CHARACTERIZATION OF SOME FUNCTION SPACES BY SQUARE FUNCTIONS

SHUICHI SATO KANAZAWA UNIVERSITY

1. Introduction

This is a survey paper. I would like to state some recent results in harmonic analysis related to characterization of function spaces by square functions. The results include the following.

(1) Characterization of L^p spaces, $1 < p < \infty$, by Littlewood-Paley functions;
(2) Characterization of Sobolev spaces by Littlewood-Paley functions;
(3) Characterization of H^1 Sobolev spaces by square functions of Lusin area integral type;
(4) Characterization of Hardy spaces H^p on homogeneous groups by Littlewood-Paley functions, where $0 < p \le 1$.

2. Mapping properties of Littlewood-Paley operators on L^p spaces

Let ψ be a function in $L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \psi(x)\, dx = 0. \tag{2.1}$$

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2}, \tag{2.2}$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. The following result is well-known.

Theorem A. *Let $\psi \in L^1(\mathbb{R}^n)$ be as in (2.1). We assume that*

$$|\psi(x)| \le C(1 + |x|)^{-n-\epsilon}, \tag{2.3}$$

$$\int_{\mathbb{R}^n} |\psi(x - y) - \psi(x)|\, dx \le C|y|^\epsilon \quad \textit{for all } y \in \mathbb{R}^n, \tag{2.4}$$

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with some positive constants C, ϵ. Then g_ψ in (2.2) is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$:

$$\|g_\psi(f)\|_p \le C_p \|f\|_p, \tag{2.5}$$

where

$$\|f\|_p = \|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.$$

This is a result of Benedek, Calderón and Panzone [2].

Let

$$m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t}.$$

Then m is a homogeneous function of degree 0. Here the Fourier transform is defined as

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i \langle x, \xi \rangle} \, dx, \quad \langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k.$$

By the Plancherel theorem, one can see that g_ψ is bounded on $L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty(\mathbb{R}^n)$.

Let

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}$$

be the Poisson kernel on the upper half space $\mathbb{R}^n \times (0, \infty)$ (see [40]) and $Q(x) = [(\partial/\partial t) P_t(x)]_{t=1}$. Then, we can see that the function Q satisfies the conditions (2.1), (2.3) and (2.4). Thus by Theorem A g_Q is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

Let $H(x) = \text{sgn}(x)\chi_{[-1,1]}(x) = \chi_{[0,1]}(x) - \chi_{[-1,0]}(x)$ on $\mathbb{R}$ (the Haar function), where χ_E denotes the characteristic function of a set E and $\text{sgn}(x)$ the signum function. Then $g_H(f)$ is the Marcinkiewicz integral

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $F(x) = \int_0^x f(y) \, dy$. We can easily see that Theorem A also implies that g_H is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

We recall a theorem of Hörmander [15] to see results about the reverse inequality of (2.5). Let $m \in L^\infty(\mathbb{R}^n)$ and define

$$T_m(f)(x) = \int_{\mathbb{R}^n} m(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi. \tag{2.6}$$

We say that m is a Fourier multiplier for L^p and write $m \in M^p$ if there exists a constant $C > 0$ such that

$$\|T_m(f)\|_p \le C \|f\|_p$$

for all $f \in L^2 \cap L^p$. Then Hörmander's result in [15] can be stated as follows (see [5] for relevant results).

Theorem B. *Let m be a bounded function on $\mathbb{R}^n$. Suppose that m is homogeneous of degree 0 and that $m \in M^p$ for all $p \in (1, \infty)$. Suppose further that m is continuous and does not vanish on $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Then, m^{-1} belongs to M^p for every $p \in (1, \infty)$.*

The idea of the proof comes from a Banach algebra technique related to Wiener-Lévy theorem on absolutely convergent Fourier series. Applying Theorem B, we can deduce the following (see [15, Theorem 3.8]).

Theorem C. *Suppose that g_ψ is bounded on L^p for every $p \in (1, \infty)$. Let $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \, dt/t$. Suppose that m is continuous and strictly positive on S^{n-1}. Then we have*

$$\|f\|_p \le c_p \|g_\psi(f)\|_p,$$

and hence $\|f\|_p \simeq \|g_\psi(f)\|_p$, $f \in L^p$, for all $p \in (1, \infty)$, where $\|f\|_p \simeq \|g_\psi(f)\|_p$ means that

$$c_1 \|f\|_p \le \|g_\psi(f)\|_p \le c_2 \|f\|_p$$

with positive constants c_1, c_2 independent of f.

We also consider a discrete parameter version of g_ψ:

$$\Delta_\psi(f)(x) = \left(\sum_{k=-\infty}^{\infty} |f * \psi_{2^k}(x)|^2 \right)^{1/2}. \tag{2.7}$$

We recall the non-degeneracy conditions

$$\sup_{t>0} |\hat{\psi}(t\xi)| > 0 \quad \text{for all } \xi \neq 0; \tag{2.8}$$

$$\sup_{k \in \mathbb{Z}} |\hat{\psi}(2^k\xi)| > 0 \quad \text{for all } \xi \neq 0, \tag{2.9}$$

where $\mathbb{Z}$ denotes the set of integers. Obviously, (2.9) implies (2.8).

We recall the weight class A_p of Muckenhoupt. A weight w belongs to A_p, $1 < p < \infty$, if

$$\sup_B \left(\fint_B w(x) \, dx \right) \left(\fint_B w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty,$$

where

$$\fint_B f(y) \, dy = \frac{1}{|B|} \int_B f(y) \, dy$$

and the supremum is taken over all balls B in $\mathbb{R}^n$ (see [13]).

The weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ with a weight w is defined to be the class of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p} < \infty.$$

Let

$$B_\epsilon(\psi) = \int_{|x|>1} |\psi(x)|\, |x|^\epsilon\, dx,$$

$$D_u(\psi) = \left(\int_{|x|<1} |\psi(x)|^u\, dx\right)^{1/u}.$$

Then the following two theorems are known (see [29]).

Theorem 2.1. *Suppose that*

(1) $B_\epsilon(\psi) < \infty$ *for some* $\epsilon > 0$;
(2) $D_u(\psi) < \infty$ *for some* $u > 1$;
(3) $H_\psi \in L^1(\mathbb{R}^n)$, *where* $H_\psi(x) = \sup_{|y|\ge|x|} |\psi(y)|$;
(4) *the non-degeneracy condition* (2.8) *holds.*

Then $\|f\|_{p,w} \simeq \|g_\psi(f)\|_{p,w}$, $f \in L^p_w$, *for all* $p \in (1,\infty)$ *and* $w \in A_p$.

Theorem 2.2. *We assume that*

(1) $B_\epsilon(\psi) < \infty$ *for some* $\epsilon > 0$;
(2) $|\hat{\psi}(\xi)| \le C|\xi|^{-\delta}$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$ *with some* $\delta > 0$;
(3) $H_\psi \in L^1(\mathbb{R}^n)$;
(4) *the non-degeneracy condition* (2.9) *holds.*

Then $\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}$, $f \in L^p_w$, *for all* $p \in (1,\infty)$ *and* $w \in A_p$.

The inequality $\|g_\psi(f)\|_{p,w} \le c\|f\|_{p,w}$ in Theorem 2.1 was established in [21] without the assumption (4). We easily see that Theorem A follows from Theorem 2.1. Also, see [34] for related results with non-isotropic dilations.

3. Characterization of the weighted Sobolev spaces by Littlewood-Paley functions of Marcinkiewicz type

Recall the function of Marcinkiewicz:

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$F(x) = \int_0^x f(y)\, dy.$$

J. Marcinkiewicz [17] introduced this square function in 1938 in the setting of periodic functions on the torus. Zygmund [45] gave proofs of results conjectured in [17]. The non-periodic analogue was established by Waterman [44].

If $2/3 < p < \infty$, then it is known that

$$\|\mu(f)\|_p \simeq \|f\|_{H^p}, \quad f \in \mathcal{S}_0(\mathbb{R}),$$

where H^p denotes the Hardy space and $\mathcal{S}_0(\mathbb{R}^n)$ is the subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of functions f with $\widehat{f}$ vanishing outside a compact set not containing the origin (see [22]), where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of rapidly decreasing smooth functions.

The equivalence $\|\mu(f)\|_p \simeq \|f\|_{H^p}$ can be rephrased as

$$\|\nu(f)\|_p \simeq \|f'\|_{H^p}, \qquad f \in \mathcal{S}_0(\mathbb{R}),$$

where

$$\nu(f)(x) = \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^2 \frac{dt}{t^3}\right)^{1/2}. \tag{3.1}$$

This may be used to characterize Sobolev spaces.

We give the definition of the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$. Let $1 < p < \infty$, $\alpha > 0$. We say that $f \in W^{\alpha,p}(\mathbb{R}^n)$ if $f \in L^p(\mathbb{R}^n)$ and $f = J_\alpha(g) = K_\alpha * g$ for some $g \in L^p(\mathbb{R}^n)$, where K_α denotes the Bessel potential whose Fourier transform is given by

$$\widehat{K}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}$$

(see [39, Chap. V]). The norm is defined to be

$$\|f\|_{p,\alpha} = \|g\|_{L^p} \quad \text{with } f = J_\alpha(g).$$

Let $n \geq 2$. Let $0 < \alpha < 2$. R. Alabern, J. Mateu and J. Verdera [1] (2012) considered

$$V_\alpha(f)(x) = \left(\int_0^\infty \left| f(x) - \fint_{B(x,t)} f(y)\, dy \right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2},$$

where $B(x,t)$ is a ball in $\mathbb{R}^n$ having center x and radius t. The article [1] proved the following.

Theorem D. *Let $1 < p < \infty$. Then, the two statements in the following are equivalent:*

(1) f belongs to $W^{1,p}(\mathbb{R}^n)$,
(2) $f \in L^p(\mathbb{R}^n)$ and $V_1(f) \in L^p(\mathbb{R}^n)$.

Furthermore,

$$\|f\|_{p,1} \simeq \|f\|_p + \|V_1(f)\|_p.$$

Since the expression $\fint_B f$ makes sense in general metric measure spaces, this result may be used to define Sobolev spaces analogous to $W^{1,p}(\mathbb{R}^n)$ in metric measure spaces.

Let $1 < p < \infty$, $\alpha > 0$ and $w \in A_p$. Since it is known that $|J_\alpha(g)| \leq CM(g)$, where M denotes the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{t>0} \fint_{B(x,t)} |f(y)|\, dy,$$

we have $J_\alpha(g) \in L^p_w$ if $g \in L^p_w$. The weighted Sobolev space $W^{\alpha,p}_w(\mathbb{R}^n)$ is defined as the collection of all the functions $f \in L^p_w(\mathbb{R}^n)$ for which we have $f = J_\alpha(g)$ for some $g \in L^p_w(\mathbb{R}^n)$. Since such g is uniquely determined, the norm is defined to be $\|f\|_{p,\alpha,w} = \|g\|_{p,w}$.

We can apply Theorems 2.1, 2.2 in characterizing the weighted Sobolev spaces $W^{\alpha,p}_w(\mathbb{R}^n)$ by square functions related to the Marcinkiewicz function including $V_\alpha(f)$ and its discrete analogue

$$\left(\sum_{k=-\infty}^{\infty} \left| f(x) - \fint_{B(x,2^k)} f(y)\, dy \right|^2 2^{-2k\alpha} \right)^{1/2}, \quad \alpha > 0.$$

For $\alpha > 0$, we define function spaces $\mathcal{M}^\alpha(\mathbb{R}^n)$. If $0 < \alpha < 1$, $\mathcal{M}^\alpha(\mathbb{R}^n)$ is the collection of functions Φ which are compactly supported, bounded on $\mathbb{R}^n$ and satisfy $\int_{\mathbb{R}^n} \Phi(x)\, dx = 1$. When $\alpha \geq 1$, we say $\Phi \in \mathcal{M}^\alpha(\mathbb{R}^n)$ if Φ further satisfies that

(3.2)
$$\int_{\mathbb{R}^n} \Phi(x) x^\gamma\, dx = 0, \quad x^\gamma = x_1^{\gamma_1} \dots x_n^{\gamma_n}, \quad \text{for all } \gamma \text{ with } 1 \leq |\gamma| \leq [\alpha],$$

where $[\alpha]$ denotes the largest integer not exceeding α and $\gamma = (\gamma_1, \dots, \gamma_n)$, $\gamma_j \in \mathbb{Z}$, $\gamma_j \geq 0$, is a multi-index and $|\gamma| = \gamma_1 + \dots + \gamma_n$. Let $\Phi \in \mathcal{M}^\alpha(\mathbb{R}^n)$ and define

$$G_\alpha(f)(x) = \left(\int_0^\infty |f(x) - \Phi_t * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0, \tag{3.3}$$

$$E_\alpha(f)(x) = \left(\sum_{k=-\infty}^{\infty} |f(x) - \Phi_{2^k} * f(x)|^2 2^{-2k\alpha} \right)^{1/2}, \quad \alpha > 0. \tag{3.4}$$

We note that

$$F = \frac{1}{|B(0,1)|} \chi_{B(0,1)} \in \mathcal{M}^\alpha, \quad 0 < \alpha < 2,$$

and that if $\Phi = F$, then $G_\alpha(f) = V_\alpha(f)$. The following results are known (see [29]).

Theorem 3.1. *Suppose that $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < n$. Let G_α be as in* (3.3)*. Then $f \in W_w^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L_w^p$ and $G_\alpha(f) \in L_w^p$; also,*

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|G_\alpha(f)\|_{p,w}.$$

Theorem 3.2. *Let $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < n$. Let E_α be as in* (3.4)*. Then $f \in W_w^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L_w^p$ and $E_\alpha(f) \in L_w^p$; furthermore,*

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|E_\alpha(f)\|_{p,w}.$$

We can find some relevant results in [14, 26]. Also, a characterization of $W_w^{2,p}(\mathbb{R}^n)$ by a square function with $\Phi \in \mathcal{M}^1$ is given in [28].

We write

$$f(x+t) + f(x-t) - 2f(x) = 2\int_{S^0} (f(x-ty) - f(x))\, d\sigma(y),$$

where $S^0 = \{-1, 1\}$ and σ is a measure on S^0 such that $\sigma(\{-1\}) = 1/2$, $\sigma(\{1\}) = 1/2$. According to this observation we generalize ν in (3.1) to higher dimensions. Let $n \geq 2$ and

$$D^\alpha(f)(x) = \left(\int_0^\infty \left| t^{-\alpha} \int_{S^{n-1}} (f(x-ty) - f(x))\, d\sigma(y)\right|^2 \frac{dt}{t}\right)^{1/2},$$

where $d\sigma$ is the Lebesgue uniform measure on S^{n-1} normalized as $\int_{S^{n-1}} d\sigma = 1$.

Let $0 < \alpha < 2$ and $S_\alpha(f) = D^\alpha(I_\alpha f)$:

$$S_\alpha(f)(x) = \left(\int_0^\infty \left| I_\alpha(f)(x) - \int_{S^{n-1}} I_\alpha(f)(x-ty)\, d\sigma(y)\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2},$$

where I_α is the Riesz potential operator defined by

$$\widehat{I_\alpha(f)}(\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi),$$

$$I_\alpha(f)(x) = L_\alpha * f(x), \quad L_\alpha(x) = C_\alpha |x|^{\alpha-n}.$$

Then the following result is known.

Theorem E. *Let $1 < p < \infty$, $n \geq 2$. Then for $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\|S_1(f)\|_p \simeq \|f\|_p.$$

This result of P. Hajłasz-Z. Liu [14] may be used to characterize the Sobolev space $W^{1,p}(\mathbb{R}^n)$.

We can give an alternative proof of Theorem E. We recall the Bochner-Riesz mean of order β on $\mathbb{R}^n$ defined by

$$S_R^\beta(f)(x) = \int_{|\xi|<R} \widehat{f}(\xi)(1 - R^{-2}|\xi|^2)^\beta \, e^{2\pi i\langle x,\xi\rangle} \, d\xi.$$

Define a Littlewood-Paley operator σ_β, $\mathrm{Re}(\beta) > 0$, by

$$\begin{aligned} \sigma_\beta(f)(x) &= \left(\int_0^\infty \left| R(\partial/\partial_R) S_R^\beta(f)(x) \right|^2 \frac{dR}{R} \right)^{1/2} \\ &= \left(\int_0^\infty \left| -2\beta \left(S_R^\beta(f)(x) - S_R^{\beta-1}(f)(x) \right) \right|^2 \frac{dR}{R} \right)^{1/2}. \end{aligned}$$

Then we have a pointwise equivalence of $\sigma_\beta(f)$ and $S_\alpha(f)$.

Theorem 3.3. *Let* $0 < \alpha < 2$, $\beta = \alpha + n/2$. *Then*

$$\sigma_\beta(f)(x) \approx D^\alpha(I_\alpha f)(x), \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

When $0 < \alpha < 1$, this is due to Kaneko-Sunouchi [16] in 1985. The range of α is extended to $(0, 2)$ by [30].

We can apply Theorem 3.3 with $\alpha = 1$ and a property of σ_β with $\beta = 1 + n/2$ to give an alternative proof of Theorem E (see [30]).

4. Characterization of H^1 Sobolev spaces by square functions of Lusin area integral type

We define H^1 Sobolev space $W^\alpha_{H^1}(\mathbb{R}^n)$, where H^1 is the Hardy space. We say that $f \in W^\alpha_{H^1}(\mathbb{R}^n)$ if $f \in H^1(\mathbb{R}^n)$ and $f = J_\alpha(h) = K_\alpha * h$ for some $h \in H^1(\mathbb{R}^n)$. Define

$$\|f\|_{W^\alpha_{H^1}} = \|h\|_{H^1}, \quad f = J_\alpha(h),$$

where $\|\cdot\|_{H^1}$ denotes the norm in H^1 (see [11]).

Let

$$\psi^{(\alpha)}(x) = L_\alpha(x) - L_\alpha * \Phi(x), \tag{4.1}$$

where $\Phi \in \mathcal{M}^\alpha$, $0 < \alpha < n$. We consider a Lusin area integral of Marcinkiewicz type:

$$\begin{aligned} S_{\psi^{(\alpha)}}(f)(x) &= \left(\int_0^\infty \int_{B(0,1)} |\psi_t^{(\alpha)} * f(x - tz)|^2 \, dz \, \frac{dt}{t} \right)^{1/2} \\ &= \left(\int_0^\infty \int_{B(x,t)} |\psi_t^{(\alpha)} * f(z)|^2 \, dz \, t^{-n} \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Also, define

$$U_\alpha(f)(x) = \left(\int_0^\infty \int_{B(0,1)} |f(x-tz) - \Phi_t * f(x-tz)|^2 \, dz \, t^{-2\alpha} \frac{dt}{t}\right)^{1/2}$$
$$= \left(\int_0^\infty \int_{B(x,t)} |f(z) - \Phi_t * f(z)|^2 \, dz \, t^{-2\alpha-n} \frac{dt}{t}\right)^{1/2}.$$

Then

$$U_\alpha(f) = S_{\psi^{(\alpha)}}(I_{-\alpha} f), \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

The H^1 Sobolev space can be characterized by U_α.

Theorem 4.1. *Suppose that $n/2 < \alpha < n$, $\Phi \in \mathcal{M}^\alpha$ and*

$$|\widehat{\Phi}(\xi)| \le C(1+|\xi|)^{-\beta}, \quad \alpha + \beta > n.$$

Then the following two statements are equivalent:

(1) $f \in W^\alpha_{H^1}(\mathbb{R}^n)$,
(2) $f \in H^1(\mathbb{R}^n)$ *and* $U_\alpha(f) \in L^1(\mathbb{R}^n)$.

Further, we have $\|f\|_{W^\alpha_{H^1}} \simeq \|f\|_{H^1} + \|U_\alpha(f)\|_1$.

In Theorem 4.1, the hypothesis $\alpha > n/2$ is optimal in the sense that if $0 < \alpha < n/2$, the estimate

$$\|U_\alpha(f)\|_1 \le C\|f\|_{W^\alpha_{H^1}}$$

does not hold.

The weighted H^1 Sobolev space $W^\alpha_{H^1_w}(\mathbb{R}^n)$ is defined as follows. Let $w \in A_1$ and set

$$H^1_w = \{f \in L^1_w : \quad f^* \in L^1_w\}, \quad \|f\|_{H^1_w} = \|f^*\|_{1,w},$$

where $f^*(x) = \sup_{t>0} |\varphi_t * f(x)|$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int \varphi \, dx = 1$. The space $W^\alpha_{H^1_w}(\mathbb{R}^n)$ is the family of functions $f \in H^1_w(\mathbb{R}^n)$ such that $f = J_\alpha(h)$ for some $h \in H^1_w(\mathbb{R}^n)$. We define $\|f\|_{W^\alpha_{H^1_w}} = \|h\|_{H^1_w}$.

We confine our attention to the one dimensional case and we have the following result.

Theorem 4.2. *Suppose that $w \in A_1$. Then the following two statements are equivalent:*

(1) $f \in W^1_{H^1_w}(\mathbb{R})$,
(2) $f \in H^1_w(\mathbb{R})$ *and* $\nu(f) \in L^1_w(\mathbb{R})$, *where ν is as in* (3.1).

Furthermore, $\|f\|_{W^1_{H^1_w}} \simeq \|f\|_{H^1_w} + \|\nu(f)\|_{1,w}$.

Theorems 4.1 and 4.2 can be found in [33].

Let

$$\mathcal{D}_\alpha(f)(x) = \left(\int_{\mathbb{R}^n} |I_\alpha(f)(x-y) - I_\alpha(f)(x)|^2 \frac{dy}{|y|^{n+2\alpha}}\right)^{1/2}.$$

It has been observed that the square function $S_{\psi^{(\alpha)}}(f)$ is closely related to $\mathcal{D}_\alpha(f)$ (see [37] and also [6]). We recall the following results for $\mathcal{D}_\alpha$.

Theorem F. *Let* $0 < \alpha < 1$ *and* $p_0 = 2n/(n+2\alpha)$. *Suppose that* $p_0 > 1$. *Then*

(1) $\mathcal{D}_\alpha$ *is bounded on* $L^p(\mathbb{R}^n)$ *if* $p_0 < p < \infty$ *(E.M. Stein* [38]*);*
(2) $\mathcal{D}_\alpha$ *is of weak type* (p_0, p_0) *(C. Fefferman* [10]*).*

In [35], this is generalized by considering analogues of $\mathcal{D}_\alpha$ with fractional integrals of mixed homogeneity in place of the Riesz potentials of Euclidean structure.

5. Sketch of proof of Theorem 3.1

Let $0 < \alpha < n$, $\Phi \in \mathcal{M}^\alpha$ and define

$$T_\alpha(f)(x) = \left(\int_0^\infty |I_\alpha(f)(x) - \Phi_t * I_\alpha(f)(x)|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}.$$

Since $\psi^{(\alpha)}(x) = L_\alpha(x) - \Phi * L_\alpha(x)$ with $\Phi \in \mathcal{M}^\alpha$, it is easy to see that $|\psi^{(\alpha)}(x)| \le C|x|^{-n+\alpha}$ for $|x| \le 1$ and $|\psi^{(\alpha)}(x)| \le C|x|^{-n+\alpha-[\alpha]-1}$ for $|x| \ge 1$. By these estimates, the conditions (1), (2) and (3) of Theorem 2.1 hold for $\psi^{(\alpha)}$. Also,

$$\widehat{\psi^{(\alpha)}}(\xi) = (2\pi|\xi|)^{-\alpha}(1 - \hat{\Phi}(\xi))$$

satisfies the non-degeneracy condition (4) of Theorem 2.1, since $\hat{\Phi}(\xi) \to 0$ as $|\xi| \to \infty$ by the Riemann-Lebesgue lemma. Further, since

$$|\widehat{\psi^{(\alpha)}}(\xi)| \le C|\xi|^{-\alpha+[\alpha]+1},$$

we have $\widehat{\psi^{(\alpha)}}(0) = 0$. We see that $T_\alpha(f) = g_{\psi^{(\alpha)}}(f)$. Thus by Theorem 2.1

$$\|T_\alpha(f)\|_{p,w} = \|g_{\psi^{(\alpha)}}(f)\|_{p,w} \simeq \|f\|_{p,w}.$$

Using this and the observation

$$T_\alpha(I_{-\alpha}f) = G_\alpha(f), \quad f \in \mathcal{S}_0(\mathbb{R}^n),$$

we have

$$\|G_\alpha(f)\|_{p,w} \simeq \|I_{-\alpha}f\|_{p,w}.$$

We can derived Theorem 3.1 from this.

6. Sketch of proof of Theorem 4.1

Theorem 4.1 follows from the next result.

Theorem 6.1. *Let $\psi^{(\alpha)}$ be defined as in* (4.1) *with Φ as in Theorem* 4.1*. Then*

$$\|S_{\psi^{(\alpha)}}(f)\|_1 \simeq \|f\|_{H^1}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

We need the following Hörmander condition in proving Theorem 6.1.

Lemma 6.2. *Let $\psi^{(\alpha)}$ be as in Theorem* 6.1*. Then*

$$\int_{|x|>2|y|} \left[\iint_{B_0\times(0,\infty)} \left|\psi_t^{(\alpha)}(x-y-tz) - \psi_t^{(\alpha)}(x-tz)\right|^2 dz\,\frac{dt}{t}\right]^{1/2} dx \le C,$$

with a constant C independent of $y \in \mathbb{R}^n$, where $B_0 = B(0,1)$.

By Lemma 6.2 and a result of [13] for vector valued singular integrals we have $\|S_{\psi^{(\alpha)}}(f)\|_1 \le C\|f\|_{H^1}$.

The reverse inequality can be deduced from the following result.

Lemma 6.3. *If $f \in \mathcal{S}_0(\mathbb{R}^n)$ and $g \in \mathrm{BMO}(\mathbb{R}^n)$, then we have*

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \le C\|g\|_{\mathrm{BMO}} \int_{\mathbb{R}^n} S_{\psi^{(\alpha)}}(f)(x)\,dx.$$

From Lemma 6.3 and duality of H^1 and BMO we see that $\|f\|_{H^1} \le C\|S_{\psi^{(\alpha)}}(f)\|_1$.

The proof of Theorem 3.1 is based on the estimates $\|g_{\psi^{(\alpha)}}(f)\|_{p,w} \simeq \|f\|_{p,w}$. If $\|g_{\psi^{(\alpha)}}(f)\|_1 \simeq \|f\|_{H^1}$, then we would be able to characterize $W^\alpha_{H^1}$ by G_α. We do not know at present if the estimate $\|f\|_{H^1} \le C\|g_{\psi^{(\alpha)}}(f)\|_1$ holds or not.

7. Sketch of proof of Theorem 4.2

The proof of Theorem 4.2 is based on the following result.

Lemma 7.1. *Let $w \in A_1$. Then we have*

$$\|\mu(f)\|_{1,w} \simeq \|f\|_{H^1_w}, \quad f \in \mathcal{S}_0(\mathbb{R}).$$

Define

$$g_0(f)(x) = \left(\int_0^\infty |(\partial/\partial x)u(x,t)|^2 t\,dt\right)^{1/2},$$

where $u(x,t)$ denotes the Poisson integral of f: $u(x,t) = P_t * f(x)$, $\widehat{P}(\xi) = e^{-2\pi|\xi|}$. Let $\widehat{R}(\xi) = 2\pi i\xi e^{-2\pi|\xi|}$. Then $g_0(f) = g_R(f)$ and we have

$$\|f\|_{H^1_w} \le C\|g_0(f)\|_{1,w}, \quad f \in \mathcal{S}_0(\mathbb{R}). \tag{7.1}$$

This can be seen from the following (the unweighted case is in [11, 43]).

Remark 7.2. Let $\psi \in L^1(\mathbb{R}^n)$. We say $\psi \in \mathcal{B}$ if

(1) $\hat{\psi} \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
(2) $\sup_{t>0} |\hat{\psi}(t\xi)| > 0$ for all $\xi \neq 0$;
(3) $\psi \in C^1(\mathbb{R}^n)$, $\partial_k \psi \in L^1(\mathbb{R}^n)$, $1 \leq k \leq n$, where $\partial_k = \partial/\partial x_k$;
(4) $|\hat{\psi}(\xi)| \leq C|\xi|^\epsilon$ for some $\epsilon > 0$;
(5) $|\partial^\gamma \hat{\psi}(\xi)| \leq C_{\gamma,\tau}|\xi|^{-\tau}$ outside a neighborhood of the origin for all multi-indices γ and $\tau > 0$, where $\partial^\gamma = \partial_1^{\gamma_1}\partial_2^{\gamma_2}\dots\partial_n^{\gamma_n}$, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$.

Let $0 < p \leq 1$, $w \in A_1$ and $\psi \in \mathcal{B}$. Then we have

$$\|f\|_{H^p_w} \leq C_p \|g_\psi(f)\|_{p,w} \tag{7.2}$$

for $f \in \mathcal{S}_0(\mathbb{R}^n)$ with a positive constant C_p independent of f. This is proved in [33]. See also [31], [32] for related results.

By applying (7.2) on $\mathbb{R}^1$ with $\psi = R$ we have (7.1).
Also, it is known that the pointwise relation

$$g_0(f) \leq C\mu(f), \quad f \in \mathcal{S}_0(\mathbb{R}), \tag{7.3}$$

holds (see [22]). Combining (7.1) and (7.3), we have

$$\|f\|_{H^1_w} \leq C\|\mu(f)\|_{1,w}.$$

To get the reverse inequality, recall that $\mu(f) = g_H(f)$,

$$H(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x).$$

We can show that

$$\left(\int_0^\infty |H_t(x-y) - H_t(x)|^2 \frac{dt}{t}\right)^{1/2} \leq C\frac{|y|^{1/2}}{|x|^{3/2}} \quad \text{for } 2|y| < |x|.$$

Using this and a result for vector valued singular integrals, we can prove the reverse inequality:

$$\|\mu(f)\|_{1,w} \leq C\|f\|_{H^1_w}.$$

This can be also shown by applying the pointwise relation between g_3^* and μ (see [22]) and the $H^1_w - L^1_w$ boundedness of g_3^* with $w \in A_1$, which can be found in [18], where

$$g_\lambda^*(f)(x) = \left(\iint_{\mathbb{R}\times(0,\infty)} \left(\frac{t}{t+|x-y|}\right)^\lambda |\nabla u(y,t)|^2 \, dy \, dt\right)^{1/2}.$$

8. Characterization of Hardy spaces on homogeneous groups by Littlewood-Paley functions

Let $\mathbb{R}^n$ be the n dimensional Euclidean space as before. Here we assume that $n \geq 2$. We also consider $\mathbb{R}^n$ as a homogeneous group $\mathbb{H}$ equipped with multiplication given by polynomial mappings. We have a dilation family $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ of the form

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n), \quad x = (x_1, \dots, x_n),$$

where real numbers $a_1, \dots, a_n$ satisfy $1 = a_1 \leq a_2 \leq \cdots \leq a_n$. We assume that each A_t is an automorphism of the group structure (see [12], [42] and [19, Section 2 of Chapter 1]). The homogeneous nilpotent Lie group structure of $\mathbb{H}$ has the following properties:

(1) Lebesgue measure is a bi-invariant Haar measure;
(2) we have $(x_1, \dots, x_n)$ as the canonical coordinates;
(3) the group law obeys the Hausdorff-Campbell formula as a nilpotent Lie group;
(4) the identity is the origin 0 and $x^{-1} = -x$;
(5) $(\alpha x)(\beta x) = \alpha x + \beta x$ for $x \in \mathbb{H}$, $\alpha, \beta \in \mathbb{R}$;
(6) $A_t(xy) = (A_t x)(A_t y)$ for $x, y \in \mathbb{H}$, $t > 0$;
(7) if $z = xy$, then $z_k = P_k(x, y)$, where $P_1(x, y) = x_1 + y_1$ and $P_k(x, y) = x_k + y_k + R_k(x, y)$ for $k \geq 2$ with a polynomial $R_k(x, y)$ depending only on $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$, which can be written as

$$R_k(x, y) = \sum_{|I| \neq 0, |J| \neq 0, a(I)+a(J)=a_k} c_{I,J}^{(k)} x^I y^J.$$

Here, $I = (i_1, i_2, \dots, i_n) \in (\mathbb{N}_0)^n$ with $\mathbb{N}_0$ denoting the set of non-negative integers and

$$a(I) = a_1 i_1 + a_2 i_2 + \cdots + a_n i_n;$$

also, $J \in (\mathbb{N}_0)^n$.

We have a norm function $\rho(x)$ which is homogeneous of degree one with respect to the dilation A_t; so we have $\rho(A_t x) = t\rho(x)$ for $t > 0$ and $x \in \mathbb{H}$. We may assume the following:

(8) ρ is continuous on $\mathbb{R}^n$ and smooth in $\mathbb{H} \setminus \{0\}$;
(9) $\rho(x + y) \leq \rho(x) + \rho(y)$ and $\rho(xy) \leq c_0(\rho(x) + \rho(y))$ for some constant $c_0 \geq 1$ and $\rho(x^{-1}) = \rho(x)$;
(10) $\rho(x) \leq 1$ if and only if $|x| \leq 1$ and if $\Sigma = \{x \in \mathbb{H} : \rho(x) = 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, then $\Sigma = S^{n-1}$.

(11) there are positive constants c_j, α_k, β_k, $1 \le j \le 4$, $1 \le k \le 2$, such that

$$c_1|x|^{\alpha_1} \le \rho(x) \le c_2|x|^{\alpha_2} \quad \text{if } \rho(x) \ge 1,$$
$$c_3|x|^{\beta_1} \le \rho(x) \le c_4|x|^{\beta_2} \quad \text{if } \rho(x) \le 1.$$

We recall the Heisenberg group $\mathbb{H}_1$ as an example of a homogeneous group. Define the multiplication

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + (x_1y_2 - x_2y_1)/2),$$

$(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$. Then this defines a group law for the Heisenberg group $\mathbb{H}_1$ with the underlying manifold $\mathbb{R}^3$, where the dilation $A_t(x_1, x_2, x_3) = (tx_1, tx_2, t^2x_3)$ is an automorphism ($\{A_t\}$ satisfies (6)).

We define the Littlewood-Paley g function on $\mathbb{H}$ by

$$g_\varphi(f)(x) = \left(\int_0^\infty |f * \varphi_t(x)|^2 \frac{dt}{t}\right)^{1/2}, \tag{8.1}$$

where $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$ satisfying $\int_{\mathbb{H}} \varphi\, dx = 0$ and $\varphi_t(x) = t^{-\gamma}\varphi(A_t^{-1}x)$ with $\gamma = a_1 + \cdots + a_n$. Here $\mathcal{S}'$ denotes the space of tempered distributions and $\mathcal{S}$ the Schwartz space, which are the same as those in the Euclidean case (see [40]). The convolution $F * G$ on $\mathbb{H}$ is defined by

$$F * G(x) = \int_{\mathbb{H}} F(xy^{-1})G(y)\, dy = \int_{\mathbb{H}} F(y)G(y^{-1}x)\, dy.$$

See [9] and [7, 8, 24, 27, 42] for the study of Littlewood-Paley operators and singular integrals, respectively, on L^p spaces on homogeneous groups, $1 \le p < \infty$. Also, see [23, 34] and [25, Section 7] for results in harmonic analysis with non-isotropic dilations.

In this section we give a characterization of Hardy spaces H^p, $0 < p \le 1$, on $\mathbb{H}$ in terms of the Littlewood-Paley g functions. We first recall related results in the Euclidean case. Let $\varphi^{(\ell)}$, $\ell = 1, 2, \ldots, M$, be functions in $\mathcal{S}(\mathbb{R}^n)$ which satisfy the non-degeneracy condition

$$\inf_{\xi \in \mathbb{R}^n \setminus \{0\}} \sup_{t>0} \sum_{\ell=1}^{M} |\mathcal{F}(\varphi^{(\ell)})(t\xi)| > c \tag{8.2}$$

for some positive constant c, where $\mathcal{F}(\varphi^{(\ell)})$ denotes the Fourier transform. In [43] the following result for the Euclidean structure can be found.

Theorem G. *Let $0 < p \le 1$. Let $\varphi^{(\ell)} \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi^{(\ell)}\, dx = 0$, $\ell = 1, 2, \ldots, M$. Suppose that the condition* (8.2) *holds. Then*

$$c_p \|f\|_{H^p} \le \sum_{\ell=1}^{M} \|g_{\varphi^{(\ell)}}(f)\|_p \le C_p \|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n)$, where $g_{\varphi^{(\ell)}}(f)$ is defined similarly to (8.1) *with the Euclidean structure (see* (2.2)*).*

See [11] for the Hardy space $H^p(\mathbb{R}^n)$. Analogous results for L^p spaces, $1 < p < \infty$, can be found in [2], [15] and [29].

Let $e_j = (e_1^{(j)}, e_2^{(j)}, \ldots, e_n^{(j)})$, $1 \le j \le n$, $e_j^{(j)} = 1$ and $e_k^{(j)} = 0$ if $k \ne j$. Define

$$X_j f(x) = \left[\frac{d}{dt} f(x(te_j))\right]_{t=0},$$
$$Y_j f(x) = \left[\frac{d}{dt} f((te_j)x)\right]_{t=0}.$$

Then X_j and Y_j are called the left-invariant and right-invariant derivatives, respectively.

Let $I = (i_1, i_2, \ldots, i_n) \in (\mathbb{N}_0)^n$. Higher order differential operators X^I and Y^I are defined as

$$X^I = X_1^{i_1} X_2^{i_2} \ldots X_n^{i_n}, \quad Y^I = Y_1^{i_1} Y_2^{i_2} \ldots Y_n^{i_n}.$$

Then $|I|$ is called the order of X^I and Y^I and $a(I)$ the homogeneous degree for them.

Let

$$P(x) = \sum c_I x^I, \quad x^I = x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}, \quad I = (i_1, i_2, \ldots i_n), \tag{8.3}$$

be a polynomial on $\mathbb{R}^n$. We may also consider $P(x)$ as a polynomial on $\mathbb{H}$. The degree of the polynomial P is $\max\{|I| : c_I \ne 0\}$. Also, the homogeneous degree of P is defined to be $\max\{a(I) : c_I \ne 0\}$.

Let $\Delta = \{a(I) : I \in (\mathbb{N}_0)^n\}$. We denote by $\mathcal{P}_a$ the space of all polynomials P in (8.3) with $a(I) \le a$ for all I.

Let

$$\|\Phi\|_{(N)} = \sup_{|I| \le N, x \in \mathbb{H}} (1 + \rho(x))^{(N+1)(\gamma+1)} |Y^I \Phi(x)|$$

(see [12, p. 35]). Define

$$B_N = \{\Phi \in \mathcal{S} : \|\Phi\|_{(N)} \le 1\}.$$

Let

$$M_{(N)}(f)(x) = \sup\{\sup_{t>0} |f * \Phi_t(x)| : \Phi \in B_N\}.$$

We define the Hardy space H^p on $\mathbb{H}$ for $p \in (0,1]$ as

$$H^p = \{f \in \mathcal{S}' : \|f\|_{H^p} = \|M_{(N_p)}(f)\|_p < \infty\},$$

with sufficiently large N_p. The number

$$\widetilde{N}_p = \min\left\{N \in \mathbb{N}_0 : N \geq \min\{a \in \Delta : a > \gamma(p^{-1} - 1)\}\right\}$$

can be taken as N_p. If $\Delta = \mathbb{N}_0$, then $\widetilde{N}_p = [\gamma(p^{-1} - 1)] + 1$ (see [12, Chap. 2]). See [11] for the definition of H^p spaces in the case of Euclidean structure.

To generalize Theorem G to the case of homogeneous groups, we recall the fact that the condition (8.2) implies the existence of functions $\eta^{(1)}, \dots, \eta^{(M)} \in \mathcal{S}(\mathbb{R}^n)$ such that each $\operatorname{supp}\mathcal{F}(\eta^{(\ell)})$ is a compact set not containing the origin and such that

$$\sum_{\ell=1}^{M} \int_0^\infty \varphi_t^{(\ell)} * \eta_t^{(\ell)} \frac{dt}{t} = \delta \quad \text{in } \mathcal{S}', \tag{8.4}$$

where δ denotes the Dirac delta function.

We employ an analogue of (8.4) as a non-degeneracy condition for $\varphi^{(1)}, \dots, \varphi^{(M)}$ on $\mathbb{H}$ and we can prove the following result analogous to Theorem G.

Theorem 8.1. *Let $0 < p \leq 1$. There exists $d \in \Delta$ having the following property. Suppose that $\{\varphi^{(\ell)} \in \mathcal{S} : 1 \leq \ell \leq M\}$ is a family of functions such that (1) and (2) below hold:*

(1)

$$\int \varphi^{(\ell)}\, dx = 0, \quad \textit{for } \ell = 1, 2, \dots, M;$$

(2) *there exist functions $\eta^{(\ell)} \in \mathcal{S}$, $1 \leq \ell \leq M$, satisfying that*

$$\sum_{\ell=1}^{M} \int_0^\infty \varphi_t^{(\ell)} * \eta_t^{(\ell)} \frac{dt}{t} = \lim_{\substack{\epsilon \to 0, \\ B \to \infty}} \sum_{\ell=1}^{M} \int_\epsilon^B \varphi_t^{(\ell)} * \eta_t^{(\ell)} \frac{dt}{t} = \delta \quad \textit{in } \mathcal{S}'$$

and that

$$\int \eta^{(\ell)} P\, dx = 0 \quad \textit{for all } P \in \mathcal{P}_d, \quad 1 \leq \ell \leq M.$$

Then

$$c_p\|f\|_{H^p} \leq \sum_{\ell=1}^{M} \|g_{\varphi^{(\ell)}}(f)\|_p \leq C_p\|f\|_{H^p} \quad \textit{for } f \in H^p \tag{8.5}$$

with positive constants c_p and C_p independent of f, where $g_{\varphi^{(\ell)}}$ is as in (8.1) and H^p is the Hardy space on $\mathbb{H}$.

Consider a stratified group $\mathbb{H}$ with a natural dilation and let h be the heat kernel on $\mathbb{H}$ (see [12]). We define $\phi^{(j)} \in \mathcal{S}$, $j = 1, 2, \ldots$, by

$$\phi^{(j)}(x) = \left[(\partial/\partial t)^j h(x,t)\right]_{t=1} = (-L)^j h(x,1),$$

where L is the sub-Laplacian of $\mathbb{H}$. We have the following result as an application of Theorem 8.1.

Corollary 8.2. *Let $f \in H^p$, $0 < p \leq 1$. Then we have*

$$c_p\|f\|_{H^p} \leq \|g_{\phi^{(j)}}(f)\|_p \leq C_p\|f\|_{H^p}$$

for any $j \geq 1$, with some positive constants c_p, C_p independent of f.

This is almost Theorem 7.28 of [12]; in [12] the first inequality is proved under the condition that $f \in \mathcal{S}'$ vanishes weakly at infinity and $g_{\phi^{(j)}}(f) \in L^p$.

We recall the Lusin area integral on the homogeneous group $\mathbb{H}$ defined by

$$S_\varphi(f)(x) = \left(\int_0^\infty \int_{\rho(x^{-1}y)<t} |f * \varphi_t(y)|^2 t^{-\gamma-1}\, dy\, dt\right)^{1/2}.$$

Then, results analogous to Theorem 8.1 were proved for $S_\varphi(f)$ in [12] (see [12, Theorem 7.11 and Corollary 7.22]), but the characterization by the Littlewood-Paley function was shown only for special Littlewood-Paley functions $g_{\phi^{(j)}}$ coming from the heat kernel.

As in the case of the Euclidean structure of Theorem G, the first inequality of (8.5) is more difficult for us to prove than the second one; the second inequality may be shown by applying a theory of vector-valued singular integrals.

In [31] an alternative proof of the first inequality of the conclusion of Theorem G is given by applying the Peetre maximal function $F^{**}_{N,R}$ of [20] defined by

$$F^{**}_{N,R}(x) = \sup_{y\in\mathbb{R}^n} \frac{|F(x-y)|}{(1+R|y|)^N}.$$

Here we would like to give some comments on the application of the Peetre maximal function in proving the first inequality of Theorem G. When $\mathcal{F}(\varphi^{(\ell)})$ each has a compact support not containing the origin, then we can prove that inequality much more easily by applying the Peetre maximal function. A reason for this is the availability of the trick similar to the one in the proof of Bernstein's inequality for the estimates of the derivatives of trigonometric polynomials.

The proof of [31] is expected to extend to some other situations. Indeed, it has been applied to characterize parabolic Hardy spaces of

Calderón-Torchinsky [3, 4] by Littlewood-Paley functions (see [32]). See also [33] for related results on weighted Hardy spaces.

The methods of [31] can be also applied to characterize Hardy spaces on the homogeneous groups by certain Littlewood-Paley functions (Theorem 8.1). In proving the theorem we apply the Peetre maximal function on $\mathbb{H}$ defined by

$$F^{**}_{N,R}(x) = \sup_{y\in\mathbb{H}} \frac{|F(xy^{-1})|}{(1+R\rho(y))^N} = \sup_{y\in\mathbb{H}} \frac{|F(y)|}{(1+R\rho(y^{-1}x))^N}. \tag{8.6}$$

and use the following lemma.

Lemma 8.3. *Let $N=\gamma/r$, $r>0$, $0<\delta\le 1$. Let $f,\varphi\in\mathcal{S}$. Then we have*

$$(f*\varphi_t)^{**}_{N,t^{-1}}(x) \le C_r\delta^{-N}M(|f*\varphi_t|^r)^{1/r}(x) + C_r\delta\sum_{j=1}^{n}(f*(X_j\varphi)_t)^{**}_{N,t^{-1}}(x)$$

for all $t>0$, where M denotes the Hardy-Littlewood maximal operator on $\mathbb{H}$ defined by

$$M(f)(x) = \sup_{t>0} t^{-\gamma}\int_{\rho(y^{-1}x)<t} |f(y)|\,dy.$$

See [36] for the details.

References

[1] R. Alabern, J. Mateu and J. Verdera, *A new characterization of Sobolev spaces on $\mathbb{R}^n$*, Math. Ann. **354** (2012), 589–626.

[2] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U. S. A. **48** (1962), 356–365.

[3] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1-64.

[4] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution. II*, Advances in Math. **24** (1977), 101-171.

[5] A. P. Calderon and A. Zygmund, *Algebras of certain singular operators*, Amer. J. Math. **78** (1956), 310–320.

[6] F. Dai, J. Liu, D. Yang and W. Yuan, *Littlewood-Paley characterizations of fractional Sobolev spaces via averages on balls*, arXiv: 1511.07598.

[7] Y. Ding and S. Sato, *Singular integrals on product homogeneous groups*, Integr. Equ. Oper. Theory **76** (2013), 55-79.

[8] Y. Ding and S. Sato, *Maximal singular integrals on product homogeneous groups*, Studia Math. **222** (2014), 41–49.

[9] Y. Ding and S. Sato, *Littlewood-Paley functions on homogeneous groups*, Forum Math. **28** (2016), 43–55.

[10] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.

[11] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

[12] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, Princeton, N.J. 1982.

[13] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, New York, Oxford, 1985.

[14] P. Hajłasz and Z. Liu, A Marcinkiewicz integral type characterization of the Sobolev space, Publ. Mat. 61 (2017), 83–104.

[15] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–139.

[16] M. Kaneko and G. Sunouchi, *On the Littlewood-Paley and Marcinkiewicz functions in higher dimensions*, Tôhoku Math. J. **37** (1985), 343–365.

[17] J. Marcinkiewicz, *Sur quelues integrales de type de Dini*, Annales de la Société Polonaise **17** (1938), 42–50.

[18] B. Muckenhoupt and R. L. Wheeden, *Norm inequalities for the Littlewood-Paley function g_λ^**, Trans. Amer. Math. Soc. **191** (1974), 95–111.

[19] A. Nagel and E. M. Stein, *Lectures on Pseudo-Differential Operators*, Mathematical Notes 24, Princeton University Press, Princeton, NJ, 1979.

[20] J. Peetre, *On spaces of Triebel-Lizorkin type*, Ark. Mat. **13** (1975), 123–130.

[21] S. Sato, *Remarks on square functions in the Littlewood-Paley theory*, Bull. Austral. Math. Soc. **58** (1998), 199–211.

[22] S. Sato, *Multiparameter Marcinkiewicz integrals and a resonance theorem*, Bull. Fac. Ed. Kanazawa Univ. Natur. Sci. **48** (1999), 1–21. (http://hdl.handle.net/2297/25017)

[23] S. Sato, *Nonisotropic dilations and the method of rotations with weight*, Proc. Amer. Math. Soc. **140** (2012), 2791–2801.

[24] S. Sato, *Estimates for singular integrals on homogeneous groups*, J. Math. Anal. Appl. **400** (2013), 311–330.

[25] S. Sato, *Boundedness of Littlewood-Paley operators*, RIMS Kokyuroku Bessatsu **49** (2014), 75–101, Research Institute for Mathematical Sciences, Kyoto University.

[26] S. Sato, *Littlewood-Paley operators and Sobolev spaces*, Illinois J. Math. **58** (2014), 1025–1039.

[27] S. Sato, *Weighted weak type (1,1) estimates for singular integrals with non-isotropic homogeneity*, Ark. Mat. **54** (2016), 157–180.

[28] S. Sato, Square functions related to integral of Marcinkiewicz and Sobolev spaces, Linear and Nonlinear Analysis 2(2) (2016), Special Issue on ISBFS2015, 237–252.

[29] S. Sato, *Littlewood-Paley equivalence and homogeneous Fourier multipliers*, Integr. Equ. Oper. Theory **87** (2017), 15–44.

[30] S. Sato, Spherical square functions of Marcinkiewicz type with Riesz potentials, Arch. Math. 108(4) (2017), 415–426.

[31] S. Sato, *Vector valued inequalities and Littlewood-Paley operators on Hardy spaces*, Hokkaido Math. J. **48** (2019), 61–84, arXiv:1608.08059v2 [math.CA].

[32] S. Sato, *Characterization of parabolic Hardy spaces by Littlewood-Paley functions*, Results Math **73** (2018), 106. https://doi.org/10.1007/s00025-018-0867-9, arXiv:1607.03645v2 [math.CA].

[33] S. Sato, *Characterization of H^1 Sobolev spaces by square functions of Marcinkiewicz type*, J Fourier Anal Appl **25** (2019), 842–873, https://doi.org/10.1007/s00041-018-9618-2.

[34] S. Sato, *Boundedness of Littlewood-Paley operators relative to non-isotropic dilations*, Czech Math. J. **69** (2019), 337-351.

[35] S. Sato, *Weak type estimates for functions of Marcinkiewicz type with fractional integrals of mixed homogeneity*, Math. Scand. 125(1) (2019), 135–162.

[36] S. Sato, *Hardy spaces on homogeneous groups and Littlewood-Paley functions*, Quart. J. Math. Published:25 January 2020, haz049, 1–26, https://doi.org/10.1093/qmath/haz049, Oxford University Press.

[37] S. Sato, F. Wang, D. Yang and W. Yuan, *Generalized Littlewood-Paley characterizations of fractional Sobolev spaces*, Communications in Contemporary Mathematics 20(7) (2018), 1750077 (48 pages).

[38] E. M. Stein, *The characterization of functions arising as potentials*, Bull. Amer. Math. Soc. **67** (1961), 102–104.

[39] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[40] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[41] J. -O. Strömberg and A. Torchinsky, Weighted Hardy Spaces, Lecture Notes in Math. 1381, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo Hong Kong, 1989.

[42] T. Tao, *The weak-type* $(1,1)$ *of* $L\log L$ *homogeneous convolution operator*, Indiana Univ. Math. J. **48** (1999), 1547–1584.

[43] A. Uchiyama, *Characterization of* $H^p(\mathbb{R}^n)$ *in terms of generalized Littlewood-Paley g-functions*, Studia Math. **81** (1985), 135–158.

[44] D. Waterman, *On an integral of Marcinkiewicz*, Trans. Amer. Math. Soc. *91* (1959), 129–138.

[45] A. Zygmund, *On certain integrals*, Trans. Amer. Math. Soc. **58** (1944), 170–204.

Department of Mathematics, Faculty of Education, Kanazawa University, Kanazawa 920-1192, Japan

E-mail address: `shuichi@kenroku.kanazawa-u.ac.jp`