)|dx'\frac{1}{|I|}\int_{I}|\psi(x_n)-\psi_I|dx_n$$
$$+\frac{1}{|I'|}\int_{I'}|g(x',0)-g(\cdot,0)_{I'}|dx'|\psi_I|$$
$$< \epsilon\|g\|_{L^\infty(\mathbb{R}^n)}+\frac{1}{|I'|}\int_{I'}|g(x',0)-g(\cdot,0)_{I'}|dx'|\psi_I|.$$

Hence, if $|I| = |I'|^{1/(n-1)} < \delta$, we get $J < \epsilon\|g\|_{L^\infty(\mathbb{R}^n)} + \epsilon$. If $I \cap [-\eta,\eta] = \emptyset$, we see trivially $J = 0$. If $I \cap [-\eta,\eta] \neq \emptyset$ and $|I| \geq \delta$, we see that

$$|\psi_I| \leq \frac{1}{|I|}\int_I |\psi(x_n)|dx_n \leq \frac{1}{|I|}\cdot 2\eta < \frac{2\eta}{\delta} < \epsilon,$$

and so we get $J < \epsilon\|g\|_{L^\infty(\mathbb{R}^n)} + 2\epsilon\|g\|_{L^\infty(\mathbb{R}^n)} = 3\epsilon\|g\|_{L^\infty(\mathbb{R}^n)}$. Modifying constants above completes the proof of Lemma 3.14. □

4. Dual spaces

Let us recall the definitions of various Hardy spaces on the upper/lower half-space in [7].

Definition 4.1. *Let f be a function on $\mathbb{R}^n_+$.*

(1) *f is said to be in $H^1_r(\mathbb{R}^n_+)$ if there exists $F \in H^1(\mathbb{R}^n)$ such that $F|_{\mathbb{R}^n_+} = f$. If $f \in H^1_r(\mathbb{R}^n_+)$, we set $\|f\|_{H^1_r(\mathbb{R}^n_+)} := \inf\left\{\|F\|_{H^1(\mathbb{R}^n)} : F|_{\mathbb{R}^n_+} = f\right\}$.*
(2) *f is said to be in $H^1_z(\mathbb{R}^n_+)$ if the function f_z belongs to $H^1(\mathbb{R}^n)$. If $f \in H^1_z(\mathbb{R}^n_+)$, we set $\|f\|_{H^1_z(\mathbb{R}^n_+)} := \|f_z\|_{H^1(\mathbb{R}^n)}$.*
(3) *f is said to be $H^1_e(\mathbb{R}^n_+)$ if $f_e \in H^1(\mathbb{R}^n)$. Moreover, $H^1_e(\mathbb{R}^n_+)$ is endowed with the norm $\|f\|_{H^1_e(\mathbb{R}^n_+)} := \|f_e\|_{H^1(\mathbb{R}^n)}$.*
(4) *f is said to be $H^1_o(\mathbb{R}^n_+)$ if $f_o \in H^1(\mathbb{R}^n)$. Moreover, $H^1_o(\mathbb{R}^n_+)$ is endowed with the norm $\|f\|_{H^1_o(\mathbb{R}^n_+)} := \|f_e\|_{H^1(\mathbb{R}^n)}$.*

Similarly one can define the spaces $H^1_r(\mathbb{R}^n_-)$, $H^1_z(\mathbb{R}^n_-)$, $H^1_e(\mathbb{R}^n_-)$ and $H^1_o(\mathbb{R}^n_-)$.

The authors in [7] proved that

$$H^1_r(\mathbb{R}^n_+) = H^1_o(\mathbb{R}^n_+) \quad \text{and} \quad H^1_z(\mathbb{R}^n_+) = H^1_e(\mathbb{R}^n_+).$$

A celebrated work of Fefferman and Stein [22] showed that $BMO(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$. Moreover, in the half-spaces setting, the duality was established in [2] as follows

$$(H^1_r(\mathbb{R}^n_+))^* = BMO_z(\mathbb{R}^n_+) \quad \text{and} \quad (H^1_z(\mathbb{R}^n_+))^* = BMO_r(\mathbb{R}^n_+).$$

As is known,

Theorem 4.2. *The dual space of $VMO_{\Delta_N}(\mathbb{R}^n)$ is $H^1_{\Delta_N}(\mathbb{R}^n)$.*

Proof. The proof can be found in Theorem 4.1 [16], in which a more general result about the operator L was given. □

Based on the duality above, let us investigate the weak*-convergence in $H^1_{\Delta_N}(\mathbb{R}^n)$.

Theorem 4.3. *Suppose that $\{f_k\}_{k\geq 1}$ is a bounded sequence in $H^1_{\Delta_N}(\mathbb{R}^n)$, and that $\lim_{k\to\infty} f_k(x) = f(x)$ a.e. $x \in \mathbb{R}^n$. Then $f \in H^1_{\Delta_N}(\mathbb{R}^n)$ and $\{f_k\}_{k\geq 1}$ weak*-converges to f, that is,*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)\phi(x)dx = \int_{\mathbb{R}^n} f(x)\phi(x)dx, \quad \forall \phi \in VMO_{\Delta_N}(\mathbb{R}^n),$$

where the integrals denote the dual form between $H^1_{\Delta_N}(\mathbb{R}^n)$ and $BMO_{\Delta_N}(\mathbb{R}^n)$ in general.

Theorem 4.4. *The dual space of $VMO_z(\mathbb{R}^n_+)$ is $H^1_r(\mathbb{R}^n_+)$.*

Theorem 4.5. *The dual space of $VMO_r(\mathbb{R}^n_+)$ is $H^1_z(\mathbb{R}^n_+)$.*

5. Compact commutators

In this section, we will characterize $VMO_{\Delta_N}(\mathbb{R}^n)$ via the compactness of commutators of Riesz transforms and the fractional integral operators associated with the Neumann Laplacian.

5.1. Compactness of $[b, R_N]$.

The Riesz transforms associated to the Neumann Laplacian are given by

$$R_N = (R_{N,1}, \dots, R_{N,n}) := \nabla \Delta_N^{-1/2}.$$

The kernel of $R_{N,j}$ was formulated in [28] as

$$R_{N,j}(x, y) = \Big(R_j(x, y) + R_j(x, \widetilde{y})\Big) H(x_n y_n),\ j = 1, \dots, n,$$

where $R_j(x, y)$ is the kernel of Riesz transform R_j:

$$R_j(x, y) = \frac{x_j - y_j}{|x - y|^{n+1}},\ j = 1, \dots, n,$$

Theorem 5.1. *Let $1 < p < \infty$ and $j = 1, \dots, n$. Then $b \in BMO_{\Delta_N}(\mathbb{R}^n)$ if and only if $[b, R_{N,j}]$ is bounded on $L^p(\mathbb{R}^n)$. Moreover, we have*

$$\|[b, R_{N,j}]\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \simeq \|b\|_{BMO_{\Delta_N}(\mathbb{R}^n)}.$$

When $p = 2$, the above result was proved in [28, Theorem 1.4]. But the proof was complicated because the authors used a weak factorization of the space $H^1_{\Delta_N}(\mathbb{R}^n)$. We present a direct and easy proof for the lower bound and the upper bound can be obtained for $1 < p < \infty$ as the case $p = 2$. To show the sufficiency we use the following.

Theorem 5.2. *Let $1 < p < \infty$ and $b \in \bigcup_{1<q<\infty} L^q_{\mathrm{loc}}(\mathbb{R}^n)$ with $b(x) = b(\tilde{x})$, $x \in \mathbb{R}^n$. Then for the Riesz transform R_i $(i = 1, \ldots, n)$ there exists a constant $A = A(n, p, R_i)$ such that*

$$\|b\|_{BMO(\mathbb{R}^n)} \le A\|[b, R_i]\|_{L^p_e(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}.$$

We prove this by modifying the corresponding proof by A. Uchiyama [34].

Theorem 5.3. *Let $1 < p < \infty$ and $j = 1, \ldots, n$. Then $b \in VMO_{\Delta_N}(\mathbb{R}^n)$ if and only if $[b, R_{N,j}]$ is a compact operator on $L^p(\mathbb{R}^n)$.*

We show the sufficiency directly, and the necessity by modifying the corresponding one by A. Uchiyama [34].

6. A BMO approximation

Let (X, d, μ) be a space of homogeneous type in the sense of Coifman-Weiss. That is, X is a topological space endowed with a Borel measure μ and a quasi-metric d, satisfying the following conditions: (a) $d(x, y) = d(y, x)$, (b) $d(x, y) > 0$ if and only if $x \neq y$ and (c) there exists a constant K such that $d(x, y) \le K[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. (d) the balls $B(x, r) = \{y \in X;\, d(x, y) < r\}$ centered at x and of radius $r > 0$ form a basis of open neighborhoods of the point x and, also, $\mu(B(x, r)) > 0$ whenever $r > 0$. Furthermore, μ satisfies the doubling condition: there exists a positive constant A such that $\mu(B(x, 2r)) \le A\,\mu(B(x, r))$.

The purpose of this section is to give an approximation for $BMO(X)$ functions by the continuous functions with bounded supports as follows. We have seen an application of such approximation in Section 4. We also believe that there will be more applications of it.

Proposition 6.1. *For any $f \in BMO(X)$ there exists a sequence of bounded, continuous and boundedly supported $\{f_j\}_{j=1}^{\infty}$ such that*

$$\|f_j\|_{BMO} \le a_1\|f\|_{BMO},$$
$$|f_j(x)| \le a_2 \mathbb{M} f(x), \quad x \in X,$$
$$\lim_{j\to\infty} f_j(x) = f(x), \quad a.e.\ x \in X,$$

where a_1 and a_2 are independent on f, and $\mathbb{M}$ is the restricted centered Hardy-Littlewood maximal function of f:

$$\mathbb{M} f(x) = \sup_{0<r<1} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

Remark 6.2. *If* (X, d, μ) *is complete as a quasi-metric space, the closure of any ball is compact, because of its total boundedness, which can be seen by using Theorem (3.1) and the claim (3.4) in* [13]. *Hence, the functions* f_j *above are compactly supported.*

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References

[1] P. Auscher and E. Russ, *Hardy spaces and divergence operators on strongly Lipschitz domain of* $\mathbb{R}^n$, J. Funct. Anal. 201 (2003), 148–184.

[2] P. Auscher, E. Russ and P. Tchamitchian, *Hardy Sobolev spaces on strongly Lipschitz domain of* $\mathbb{R}^n$, J. Funct. Anal. 218 (2005), 54–109.

[3] Á. Bényi, W. Damián, K. Moen and R. H. Torres, *Compactness properties of commutators of bilinear fractional integrals*, Math. Z. 280 (2015), 569–582.

[4] Á. Bényi and R. H. Torres, *Compact bilinear operators and commutators*, Proc. Amer. Math. Soc. 141 (2013), 3609–3621.

[5] M. Cao and K. Yabuta, *Characterizations of* $VMO_{\Delta_N}(\mathbb{R}^n)$ *space*, Preprint.

[6] D.-C. Chang, *The dual of Hardy spaces on a bounded domain in* $\mathbb{R}^n$, Forum Math. 6 (1994), 65–81.

[7] D.-C. Chang, S. G. Krantz and E. M. Stein, H^p *theory on a smooth domain in* $\mathbb{R}^N$ *and elliptic boundary value problems*, J. Funct. Anal. 114 (1993), 286–347.

[8] J. Chen, Y. Chen and G. Hu, *Compactness for the commutators of singular integral operators with rough variable kernels*, J. Math. Anal. Appl. 431 (2015), 597–621.

[9] Y. Chen and Y. Ding, *Compactness of the commutators of parabolic singular integrals*, Sci. China Math. 53 (2010), 2633–2648.

[10] Y. Chen, Y. Ding and X. Wang, *Compactness of commutators of Riesz potential on Morrey spaces*, Potential Anal. 30 (2009), 301–313.

[11] A. Clop and V. Cruz, *Weighted estimates for Beltrami equations*, Ann. Acad. Sci. Fenn. Math. 38 (2013), 91–113.

[12] R. Coifman, P. L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. 72 (1993), 247–286.

[13] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[14] R. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. Math. 103 (1976), 611–635.

[15] D. G. Deng, X. T. Duong, A. Sikora and L. X. Yan, *Comparison of the classical BMO with the BMO spaces associated with operators and applications*, Rev. Mat. Iberoam. 24 (2008), 267–296.

[16] D. G. Deng, X. T. Duong, L. Song, C. Tan and L. Yan, *Functions of vanishing mean oscillation associated with operators and applications*, Michigan Math. J. 56 (2008), 529–550.

[17] X. T. Duong, I. Holmes, J. Li, B. D. Wick and D. Yang, *Two weight commutators in the Dirichlet and Neumann Laplacian settings*, J. Funct. Anal. 276 (2019), 1007–1060.

[18] X. Duong, J. Li, S. Mao, H. Wu and D. Yang, *Compactness of Riesz transform commutator associated with Bessel operators*, J. Anal. Math. 135 (2018), 639–673.

[19] X. T. Duong and L. X. Yan, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. 58 (2005), 1375–1420.

[20] X. T. Duong and L. X. Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. 18 (2005), 943–973.

[21] N. Dunford and J. Schwartz, *Linear operators. I*, Interscience, New York and London, 1964.

[22] C. Fefferman, and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[23] I. Holmes, R. Rahm and S. Spencer, *Commutators with fractional integral operators*, Studia Math. 233 (2016), 279–291.

[24] T. Iwaniec, *Nonlinear commutators and Jacobians*, J. Fourier Anal. Appl. 3 (2007), 775–796.

[25] T. Iwaniec and C. Sbordone, *Riesz transform and elliptic PDEs with VMO coefficients*, J. Anal. Math. 74 (1998), 183–212.

[26] P. W. Jones and J-L. Journé, *On weak convergence in $H^1(\mathbb{R}^n)$*, Proc. Amer.Math. Soc. 120 (1994), 137–138.

[27] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415–426.

[28] J. Li and B. D. Wick, *Characterizations of $H^1_{\Delta_N}(\mathbb{R}^n)$ and $BMO_{\Delta_N}(\mathbb{R}^n)$ via weak factorizations and commutators*, J. Funct. Anal. 272 (2017), 5384–5416.

[29] R. A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. Math. 33 (1979), 257–270.

[30] D. Palagachev and L. Softova, *Singular integral operators, Morrey spaces and fine regularity of solutions to PDEs*, Potential Anal. 20 (2004), 237–263.

[31] M. Reed and B. Simon, *Methods of modern mathematical physics I: functional analysis*, Academic Press, New York 1980.

[32] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. 207 (1975), 391–405.

[33] W. A. Strauss, *Partial differential equation: An introduction*, John Wiley Sons, Inc., New York, 2008.

[34] A. Uchiyama, *On the compactness of operators of the Hankel type*, Tohoku Math. J. 30 (1978), 163–171.

[35] S. L. Wang, *Compactness of commutators of fractional integrals* (Chinese), an English summary appears in Chinese Ann. Math. Ser. B 8 (1987), no. 4, 493, Chinese Ann. Math. Ser. A 8 (1987), 475–482.

[36] H. Wu and D. Yang, *Characterizations of weighted compactness of commutators via $CMO(\mathbb{R}^n)$*, Proc. Amer. Math. Soc. 146 (2018), 4239–4254.

[37] K. Yabuta, *Singular Integrals* (in Japanese), Iwanami, 2010.

[38] K. Yosida, *Functional Analysis*, Springer, Berlin (1995).

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