Pointwise multipliers and generalized Campanato spaces with variable growth condition

茨城大学 理工学研究科（理学野） 中井 英一 *
Eiichi Nakai
Department of Mathematics, Ibaraki University

1 Introduction

Let $E = L^p(\mathbb{R}^n)$, $0 < p \le \infty$. If $g \in L^\infty(\mathbb{R}^n)$, then $fg \in E$ for all $f \in E$. Conversely, if a measurable function g satisfies that $fg \in E$ for all $f \in E(\mathbb{R}^n)$, then $g \in L^\infty(\mathbb{R}^n)$. However, this property does not hold for $E = \mathrm{BMO}(\mathbb{R}^n)$. Actually, $\log|x|$ is in $\mathrm{BMO}(\mathbb{R})$, but $\mathrm{sign}(x)\log|x|$ is not in $\mathrm{BMO}(\mathbb{R})$. The generalized Campanato space $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ was introduced by Nakai and Yabuta [54] (1985) to characterize pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$.

Twenty years later, the result in [54] was used by Lerner [26] (2005) to study the class $\mathcal{P}(\mathbb{R}^n)$ of functions $p(\cdot)$ for which the Hardy-Littlewood maximal operator M is bounded on the Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent, and positively solve a conjecture by Diening [13] saying that there are discontinuous functions belonging to $\mathcal{P}(\mathbb{R}^n)$. As the same application, using the pointwise multiplier for martingale BMO, Nakai and Sadasue [49] give a sufficient condition for the boundedness of maximal operator on $L^{p(\cdot)}$ on probability spaces. Note that we cannot use the log-Hölder continuity on probability spaces, because the probability spaces have not topology in general. We also use the pointwise multipliers on BMO to prove that the boundedness "$M : \mathrm{BMO}(Q) \to \mathrm{BLO}(Q)$" is real improvement of the boundedness "$M : \mathrm{BMO}(Q) \to \mathrm{BMO}(Q)$", where Q is a cube in $\mathbb{R}^n$. The function Mf is bounded from below and we cannot construct easily a function in $\mathrm{BMO}(Q) \setminus \mathrm{BLO}(Q)$ which is bounded from below. See also [28, 51] (2011, 2017) for other applications.

*eiichi.nakai.math@vc.ibaraki.ac.jp

Later, Nakai and Sawano [52] (2012) proved that $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is the dual space of the Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent. In general the predual is not unique. See [43, 47] (2008, 2017) for another predual of $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$.

For the relation among Campanato, Morrey and Hölder (Lipschitz) spaces with variable growth condition, see [41, 43, 44] (2006, 2008, 2010). For the boundedness of singular and fractional integral operators and the convolution operator with the heat kernel on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$, see [44, 57] (2010, 2019). The characterization of $b \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$ by using the commutator $[b, T]$ or $[b, I_\rho]$ is in [1, 2, 3] (2018, 2019, 2020).

The organization of this paper is as follows. In Section 2 we give the definition of the pointwise multipliers and a hisotry of the pointwise multipliers on BMO. Then we state topics related to the pointwise multipliers on BMO with several basic calculations from Section 3 to Section 8. The definition of generalized Campanato spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ with variable growth condition is in Section 4. In Section 9 we give proofs of the relation among Campanato, Morrey and Hölder (Lipschitz) spaces with variable growth condition. From Section 10 to Section 13 we state some operators on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$, pointwise multiplies, singular integral operators and the convolution with the heat kernel, and then an application to the Navier-Stokes equation. In Sections 14 and 15 we state generalized fractional integral operators I_ρ and commutators $[b, T]$ and $[b, I_\rho]$ with $b \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$. From Section 10 to Section 15 are subsets of the paper [48].

2 Pointwise multipliers

let $\Omega = (\Omega, \mu)$ be a complete σ-finite measure space. We denote by $L^0(\Omega)$ the set of all measurable functions from Ω to $\mathbb{R}$ or $\mathbb{C}$. Then $L^0(\Omega)$ is a linear space under the usual sum and scalar multiplication. Let $E_1, E_2 \subset L^0(\Omega)$ be subspaces. We say that a function $g \in L^0(\Omega)$ is a pointwise multiplier from E_1 to E_2, if the pointwise multiplication fg is in E_2 for any $f \in E_1$. We denote by $\mathrm{PWM}(E_1, E_2)$ the set of all pointwise multipliers from E_1 to E_2. We abbreviate $\mathrm{PWM}(E, E)$ to $\mathrm{PWM}(E)$.

It is well known as Hölder's inequality that

$$\|fg\|_{L^{p_2}(\Omega)} \le \|f\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_3}(\Omega)},$$

for $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0, \infty]$, $i = 1, 2, 3$. This shows that

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \supset L^{p_3}(\Omega).$$

Conversely, we can show the reverse inclusion by using the uniform boundedness theorem or the closed graph theorem. That is,

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) = L^{p_3}(\Omega). \tag{2.1}$$

If $p_1 = p_2 = p$, then

$$\mathrm{PWM}(L^p(\Omega)) = L^\infty(\Omega). \tag{2.2}$$

See Section 3, for the proof of (2.2) with $\Omega = \mathbb{R}^n$. However,

$$\mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n)) \neq L^\infty(\mathbb{R}^n). \tag{2.3}$$

In 1976 Stegenga [63] and Janson [21] gave the characterization of the pointwise multipliers on $\mathrm{BMO}(\Omega)$ for $\Omega = \mathbb{T}$ and $\Omega = \mathbb{T}^n$, respectively. After then the history is the following:

- Nakai and Yabuta [54] (1985) for $\mathrm{BMO}(\mathbb{R}^n)$ and local $\mathrm{BMO}(\mathbb{R}^n)$.
- Yabuta [65] (1993) for weighted dyadic BMO on $\mathbb{R}^n$.
- Nakai [33] (1993) for Campanato spaces on $\mathbb{R}^n$.
- Nakai and Yabuta [55] (1997) and Nakai [35] (1997) for BMO and Campanato spaces on spaces of homogeneous type (Ω, d, μ).
- Lin and Da. Yang [29] (2014) for local BMO and local Campanato spaces on RD spaces.
- Liu and Da. Yang [30] (2014) for $\mathrm{BMO}(\mathbb{R}^n, \mu)$ with the Gauss measure.
- Nakai and Sadasue [50] (2014) for martingale BMO and Campanato spaces on probability spaces $(\Omega, \mathcal{F}, P)$.
- Li, Nakai and Do. Yang [27] (2018) for $\mathrm{RBMO}(\mathbb{R}^n, \mu)$ with non-doubling measures.

3 Proof of $\mathrm{PWM}(L^p(\mathbb{R}^n)) = L^\infty(\mathbb{R}^n)$

In this section we state a proof of the following theorem (see [31, 46]).

Theorem 3.1. *Let $0 < p \le \infty$. Then*

$$\mathrm{PWM}(L^p(\mathbb{R}^n)) = L^\infty(\mathbb{R}^n). \tag{3.1}$$

Proof. We first show that, if $g \in L^\infty(\mathbb{R}^n)$, then $g \in \mathrm{PWM}(L^p(\mathbb{R}^n))$ and

$$\|g\|_{\mathrm{PWM}(L^p)} = \|g\|_{L^\infty}.$$

We may assume that $g \not\equiv 0$. From the inequality $\|fg\|_{L^p} \le \|g\|_{L^\infty}\|f\|_{L^p}$, we get $g \in \mathrm{PWM}(L^p(\mathbb{R}^n))$ and $\|g\|_{\mathrm{PWM}(L^p)} \le \|g\|_{L^\infty}$. Moreover, for any $\eta \in (0, \|g\|_{L^\infty})$, let $A_\eta = \{x : |g(x)| > \eta\}$ and take a ball B_η such that $|A_\eta \cap B_\eta| > 0$. Then $\chi_{A_\eta \cap B_\eta} \in L^p(\mathbb{R}^n)$ and $\eta\chi_{A_\eta \cap B_\eta} \le |g|\chi_{A_\eta \cap B_\eta}$, which shows

$$\eta \le \frac{\|g\chi_{A_\eta \cap B_\eta}\|_{L^p}}{\|\chi_{A_\eta \cap B_\eta}\|_{L^p}} \le \|g\|_{\mathrm{PWM}(L^p)}.$$

Therefore, we have $\|g\|_{\mathrm{PWM}(L^p)} = \|g\|_{L^\infty}$.

There are two ways to prove $\mathrm{PWM}(L^p(\mathbb{R}^n)) \subset L^\infty(\mathbb{R}^n)$.

(i) Let $g \in \mathrm{PWM}(L^p(\mathbb{R}^n))$. For $j \in \mathbb{N}$, let

$$g_j(x) = \begin{cases} 0 & (|x| \ge j), \\ |g(x)| & (|x| < j,\ |g(x)| \le j), \\ j & (|x| < j,\ |g(x)| < j). \end{cases} \tag{3.2}$$

Then $g_j \in L^\infty(\mathbb{R}^n)$. By the first part of the proof we have $\|g_j\|_{\mathrm{PWM}(L^p)} = \|g_j\|_{L^\infty}$. For any $f \in L^p(\mathbb{R}^n)$, $fg \in L^p(\mathbb{R}^n)$ and $\sup_j \|fg_j\|_{L^p} \le \|fg\|_{L^p}$. From the uniform boundedness principle it follows that $\sup_j \|g_j\|_{\mathrm{PWM}(L^p)} < \infty$, which implies $\sup_j \|g_j\|_{L^\infty} < \infty$. Therefore, we have $g \in L^\infty(\mathbb{R}^n)$.

(ii) Let $g \in \mathrm{PWM}(L^p(\mathbb{R}^n))$. Then it is easy to see that g is a closed operator. Hence g is a bounded operator by the closed graph theorem. For $j \in \mathbb{N}$, let g_j be as in (3.2). Then $g_j \in L^\infty(\mathbb{R}^n)$ and $\|g_j\|_{L^\infty} = \|g_j\|_{\mathrm{PWM}(L^p)} \le \|g\|_{\mathrm{PWM}(L^p)}$. Therefore, we have $g \in L^\infty(\mathbb{R}^n)$. □

4 Campanato spaces

For $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r)$ be a ball centered at x and radius r, or, a cube centered at x and sidelength r. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and for a ball (cube) B, let

$$f_B = \fint_B f = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy.$$

where $|B|$ is the Lebesgue measure of B. In the following we use the symbol Q for the cube.

Definition 4.1. For $p \in [1,\infty)$ and $\lambda \in [-n/p, 1]$, let $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\lambda}(\mathbb{R}^n)} = \sup_{B=B(x,r)} \frac{1}{r^\lambda} \left(\fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

We regard $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ as a space of functions modulo null-functions and constant functions. Then $\|f\|_{\mathcal{L}_{p,\lambda}(\mathbb{R}^n)}$ is a norm and thereby $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ is a Banach space. We can define an equivalent norm by using cubes Q instead of balls B in Definition 4.1.

If $p = 1$ and $\lambda = 0$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $p = 1$ and $\lambda \in (0,1]$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ coincides with $\mathrm{Lip}_\lambda(\mathbb{R}^n)$ modulo null-functions. By the John-Nirenberg inequality [23] we conclude that, if $\lambda \in [0,1]$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathcal{L}_{1,\lambda}(\mathbb{R}^n)$ with equivalent norms for each p. If $\lambda \in [-n/p, 0)$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ coincides with the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ modulo constant functions, which is defined by the norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{B=B(x,r)} \frac{1}{r^\lambda} \left(\fint_B |f(y)|^p \, dy \right)^{1/p}.$$

If $\lambda = -n/p$, then $L_{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. For these relations, see [7, 23, 32, 59].

The generalized Campanato space $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ with variable growth condition is defined as follows: For a variable growth function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and a ball $B = B(x,r)$ we write $\phi(B) = \phi(x,r)$.

Definition 4.2. For $p \in [1,\infty)$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)} = \sup_B \frac{1}{\phi(B)} \left(\fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

We regard $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ as a space of functions modulo null-functions and constant functions. Then $\|f\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)}$ is a norm and thereby $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is a Banach space. If $\phi(x,r) = r^\lambda$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is the usual Campanato space $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$. For example, let $\lambda(\cdot) : \mathbb{R}^n \to [-n/p, 1]$ be a continuous function and

$$\phi(x,r) = r^{\lambda(x)}, \quad \lambda(x) = \begin{cases} 0 & \text{on } B_1, \\ 1 & \text{on } B_2, \\ -n/p & \text{on } B_3. \end{cases}$$

In this case, if $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$, then f is a BMO function on B_1, a Lipschitz function on B_2 and an L^p function on B_3.

In 1985 the generalized Campanato space $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ was introduced by Nakai and Yabuta [54] to characterize pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$.

For a function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let

$$\phi_*(x,r) = \int_r^1 \frac{\phi(x,t)}{t}\,dt.$$

Then we have the following lemma:

Lemma 4.1 ([33, Lemma 3.1]). *Let $p \in [1,\infty)$. Assume that $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfies the following properties:*

$$\frac{1}{A_1} \le \frac{\phi(x,s)}{\phi(x,r)} \le A_1, \qquad \text{if } \frac{1}{2} \le \frac{s}{r} \le 2, \tag{4.1}$$

$$\frac{1}{A_2} \le \frac{\phi(x,r)}{\phi(y,r)} \le A_2, \qquad \text{if } |x-y| \le r, \tag{4.2}$$

$$\frac{\phi(x,s)}{s} \le A_3 \frac{\phi(x,r)}{r}, \qquad \text{if } r < s, \tag{4.3}$$

$$\int_0^r \frac{t^{n/p}\phi(x,t)}{t} \le A_4 r^{n/p}\phi(x,r), \tag{4.4}$$

where A_i $(i = 1,2,3,4)$ are positive constants independent of $x, y \in \mathbb{R}^n$ and $r, s \in (0,\infty)$. Then $f_a(x) = \phi_*(a, |x-a|)$ is in $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\|f_a\|_{\mathcal{L}_{p,\phi}} \le C$ independently of $a \in \mathbb{R}^n$.

In the above theorem, if $\phi \equiv 1$, then $f_a(x) = -\log|x-a| \in \mathrm{BMO}(\mathbb{R}^n)$.

In the rest of this section we give simple properties of the mean oscilation with basic calculations. For a function f and a ball B, let

$$\mathrm{MO}(f,B) = \fint_B |f - f_B|.$$

Then

$$0 \le |f|_B - |f_B| = \fint_B |f - f_B + f_B| - |f_B| \le \mathrm{MO}(f,B),$$

and

$$\big||f| - |f|_B\big| \le \big||f| - |f_B|\big| + \big||f|_B - |f_B|\big| \le |f - f_B| + \mathrm{MO}(f,B).$$

That is,

$$\mathrm{MO}(|f|,B) \le 2\mathrm{MO}(f,B).$$

If f and g are real valued, from the above inequality and the relations

$$\max(f,g) = \frac{f+g+|f-g|}{2}, \quad \min(f,g) = \frac{f+g-|f-g|}{2},$$

it follows that

$$\|\max(f,g)\|_{\mathcal{L}_{p,\phi}}, \|\min(f,g)\|_{\mathcal{L}_{p,\phi}} \le \frac{3}{2}\left(\|f\|_{\mathcal{L}_{p,\phi}} + \|g\|_{\mathcal{L}_{p,\phi}}\right). \tag{4.5}$$

For two balls B_1 and B_2, if $B_1 \subset B_2$, then

$$|f_{B_1} - f_{B_2}| \le \fint_{B_1} |f - f_{B_2}| \le \frac{|B_2|}{|B_1|}\mathrm{MO}(f, B_2). \tag{4.6}$$

Lemma 4.2. *Let $p \in [1,\infty)$. Assume that $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfies* (4.1). *Let $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for $a \in \mathbb{R}^n$ and $0 < r < s < \infty$,*

$$\left|f_{B(a,r)} - f_{B(a,s)}\right| \le C\left(\int_r^{2s} \frac{\phi(a,t)}{t}\,dt\right)\|f\|_{\mathcal{L}_{p,\phi}}. \tag{4.7}$$

Proof. If $r < s \le 2r$, then by (4.6) we have

$$\begin{aligned}\left|f_{B(a,r)} - f_{B(a,s)}\right| &\le 2^n\mathrm{MO}(f, B(a,s)) \le 2^n\phi(a,s)\|f\|_{\mathcal{L}_{p,\phi}} \\ &\sim \left(\int_r^{2r} \frac{\phi(a,t)}{t}\,dt\right)\|f\|_{\mathcal{L}_{p,\phi}}. \end{aligned} \tag{4.8}$$

If $2^k r < s \le 2^{k+1} r$ for some $k \in \mathbb{N}$, than, by the same calculation as (4.8), we have

$$\begin{aligned}\left|f_{B(a,r)} - f_{B(a,s)}\right| &\le \sum_{j=1}^{k}\left|f_{B(a,2^{j-1}r)} - f_{B(a,2^j r)}\right| + \left|f_{B(a,2^k r)} - f_{B(a,s)}\right| \\ &\lesssim \sum_{j=1}^{k+1}\left(\int_{2^{j-1}r}^{2^j r} \frac{\phi(a,t)}{t}\,dt\right)\|f\|_{\mathcal{L}_{p,\phi}} = \left(\int_r^{2^{k+1}r} \frac{\phi(a,t)}{t}\,dt\right)\|f\|_{\mathcal{L}_{p,\phi}},\end{aligned}$$

which shows (6.1). □

5 Pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$

To consider the pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$ we introduce a norm

$$\|f\|_{\mathrm{BMO}^\natural(\mathbb{R}^n)} = \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |f_{B(0,1)}|.$$

Then $\mathrm{BMO}^\natural(\mathbb{R}^n) = (\mathrm{BMO}(\mathbb{R}^n), \|\cdot\|_{\mathrm{BMO}^\natural(\mathbb{R}^n)})$ is a Banach space not modulo constant functions. We have the following theorem, whose proof will be given in Section 6.

Theorem 5.1 ([54] (1985))**.** *Let*

$$\phi(x,r) = \frac{1}{\log(r + 1/r + |x|)}, \quad x \in \mathbb{R}^n,\ r \in (0,\infty). \tag{5.1}$$

Then $\mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n)) = \mathcal{L}_{1,\phi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *In this case, the operator norm of $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ is comparable to $\|g\|_{\mathcal{L}_{1,\phi}(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)}$.*

Remark 5.1. Let $\phi(r) = 1/\log(1/r)$. Then $\mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{T}^n)) = \mathcal{L}_{1,\phi}(\mathbb{T}^n) \cap L^\infty(\mathbb{T}^n)$. In this case we don't need variable growth functions.

The following proposition is a special case of [54, Proposition 5.1], which gives a sufficient condition for $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$.

Proposition 5.2 ([54] (1985))**.** *Assume that a function g satisfies*

$$|g(x) - g(y)| \le \frac{C_1}{\log(e/|x-y|)}, \quad \textit{if} \;\; |x-y| \le 1, \tag{5.2}$$

$$|g(x) - g_\infty| \le \frac{C_2}{\log(e+|x|)} \tag{5.3}$$

for some constants C_1, C_2 and g_∞ independent of $x, y \in \mathbb{R}^n$. Then g is a pointwise multiplier on $\mathrm{BMO}(\mathbb{R}^n)$*.*

Nowadays the conditions (5.2) and (5.3) are called the log-Hölder conditions. Note that, if $p(x)$ satisfies (5.2), then $\phi(x,r) = r^{p(x)}$ satisfies (4.2). Diening [12] and Crutz-Uribe et al. [11] proved that, if

$$1 < \inf_{x\in\mathbb{R}^n} p(x) \le \sup_{x\in\mathbb{R}^n} p(x) < \infty \tag{5.4}$$

and $p(\cdot)$ satisfies (5.2) and (5.3), then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

However, the continuity condition (5.2) and the existance of the limit $g_\infty = \lim_{|x|\to\infty} g(x)$ in (5.3) are not necessary for the boundedness of the operator M on $L^{p(\cdot)}(\mathbb{R}^n)$. Lerner [26] (2005) proved the following theorem by using Theorem 5.1.

Theorem 5.3 ([26] (2005))**.** *Let $p(\cdot)$ be a real valued measurable function. If $p(\cdot) \in$* $\mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$*, then there exists a positive constant α such that the Hardy-Littlewood maximal operator M is bounded on $L^{\alpha+p(\cdot)}(\mathbb{R}^n)$.*

Here we give critical examples of $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ which don't satisfies the log-Hölder conditions. In Theorem 5.1

$$\phi(a,r) \sim \begin{cases} \dfrac{1}{\log(1/r+|a|)} & \text{for small } r, \\ \dfrac{1}{\log(r+|a|)} & \text{for large } r. \end{cases}$$

Let $a = 0$. Then

$$\max(0, \phi_*(0,|x|) - \phi_*(0,1/e)) = \int_{\min(1/e,|x|)}^{1/e} \frac{dt}{t\log(1/t)} = \chi_{B(0,1/e)}(x) \log\log(|x|^{-1}) \tag{5.5}$$

$$\max(0, -\phi_*(0,|x|) + \phi_*(0,e)) = \int_e^{\max(e,|x|)} \frac{dt}{t\log(t)} = \chi_{B(0,e)^\complement}(x) \log\log|x| \tag{5.6}$$

are in $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ by Lemma 4.1 and (4.5). Let

$$g_1(x) = \sin\left(\chi_{B(0,1/e)}(x)\log\log(|x|^{-1})\right) \tag{5.7}$$
$$g_2(x) = \sin\left(\chi_{B(0,e)^\complement}(x)\log\log|x|\right). \tag{5.8}$$

Then $g_i \in \mathcal{L}_{1,\phi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) = \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ for $i = 1, 2$. Note that g_1 is not continuous and that $\lim_{|x|\to\infty} g_2(x)$ does not exist. The example (5.7) was given by Janson [21] (1976) and Stegenga [63] (1976) on the Torus $\mathbb{T}^n$, and the example (5.8) by [54] on $\mathbb{R}^n$.

南雲恵 (2008) proved the following theorem in her Master's thesis.

Theorem 5.4 ([58]). *If $p(\cdot)$ satisfies* (5.4) *and is a constant outside some ball B, and if*

$$\sup_{a\in\mathbb{R}^d,\,0<r<1/2} \frac{1}{1/\log(1/r)} \fint_{B(a,r)} |1/p - (1/p)_B| < \infty, \tag{5.9}$$

then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

In the above theorem $1/p(\cdot)$ is bounded. We can also conclude that $1/p(\cdot) \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ for ϕ in (5.1). That is, $1/p(\cdot) \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$. Moreover, the following theorem is known.

Theorem 5.5 ([14]). *Let $p(\cdot)$ satisfy*

$$1 < \inf_{x\in\mathbb{R}^n} p(x) \le \sup_{x\in\mathbb{R}^n} p(x) \le \infty. \tag{5.10}$$

If $1/p(\cdot)$ satisfies the log-Hölder conditions (5.2) *and* (5.3)*, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

In the above theorem $1/p(\cdot)$ is also in $\mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ by Proposition 5.2.

6 Proof of Theorem 5.1

First we note that, if $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$, then g is a closed operator, and then g is a bounded operator by the closed graph theorem.

Nest we state several lemmas.

Lemma 6.1. *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. If $B(a,r) \subset B(b,s)$, then*

$$\left|f_{B(a,r)} - f_{B(b,s)}\right| \lesssim \left(\int_r^{2s} \frac{1}{t}\,dt\right) \|f\|_{\mathrm{BMO}}. \tag{6.1}$$

Proof. If $r < s \le 2r$ and $B(a,r) \subset B(b,s)$, then by (4.6) we have

$$\left|f_{B(a,r)} - f_{B(b,s)}\right| \le 2^n \|f\|_{\mathrm{BMO}} = \frac{2^n}{\log 2}\left(\int_r^{2r} \frac{1}{t}\,dt\right)\|f\|_{\mathrm{BMO}}. \tag{6.2}$$

If $2^k r < s \le 2^{k+1} r$ for some $k \in \mathbb{N}$, then take balls B_j of radius $2^j r$, $j = 1, 2, \cdots k$, such that

$$B(a,r) = B_0 \subset B_1 \subset \cdots \subset B_k \subset B(b,s).$$

Than, by the same calculation as (6.2), we have

$$\begin{aligned}\left|f_{B(a,r)} - f_{B(b,s)}\right| &\le \sum_{j=1}^{k} \left|f_{B_{j-1}} - f_{B_j}\right| + \left|f_{B_k} - f_{B(b,s)}\right| \\ &\lesssim \sum_{j=1}^{k+1} \left(\int_{2^{j-1}r}^{2^j r} \frac{1}{t}\,dt\right)\|f\|_{\mathrm{BMO}} = \left(\int_r^{2^{k+1}r} \frac{1}{t}\,dt\right)\|f\|_{\mathrm{BMO}},\end{aligned}$$

which shows (6.1). □

Lemma 6.2. *There exists a positive constant C such that, for all $f \in \mathrm{BMO}^{\natural}(\mathbb{R}^n)$ and balls $B(a,r)$,*

$$\left|f_{B(a,r)}\right| \le C\Big(\log(r + 1/r + |a|)\Big)\|f\|_{\mathrm{BMO}^{\natural}}. \tag{6.3}$$

Proof. First note that $B(a,r), B(0,1) \subset B(0, r + 1/r + |a|)$, since $r + 1/r \ge 2$. By Lemma 6.1 we have

$$\begin{aligned}\left|f_{B(a,r)} - f_{B(0,1)}\right| &\le \left|f_{B(a,r)} - f_{B(0,r+1/r+|a|)}\right| + \left|f_{B(0,1)} - f_{B(0,r+1/r+|a|)}\right| \\ &\lesssim \left(\int_r^{2(r+1/r+|a|)} \frac{1}{t}\,dt + \int_1^{2(r+1/r+|a|)} \frac{1}{t}\,dt\right)\|f\|_{\mathrm{BMO}} \\ &= \Big(2\log(2(r + 1/r + |a|)) + \log(1/r)\Big)\|f\|_{\mathrm{BMO}} \\ &\lesssim \Big(\log(r + 1/r + |a|)\Big)\|f\|_{\mathrm{BMO}},\end{aligned}$$

which shows (6.3). □

Let

$$h^*(r) = \max(1, \log r),\ \ h_*(r) = \max(1, -\log r), \quad r > 0.$$

Then $h^*(|\cdot|), h_*(|\cdot|) \in \mathrm{BMO}^{\natural}(\mathbb{R}^n)$ by (4.5) and $\|h_*(|\cdot - a|)\|_{\mathrm{BMO}^{\natural}} \le C$ for some constant independent of $a \in \mathbb{R}^n$. Moreover, we can easily check that

$$h^*(|a|) + h^*(r) + h_*(r) \sim \log(r + 1/r + |a|).$$

Lemma 6.3. *There exsits a positive constant c such that, for any ball $B(a,r)$,*

$$\fint_{B(a,r)} h^*(|x|)\,dx \ge c\Big(h^*(|a|) + h^*(r)\Big), \tag{6.4}$$

$$\fint_{B(a,r)} h_*(|x-a|)\,dx \ge h_*(r). \tag{6.5}$$

Proof. From the monotonicity of h^* and h_* it follows that

$$\fint_{B(a,r)} h^*(|x|) \ge \frac{1}{|B(a,r)|}\int_{B(a,r)\setminus B(0,|a|)} h^*(|x|) \gtrsim h^*(|a|),$$
$$\fint_{B(a,r)} h^*(|x|) \ge \fint_{B(0,r)} h^*(|x|) \ge \frac{1}{|B(0,r)|}\int_{B(0,r)\setminus B(0,r/2)} h^*(|x|) \gtrsim h^*(r/2) \sim h^*(r),$$
$$\fint_{B(a,r)} h_*(|x-a|) = \fint_{B(0,r)} h_*(|x|) \ge h_*(r),$$

which show the conclusion. □

Lemma 6.4. *If $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$, then $g \in L^\infty(\mathbb{R}^n)$ and, for some constant C independent of g, $\|g\|_{L^\infty} \le C\|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}$.*

Proof. For any ball $B(a,r)$ with $r < 1/2$, let

$$h(x) = \max(0, -\log|x-a| + \log r) = \begin{cases} 0 & (|x-a| \ge r) \\ -\log|x-a| + \log r & (|x-a| < r). \end{cases}$$

Then $\|h\|_{\mathrm{BMO}}$ is independent of $B(a,r)$ and $|h_{B(0,1)}| \le C$. Moreover, we have

$$|h(x)| > -\log(r/2) + \log r = \log 2 \quad \text{for} \quad x \in B(a,r/2).$$

Now, for $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$, let $\sigma = (gh)_{B(a,2r)}$. Then

$$\begin{aligned}\int_{B(a,2r)} |gh-\sigma| &\ge \int_{B(a,r/2)} |gh-\sigma| + \int_{B(a,2r)\setminus B(a,r)} |\sigma| \\ &\ge \int_{B(a,r/2)} (|gh-\sigma| + |\sigma|) \\ &\ge \int_{B(a,r/2)} |gh| \ge \log 2 \int_{B(a,r/2)} |g|,\end{aligned}$$

and

$$\fint_{B(a,r/2)} |g| \lesssim \fint_{B(a,2r)} |gh-\sigma| \le \|gh\|_{\mathrm{BMO}^\natural} \lesssim \|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}.$$

Letting $r \to 0$, we get $|g(a)| \lesssim \|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}$ a.e. $a \in \mathbb{R}^n$ by Lebesgue's differentiation theorem. This shows that $\|g\|_{L^\infty} \lesssim \|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}$. □

Lemma 6.5. *Let $f \in \mathrm{BMO}^\natural(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$. Then $fg \in \mathrm{BMO}^\natural(\mathbb{R}^n)$ if and only if $\sup_B |f_B| \mathrm{MO}(g, B) < \infty$. In this case*

$$\Big| \|fg\|_{\mathrm{BMO}} - \sup_B |f_B| \mathrm{MO}(g, B) \Big| \le 2\|f\|_{\mathrm{BMO}} \|g\|_{L^\infty}. \tag{6.6}$$

Proof.

$$\begin{aligned}
\Big| \mathrm{MO}(fg, B) - |f_B| \mathrm{MO}(g, B) \Big| &= \Big| \fint_B |fg - (fg)_B| - |f_B| \fint_B |g - g_B| \Big| \\
&\le \fint_B \Big| (fg - (fg)_B) - (f_B(g - g_B)) \Big| \\
&= \fint_B \Big| (f - f_B)g - ((fg)_B - f_B g_B) \Big| \\
&\le \fint_B \Big| (f - f_B)g \Big| + \Big| (fg)_B - f_B g_B \Big| \\
&= \fint_B \Big| (f - f_B)g \Big| + \Big| \fint_B (f - f_B)g \Big| \\
&\le 2\|f\|_{\mathrm{BMO}} \|g\|_{L^\infty},
\end{aligned}$$

which implies

$$\sup_B \mathrm{MO}(fg, B) < \infty \;\Leftrightarrow\; \sup_B |f_B| \mathrm{MO}(g, B) < \infty.$$

In this case we have (6.6). □

Proof of Theorems 5.1. Let $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ and $f \in \mathrm{BMO}^\natural(\mathbb{R}^n)$. Then $fg \in \mathrm{BMO}^\natural(\mathbb{R}^n)$ and $\|fg\|_{\mathrm{BMO}^\natural} \le \|f\|_{\mathrm{BMO}^\natural} \|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}$. By Lemma 6.4 we have $g \in L^\infty(\mathbb{R}^n)$ and $\|g\|_{L^\infty} \lesssim \|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}$. By Lemma 6.5 we have

$$\sup_B |f_B| \mathrm{MO}(g, B) \le \|fg\|_{\mathrm{BMO}} + 2\|f\|_{\mathrm{BMO}} \|g\|_{L^\infty} \lesssim \|f\|_{\mathrm{BMO}^\natural} \|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}. \tag{6.7}$$

Take $h^*(|\cdot|)$ or $h_*(|\cdot - a|)$ as f in (6.7). Then, by Lemma 6.3, we have

$$\sup_{B(a,r)} \Big(h^*(|a|) + h^*(r) + h_*(r) \Big) \mathrm{MO}(g, B(a, r)) \lesssim \|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)},$$

which shows $g \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and $\|g\|_{\mathcal{L}_{1,\phi}} \lesssim \|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}$, since $h^*(|a|) + h^*(r) + h_*(r) \sim \log(r + 1/r + |a|) = 1/\phi(a, r)$.

Conversely, let $g \in \mathcal{L}_{1,\phi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. For any $f \in \mathrm{BMO}^\natural(\mathbb{R}^n)$, by Lemma 6.2 we have

$$|f_B| \lesssim \frac{1}{\phi(B)} \|f\|_{\mathrm{BMO}^\natural} \quad \text{for any ball } B,$$

and

$$\sup_B |f_B| \mathrm{MO}(g, B) \lesssim \sup_B \|f\|_{\mathrm{BMO}^\natural} \frac{\mathrm{MO}(g, B)}{\phi(B)} \le \|f\|_{\mathrm{BMO}^\natural} \|g\|_{\mathcal{L}_{1,\phi}}.$$

By Lemma 6.5 we have $fg \in \mathrm{BMO}^\natural(\mathbb{R}^n)$ and

$$\|fg\|_{\mathrm{BMO}^\natural} \lesssim \sup_B |f_B| \mathrm{MO}(g,B) + 2\|f\|_{\mathrm{BMO}} \|g\|_{L^\infty} \lesssim \|f\|_{\mathrm{BMO}^\natural} \Big(\|g\|_{\mathcal{L}_{1,\phi}} + \|g\|_{L^\infty} \Big).$$

That is, $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ and $\|g\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)} \lesssim \|g\|_{\mathcal{L}_{1,\phi}} + \|g\|_{L^\infty}$. □

7 Proof of Theorem 5.3

In this section we prove Theorem 5.3 by Lerner's idea in [26]. We use the following theorems.

Theorem 7.1 (John and Nirenberg [23] (1961))**.** *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$ and Q a cube in $\mathbb{R}^n$. Then, for $\lambda > 0$,*

$$\Big|\{x \in Q : |f(x) - f_Q| > \lambda\}\Big| \le e|Q| e^{-A\lambda/\|f\|_{\mathrm{BMO}}}, \tag{7.1}$$

where $A = (2^n e)^{-1}$.

For the constant $A = (2^n e)^{-1}$, see Grafakos [19, p. 160]

Theorem 7.2 (Coifman and Rochberg [9] (1980))**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $Mf < \infty$ a.e. Then $\log(Mf) \in \mathrm{BMO}(\mathbb{R}^n)$ and $\|\log(Mf)\|_{\mathrm{BMO}} \le \gamma_n$, where $\gamma_n > 1$ depends only on n.*

We note that, if the operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then M is also bounded on $L^{rp(\cdot)}(\mathbb{R}^n)$ for any $r > 1$. Indeed, by Hölder's inequality,

$$\|Mf\|_{L^{rp(\cdot)}} \le \|(M|f|^r)^{1/r}\|_{L^{rp(\cdot)}} = \|M|f|^r\|_{L^{p(\cdot)}}^{1/r} \lesssim \||f|^r\|_{L^{p(\cdot)}}^{1/r} = \|f\|_{L^{rp(\cdot)}}.$$

Next we recall the Muckenhoupt A_p condition, $1 < p < \infty$. A weight w satisfies the A_p condition ($w \in A_p$), if

$$\|w\|_{A_p} = \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{-1/(p-1)} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q. Muckenhoupt proved that

$$\int_{\mathbb{R}^n} (Mf)^p w \le c \int_{\mathbb{R}^n} |f|^p w,$$

for any $f \in L^p(\mathbb{R}^n, w)$ if and only if $w \in A_p$. Moreover, the constant c is depends only on n, p and $\|w\|_{A_p}$

Lemma 7.3. *If $\|f\|_{\mathrm{BMO}} \le (2^n e)^{-1}$, then $\|e^f\|_{A_2} \le 4$.*

Proof. By Theorem 7.1,

$$\begin{aligned}\int_Q e^{|f-f_Q|} &\le |Q| + \int_1^\infty \Big|\{x\in Q : |f(x)-f_Q|>\lambda\}\Big|\,d\lambda \\ &\le |Q| + \int_1^\infty e|Q|e^{-A\lambda/\|f\|_{\mathrm{BMO}}}\,d\lambda \le 2|Q|,\end{aligned}$$

and

$$\begin{aligned}\|e^f\|_{A_2} &= \sup_Q \left(⨍_Q e^f\right)\left(⨍_Q e^{-f}\right) = \sup_Q \left(⨍_Q e^{f-f_Q}\right)\left(⨍_Q e^{-f+f_Q}\right) \\ &\le \sup_Q \left(⨍_Q e^{|f-f_Q|}\right)^2 \le 4.\end{aligned}$$

□

Take a cube Q_0 with measure 1 and let

$$\|f\|_{\mathrm{BMO}^\natural} = \|f\|_{\mathrm{BMO}} + |f_{Q_0}|$$

Let f^* be the non-increasing rearrangement on Q_0 and $f^{**} = t^{-1}\int_0^t f^*$. We use the following facts (see e.g. [5, p. 122 and p. 53]);

$$(Mf)^*(t) \le 3^n f^{**}(t), \quad t>0, \tag{7.2}$$

$$\int_0^t f^*(\tau)\,d\tau = \sup_{|E|=t}\int_E |f(x)|\,dx, \tag{7.3}$$

where the supremum is taken over all measurable sets E with $|E|=t$.

Proof of Theorem 5.3. For $p(\cdot)\in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$, take small $\epsilon>0$ and large $\alpha>0$ such that $0<2-\epsilon\alpha-\epsilon p(\cdot)<1/2$, and

$$\|2-\epsilon\alpha-\epsilon p(\cdot)\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)} \le (2^n e)^{-1}(\gamma_n+3n)^{-1}.$$

Let $q(\cdot) = 2-\epsilon(\alpha+p(\cdot))$. Then $3/2\le 2-q(\cdot)\le 2$. We will show that the operator M is bounded on $L^{2-q(\cdot)}(\mathbb{R}^n) = L^{\epsilon(\alpha+p(\cdot))}(\mathbb{R}^n)$. Then the operator M is also bounded on $L^{\alpha+p(\cdot)}(\mathbb{R}^n)$.

Let $\|f\|_{L^{2-q(\cdot)}}=1$ and $f\ge 0$. Let $\tilde f = f+\chi_{Q_0}$. Then $\log(M\tilde f)\ge 0$ on Q_0 and

$$\begin{aligned}\int_{Q_0}\log(M\tilde f) \le \int_{Q_0}\log(1+Mf) &= \int_0^1 \log(1+(Mf)^*(t))\,dt \\ &\le \int_0^1 \log(1+3^n f^{**}(t))\,dt.\end{aligned}$$

From

$$\sup_{|E|=t}\int_E |f| \le 2\|f\|_{L^{2-q(\cdot)}}\|\chi_{Q_0}\|_{L^{(2-q(\cdot))'}} \le 2,$$

it follows that $f^{**}(t) \le 2/t$. Then

$$\int_{Q_0} \log(M\tilde{f}) \le \int_0^1 \log(1 + 3^n f^{**}(t))\,dt \le \int_0^1 \log(1 + 2 \cdot 3^n/t)\,dt \le 3n.$$

By Theorem 7.2 we have $\|\log(M\tilde{f})\|_{\mathrm{BMO}^\natural} \le \gamma_n + 3n$. Then

$$\| - q(\cdot)\log(M\tilde{f})\|_{\mathrm{BMO}^\natural} \le \|q(\cdot)\|_{\mathrm{PWM}(\mathrm{BMO}^\natural)}\|\log(M\tilde{f})\|_{\mathrm{BMO}^\natural} \le (2^n e)^{-1}.$$

By Lemma 7.3 we have $\|(M\tilde{f})^{-q(\cdot)}\|_{A_2} \le 4$. Then

$$\int_{\mathbb{R}^n} (Mf)^{2-q(\cdot)} \le \int_{\mathbb{R}^n} (M\tilde{f})^{2-q(\cdot)} = \int_{\mathbb{R}^n} (M\tilde{f})^2 (M\tilde{f})^{-q(\cdot)} \lesssim \int_{\mathbb{R}^n} |\tilde{f}|^2 (M\tilde{f})^{-q(\cdot)}.$$

Since $q(\cdot) > 0$, $(M\tilde{f})^{-q(\cdot)} \le |\tilde{f}|^{-q(\cdot)}$. Then

$$\int_{\mathbb{R}^n} (Mf)^{2-q(\cdot)} \le \int_{\mathbb{R}^n} |\tilde{f}|^2 (M\tilde{f})^{-q(\cdot)} \le \int_{\mathbb{R}^n} |\tilde{f}|^{2-q(\cdot)} \lesssim 1.$$

This shows that the operator M is bounded on $L^{2-q(\cdot)}(\mathbb{R}^n)$. $\square$

8 Another application of Theorem 5.1

For a cube $Q \subset \mathbb{R}^n$, let $\mathrm{BMO}(Q)$ and $\mathrm{BLO}(Q)$ be the set of all functions f such that the following functionals are finite, respectively.

$$\|f\|_{\mathrm{BMO}(Q)} = \sup_{P \subset Q} \fint_P |f - f_P|,$$
$$\|f\|_{\mathrm{BLO}(Q)} = \sup_{P \subset Q} \fint_P (f - \inf_P f),$$

where the suprema are taken over all cubes P containing Q.

Bennett, DeVore and Sharpley [6] (1981) proved that the Hardy-Littlewood maximal operator M is bounded on $\mathrm{BMO}(Q)$. Then Bennett [4] (1982) proved that the operator M is bounded from $\mathrm{BMO}(Q)$ to $\mathrm{BLO}(Q)$. The latter seems to be improvement upon the former, since $\mathrm{BLO}(Q) \subsetneq \mathrm{BMO}(Q)$. However, if

$$\mathrm{BMO}(Q) \cap \{f : f \text{ is bounded from below}\} = \mathrm{BLO}(Q),$$

then the latter is only a corollary of the former, since $Mf \ge 0$. Therefore, we need to find a function in $\mathrm{BMO}(Q) \setminus \mathrm{BLO}(Q)$ which is bounded from below. To do this, we use the pointwise multiplier on $\mathrm{BMO}(Q)$. This idea is in [28].

We may assume that $Q = [-1, 1]^n$. Let

$$f(x) = \max(0, -\log|x|) \quad \text{and} \quad g(x) = \sin\big(\chi_{B(0,1/e)}(x) \log\log(|x|^{-1})\big),$$

where g is the same as g_1 in (5.7). Then $f \in \mathrm{BMO}^{\natural}(Q)$ and $g \in \mathrm{PWM}(\mathrm{BMO}^{\natural}(Q))$. Hence $|fg| \in \mathrm{BMO}(Q)$. We will show that $|fg| \notin \mathrm{BLO}(Q)$. Choose $r_k > 0$ such that

$$\log\log(r_k^{-1}) = (\pi/4)k, \quad k \in \mathbb{N}.$$

Then $r_k \searrow 0$ as $k \to \infty$. If $r_{8m+4} \le r \le r_{8m+3}$, $m \in \mathbb{N}$, then

$$(2m + 3/4)\pi \le \log\log(r^{-1}) \le (2m+1)\pi,$$

and then

$$\sin(\log\log(r^{-1})) \ge 0, \quad \cos(\log\log(r^{-1})) < 0$$

and

$$\sin(\log\log(r^{-1})) + \cos(\log\log(r^{-1})) < 0.$$

Letting $h(r) = (-\log r)\sin(\log\log(r^{-1}))$, we have

$$\begin{aligned}
h'(r) &= -\frac{1}{r}\sin(\log\log(r^{-1})) + \log(r^{-1})\cos(\log\log(r^{-1}))\left(-\frac{1}{r\log(r^{-1})}\right) \\
&= -\frac{1}{r}\Big(\sin(\log\log(r^{-1})) + \cos(\log\log(r^{-1}))\Big) > 0, \\
h''(r) &= \frac{1}{r^2}\Big(\sin(\log\log(r^{-1})) + \cos(\log\log(r^{-1}))\Big) \\
&\quad - \frac{1}{r}\Big(\cos(\log\log(r^{-1})) - \sin(\log\log(r^{-1}))\Big)\left(-\frac{1}{r\log(r^{-1})}\right) < 0.
\end{aligned}$$

r	$\cdots$	r_{8m+4}	$\cdots$	r_{8m+3}	$\cdots$
$h'(r)$			$+$		
$h''(r)$			$-$		
$h(r)$		0	$\nearrow$	$(\sqrt{2}/2)\log(1/r_{8m+3})$	

Since h is concave in this interval,

$$\frac{1}{r_{8m+3} - r_{8m+4}} \int_{r_{8m+4}}^{r_{8m+3}} \Big(h - \inf h\Big) \ge \frac{1}{2}\frac{\sqrt{2}}{2}\log(1/r_{8m+3}) \to \infty \quad \text{as} \quad m \to \infty.$$

This shows that $|fg|$ is not in $\mathrm{BLO}(Q)$.

For the martingale maximal function on probability spaces, the same application is in [51].

9 Related function spaces

In this section, we state the relations between $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and other function spaces with variable growth condition.

Definition 9.1. For $1 \le p < \infty$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, the function spaces $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$ are the sets of all functions f such that

$$\|f\|_{\mathcal{L}^\natural_{p,\phi}} = \|f\|_{\mathcal{L}_{p,\phi}} + |f_{B(0,1)}| < \infty,$$
$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\fint_B |f(x)|^p \, dx \right)^{1/p} < \infty,$$

respectively.

We regard $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$ as spaces of functions modulo null-functions. Then these functionals are also norms and thereby these spaces are Banach spaces. If $\phi(B) = |B|^{-1/p}$ for all balls B, then

$$\|f\|_{L_{p,\phi}} = \|f\|_{L^p}.$$

From the definition it follows that

$$\|f\|_{\mathcal{L}_{p,\phi}} \le 2\|f\|_{L_{p,\phi}}, \quad \|f\|_{\mathcal{L}^\natural_{p,\phi}} \le (2 + \phi(0,1))\|f\|_{L_{p,\phi}}. \tag{9.1}$$

Definition 9.2. For $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, the function spaces $\Lambda_\phi(\mathbb{R}^n)$ and $\Lambda^\natural_\phi(\mathbb{R}^n)$ are the sets of all functions f such that

$$\|f\|_{\Lambda_\phi} = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{2|f(x) - f(y)|}{\phi(x,|x-y|) + \phi(y,|y-x|)} < \infty,$$
$$\|f\|_{\Lambda^\natural_\phi} = \|f\|_{\Lambda_\phi} + |f(0)| < \infty,$$

respectively.

We regard $\Lambda^\natural_\phi(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$, and $\Lambda_\phi(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$ modulo constant functions. Then these functionals are also norms and thereby these spaces are Banach spaces. For $\phi(x,r) = r^\alpha$, $0 < \alpha \le 1$, we denote $\Lambda_{r^\alpha}(\mathbb{R}^n)$ and $\Lambda^\natural_{r^\alpha}(\mathbb{R}^n)$ by $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ and $\mathrm{Lip}^\natural_\alpha(\mathbb{R}^n)$, respectively. In this case,

$$\|f\|_{\mathrm{Lip}_\alpha} = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \quad \text{and} \quad \|f\|_{\mathrm{Lip}^\natural_\alpha} = \|f\|_{\mathrm{Lip}_\alpha} + |f(0)|.$$

If $\phi(x,r) = \min(r^\alpha, 1)$, $0 < \alpha \le 1$, then

$$\|f\|_{\Lambda^\natural_\phi} \sim \|f\|_{\mathrm{Lip}_\alpha} + \|f\|_{L^\infty}.$$

For two variable growth functions ϕ_1 and ϕ_2, we write $\phi_1 \sim \phi_2$ if there exists a positive constant C such that

$$C^{-1}\phi_1(B) \le \phi_2(B) \le C\phi_1(B) \quad \text{for all balls } B.$$

In this case, two function spaces defined by ϕ_1 and by ϕ_2 coincide with equivalent norms.

We consider the conditions (4.1) and (4.2) on variable growth function ϕ. We also use the following conditions.

$$\phi(x,r) \le C\phi(x,s), \qquad \text{if } r < s, \tag{9.2}$$

$$C\phi(x,r) \ge \phi(x,s), \qquad \text{if } r < s, \tag{9.3}$$

where C is positive constant independent of $x, y \in \mathbb{R}^n$ and $r, s \in (0,\infty)$. The conditions (4.1), (9.2) and (9.3) are called the doubling, almost increasingness and almost decreasingness conditions, respectively. The condition (4.2) is introduced in [33] and studied in [52] precisely. In this paper, we call it the nearness condition.

Note that (4.2) and (9.2) imply that there exists a positive constant C such that, for all $x, y \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\phi(x,r) \le C\phi(y,s) \quad \text{if} \quad B(x,r) \subset B(y,s).$$

Then we have the following three theorems:

Theorem 9.1 ([43] (2008)). *If ϕ satisfies (4.1), (4.2) and (9.2), then, for every $1 \le p < \infty$, $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}^{\natural}_{1,\phi}(\mathbb{R}^n)$ with equivalent norms, respectively.*

The abave theorem was proved by using the John-Nirenberg inequality [23].

Theorem 9.2 ([41] (2006)). *If ϕ satisfies (4.1), (4.2) and (9.2), and if there exists a positive constant C such that*

$$\int_0^r \frac{\phi(x,t)}{t}\,dt \le C\phi(x,r), \quad x \in \mathbb{R}^n,\ r \in (0,\infty), \tag{9.4}$$

then, for every $1 \le p < \infty$, each element in $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ can be regarded as a continuous function, (that is, each element is equivalent to a continuous function modulo null-functions) and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \Lambda^{\natural}_\phi(\mathbb{R}^n)$ with equivalent norms, respectively. In particular, if $\phi(x,r) = r^\alpha$, $0 < \alpha \le 1$, then, for every $1 \le p < \infty$, $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}^{\natural}_\alpha(\mathbb{R}^n)$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$ with equivalent norms, respectively.

Proof. Let $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$. For $x, y \in \mathbb{R}^n$ and $r > 0$, if $r < |x-y|$, then

$$B(x,r), B(y,|x-y|) \subset B(x,2|x-y|), \quad B(y,r) \subset B(y,|x-y|).$$

By Lemma 4.2 and inequalities (4.6) and (9.4) we have

$$\begin{aligned}
&\left|f_{B(x,r)} - f_{B(y,r)}\right| \\
&\le \left|f_{B(x,r)} - f_{B(x,2|x-y|)}\right| + \left|f_{B(x,2|x-y|)} - f_{B(y,|x-y|)}\right| + \left|f_{B(y,|x-y|)} - f_{B(y,r)}\right| \\
&\lesssim \left(\int_r^{4|x-y|} \frac{\phi(x,t)}{t}\,dt + \phi(x,2|x-y|) + \int_r^{2|x-y|} \frac{\phi(y,t)}{t}\,dt\right)\|f\|_{\mathcal{L}_{p,\phi}} \\
&\lesssim \Big(\phi(x,4|x-y|) + \phi(x,2|x-y|) + \phi(y,2|x-y|)\Big)\|f\|_{\mathcal{L}_{p,\phi}}.
\end{aligned}$$

By the doubling condition (4.1) we have

$$\left|f_{B(x,r)} - f_{B(y,r)}\right| \lesssim \Big(\phi(x,|x-y|) + \phi(y,|x-y|)\Big)\|f\|_{\mathcal{L}_{p,\phi}}.$$

Letting $r \to 0$, we have

$$|f(x) - f(y)| \lesssim \Big(\phi(x,|x-y|) + \phi(y,|x-y|)\Big)\|f\|_{\mathcal{L}_{p,\phi}} \quad \text{a.e. } x, y,$$

which shows $f \in \Lambda_\phi(\mathbb{R}^n)$ and $\|f\|_{\Lambda_\phi} \lesssim \|f\|_{\mathcal{L}_{p,\phi}}$.

Conversely, let $f \in \Lambda_\phi(\mathbb{R}^n)$. For $x, y \in B(a,r)$, by the almost increasingness (9.2) and the nearness condition (4.2) of ϕ we have

$$\begin{aligned}
|f(x) - f(y)| &\le \Big(\phi(x,|x-y|) + \phi(y,|x-y|)\Big)\|f\|_{\Lambda_\phi} \\
&\lesssim \Big(\phi(x,r) + \phi(y,r)\Big)\|f\|_{\Lambda_\phi} \lesssim \phi(a,r)\|f\|_{\Lambda_\phi}.
\end{aligned}$$

Then

$$\begin{aligned}
\left(\fint_{B(a,r)} |f - f_{B(a,r)}|^p\right)^{1/p} &\le \left(\fint_{B(a,r)} \left(\fint_{B(a,r)} |f(x) - f(y)|\,dy\right)^p dx\right)^{1/p} \\
&\lesssim \phi(a,r)\|f\|_{\Lambda_\phi},
\end{aligned}$$

which shows $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{L}_{p,\phi}} \lesssim \|f\|_{\Lambda_\phi}$. $\square$

Theorem 9.3 ([41] (2006))**.** *Let $1 \le p < \infty$. If ϕ satisfies (4.1) and (4.2), and if there exists a positive constant C such that*

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le C\phi(x,r), \quad x \in \mathbb{R}^n,\ r \in (0,\infty), \tag{9.5}$$

then, for $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$, the limit $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}$ exists and

$$\|f\|_{\mathcal{L}_{p,\phi}} \sim \|f - \sigma(f)\|_{L_{p,\phi}}.$$

That is, the mapping $f \mapsto f - \sigma(f)$ is bijective and bicontinuous from $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ (modulo constants) to $L_{p,\phi}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$. By (9.4) and (4.7) we have

$$\left|f_{B(0,r)} - f_{B(0,s)}\right| \le \int_r^{2s} \frac{\phi(0,t)}{t}\,dt \to 0 \quad \text{as } r,s \to \infty \text{ with } r < s.$$

Hence, $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}$ exists. By (4.6) we have that, for any $a \in \mathbb{R}^n$,

$$\begin{aligned}\left|f_{B(a,s)} - \sigma(f)\right| &\lesssim \left|f_{B(a,s)} - f_{B(0,2s)}\right| + \left|f_{B(0,2s)} - \sigma(f)\right| \\ &\lesssim 2^n\phi(0,2s)\|f\|_{\mathcal{L}_{p,\phi}} + \left|f_{B(0,2s)} - \sigma(f)\right| \to 0 \quad \text{as } s \to \infty.\end{aligned}$$

Letting $s \to \infty$ in (4.7), we have

$$\left|f_{B(a,r)} - \sigma(f)\right| \lesssim \int_r^{\infty} \frac{\phi(a,t)}{t}\,dt\|f\|_{\mathcal{L}_{p,\phi}} \lesssim \phi(a,r)\|f\|_{\mathcal{L}_{p,\phi}}.$$

Then

$$\begin{aligned}\left(\fint_{B(a,r)} |f-\sigma(f)|^p\right)^{1/p} &\le \left(\fint_{B(a,r)} |f - f_{B(a,r)}|^p\right)^{1/p} + \left|f_{B(a,r)} - \sigma(f)\right| \\ &\lesssim \phi(a,r)\|f\|_{\mathcal{L}_{p,\phi}}.\end{aligned}$$

This shows that $f - \sigma(f)$ is in $L_{p,\phi}(\mathbb{R}^n)$ and

$$\|f-\sigma(f)\|_{L_{p,\phi}} \lesssim \|f\|_{\mathcal{L}_{p,\phi}}.$$

From (9.1) it follows that

$$\|f\|_{\mathcal{L}_{p,\phi}} = \|f-\sigma(f)\|_{\mathcal{L}_{p,\phi}} \le 2\|f-\sigma(f)\|_{L_{p,\phi}}.$$

Conversely, let $f \in L_{p,\phi}(\mathbb{R}^n)$. Then $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\sigma(f) = 0$, since $|f_{B(0,r)}| \le \phi(0,r)\|f\|_{L_{p,\phi}} \to 0$ as $r \to \infty$. □

These theorems are valid for spaces of homogeneous type, see [41, 43].

10 Pointwise multipliers on Campanato spaces

In this section, we investigate the pointwise multipliers on generalized Campanato spaces $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$.

Theorem 10.1 ([33] (1993))**.** *Let $p \in [1,\infty)$. Assume that $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfies the same conditions as Lemma 4.1. Let*

$$\Phi^*(x,r) = \int_1^{\max(2,|x|,r)} \frac{\phi(0,t)}{t}\,dt, \;\; \Phi^{**}(x,r) = \int_r^{\max(2,|x|,r)} \frac{\phi(x,t)}{t}\,dt, \tag{10.1}$$

and let $\psi = \phi/(\Phi^* + \Phi^{**})$*. Then*

$$\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)) = \mathcal{L}_{p,\psi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

and

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)},$$

where $\|g\|_{\mathrm{Op}}$ *is the operator norm of* $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n))$*.*

For ϕ_i $(i = 1, 2)$, we define Φ^*_i and Φ^{**}_i by (10.1).

Theorem 10.2 ([35] (1997))**.** *Let* $1 < p_2 < p_1 < \infty$ *and* $p_1 + p_2 \le p_1 p_2$*. Assume that* ϕ_i $(i = 1, 2)$ *satisfy* (4.1), (4.2), (9.2) *and* (4.4) *with* $p = p_i$*. Assume also that*

$$\int_1^r \frac{\phi_2(x,t)}{\phi_1(x,t)} t^{n/p_2 - 1}\, dt \le A \frac{\phi_2(x,r)}{\phi_1(x,r)} r^{n/p_2}, \tag{10.2}$$

and that $(\Phi^*_2 + \Phi^{**}_2)/\phi_2 \le C(\Phi^*_1 + \Phi^{**}_1)/\phi_1$*. If* $\phi_3 = \phi_2/(\Phi^*_1 + \Phi^{**}_1)$ *is almost increasing, then*

$$\mathrm{PWM}(\mathcal{L}^\natural_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^\natural_{p_2,\phi_2}(\mathbb{R}^n)) = \mathcal{L}_{1,\phi_3}(\mathbb{R}^n) \cap L_{1,\phi_2/\phi_1}(\mathbb{R}^n)$$

and

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\mathcal{L}_{1,\phi_3}(\mathbb{R}^n)} + \|g\|_{L_{1,\phi_2/\phi_1}(\mathbb{R}^n)},$$

where $\|g\|_{\mathrm{Op}}$ *is the operator norm of* $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^\natural_{p_2,\phi_2}(\mathbb{R}^n))$*.*

In the above $L_{1,\phi_2/\phi_1}(\mathbb{R}^n)$ is the Morrey space.

Proposition 10.3 ([35] (1997))**.** *Suppose that* ϕ_1 *and* ϕ_2 *satisfy the doubling condition* (4.1)*. Let* $\phi_3 = \phi_2/(\Phi^*_1 + \Phi^{**}_1)$*. If* $1 \le p_2 < p_1 < \infty$ *and* $p_4 \ge p_1 p_2/(p_1 - p_2)$*, then*

$$\mathrm{PWM}(\mathcal{L}^\natural_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^\natural_{p_2,\phi_2}(\mathbb{R}^n)) \supset \mathcal{L}^\natural_{p_2,\phi_3}(\mathbb{R}^n) \cap L_{p_4,\phi_2/\phi_1}(\mathbb{R}^n)$$

and

$$\|g\|_{\mathrm{Op}} \le C(\|g\|_{\mathcal{L}_{p_2,\phi_3}} + \|g\|_{L_{p_4,\phi_2/\phi_1}}),$$

where $\|g\|_{\mathrm{Op}}$ *is the operator norm of* $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^\natural_{p_2,\phi_2}(\mathbb{R}^n))$*.*

Corollary 10.4 ([57] (2019))**.** *Let* $1 \le p_2 < p_1 < \infty$ *and* $1/p_4 = 1/p_2 - 1/p_1$*. Suppose that* ϕ *satisfies the doubling condition* (4.1) *and that there exists a positive constant* C_ϕ *such that*

$$\int_0^\infty \frac{\phi(x,t)}{t}\, dt \le C_\phi \quad \text{for all } x \in \mathbb{R}^n, \tag{10.3}$$

$$\int_r^\infty \frac{\phi(x,t)}{t}\, dt \le C_\phi\, \phi(x,r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r \ge 1. \tag{10.4}$$

Let

$$\psi(x,r)=\begin{cases}\phi(x,r) & r<1,\\ \phi(x,r)^2 & r\ge 1.\end{cases}$$

If $f\in\mathcal{L}^{\natural}_{p_1,\phi}(\mathbb{R}^n)$, $g\in\mathcal{L}^{\natural}_{p_4,\phi}(\mathbb{R}^n)$ *and* $\sigma(f)=\sigma(g)=0$, *then* $fg\in\mathcal{L}^{\natural}_{p_2,\psi}(\mathbb{R}^n)$, $\sigma(fg)=0$ *and*

$$\|fg\|_{\mathcal{L}^{\natural}_{p_2,\psi}}\le C\|f\|_{\mathcal{L}^{\natural}_{p_1,\phi}}\|g\|_{\mathcal{L}^{\natural}_{p_4,\phi}}. \tag{10.5}$$

For example, we can take $p_1=p_4=4$ and $p_2=2$.

11 Singular integral operators

Let $0<\kappa\le 1$. We shall consider a singular integral operator T with kernel K on $\mathbb{R}^n\times\mathbb{R}^n\setminus\{(x,x):x\in\mathbb{R}^n\}$ satisfying the following properties:

$$|K(x,y)|\le\frac{C}{|x-y|^n}\quad\text{for}\quad x\ne y, \tag{11.1}$$

$$|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)|\le\frac{C}{|x-y|^n}\left(\frac{|x-z|}{|x-y|}\right)^{\kappa}$$
$$\text{for}\quad |x-y|\ge 2|x-z|, \tag{11.2}$$

$$\int_{r\le|x-y|<R}K(x,y)\,dy=\int_{r\le|x-y|<R}K(y,x)\,dy=0$$
$$\text{for } 0<r<R<\infty \text{ and } x\in\mathbb{R}^n, \tag{11.3}$$

where C is a positive constant independent of $x,y,z\in\mathbb{R}^n$.

For $\eta>0$, let

$$T_\eta f(x)=\int_{|x-y|\ge\eta}K(x,y)f(y)\,dy.$$

Then $T_\eta f(x)$ is well defined for $f\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. We assume that, for all $1<p<\infty$, there exists positive constant C_p independently $\eta>0$ such that,

$$\|T_\eta f\|_{L^p}\le C_p\|f\|_{L^p}\quad\text{for}\quad f\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n),$$

and $T_\eta f$ converges to Tf in $L^p(\mathbb{R}^n)$ as $\eta\to 0$. By this assumption, the operator T can be extended as a continuous linear operator on $L^p(\mathbb{R}^n)$. We shall say the operator T satisfying the above conditions is a singular integral operator of type κ.

Now, to define T for functions f in Campanato spaces we first define the modified version of T_η by

$$\tilde{T}_\eta f(x)=\int_{|x-y|\ge\eta}f(y)\big[K(x,y)-K(0,y)(1-\chi_{B(0,1)}(y))\big]\,dy. \tag{11.4}$$

If ϕ satisfies (4.1) and $\int_1^\infty \frac{\phi(x,t)}{t^2}\,dt < \infty$, then we can show that the integral in the definition above converges absolutely for each x and that $\tilde{T}_\eta f$ converges in $L^p(B)$ as $\eta \to 0$ for each ball B. We denote the limit by $\tilde{T}f$. If both $\tilde{T}f$ and Tf are well defined, then the difference is a constant.

Theorem 11.1 ([44] (2010)). *Let $0 < \kappa \le 1$ and $1 < p < \infty$. Assume that ϕ satisfies* (4.1) *and that there exists a positive constant A such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$r^\kappa \int_r^\infty \frac{\phi(x,t)}{t^{1+\kappa}}\,dt \le A\phi(x,r). \tag{11.5}$$

If T is a singular integral operator of type κ, then $\tilde{T}$ is bounded on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and on $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$, that is, there exists a positive constants C such that

$$\|\tilde{T}f\|_{\mathcal{L}_{p,\phi}} \le C\|f\|_{\mathcal{L}_{p,\phi}}, \quad \|\tilde{T}f\|_{\mathcal{L}^\natural_{p,\phi}} \le C\|f\|_{\mathcal{L}^\natural_{p,\phi}}.$$

Moreover, if ϕ satisfies (4.2) *and* (9.2) *also, then $\tilde{T}$ is bounded on $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and on $\mathcal{L}^\natural_{1,\phi}(\mathbb{R}^n)$.*

For example, $\phi(x,r) = r^{\lambda(x)}$ with $-n/p \le \inf_{x\in\mathbb{R}^n} \lambda(x) \le \sup_{x\in\mathbb{R}^n} \lambda(x) < 1$ satisfies the condition (11.5).

For Morrey spaces $L_{p,\phi}(\mathbb{R}^n)$, we have the following.

Theorem 11.2 ([34] (1994)). *Let $1 < p < \infty$. Assume that ϕ satisfies* (4.1) *and that there exists a positive constant A such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le A\phi(x,r).$$

If T is a singular integral operator with kernel satisfying (11.1), *and if T is bounded on $L^p(\mathbb{R}^n)$, then T can be extended to a bounded operator on $L_{p,\phi}(\mathbb{R}^n)$.*

For example, $\phi(x,r) = r^{\lambda(x)}$ with $-n/p \le \inf_{x\in\mathbb{R}^n} \lambda(x) \le \sup_{x\in\mathbb{R}^n} \lambda(x) < 0$ satisfies the above condition.

Next we state the boundedness of the Riesz transforms particularly, which are singular integral operators of type 1. For $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ the Riesz transforms of f are defined by

$$R_j f(x) = \lim_{\varepsilon\to 0} R_{j,\varepsilon} f(x), \quad j = 1,\dots,n,$$

where

$$R_{j,\varepsilon} f(x) = c_n \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy, \quad c_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}.$$

Then it is known that, for all $1 < p < \infty$, there exists a positive constant C_p independently $\varepsilon > 0$ such that,

$$\|R_{j,\varepsilon} f\|_{L^p} \le C_p \|f\|_{L^p} \quad \text{for} \quad f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n),$$

and $R_{j,\varepsilon} f$ converges to $R_j f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$. That is, the operator R_j can be extended as a continuous linear operator on $L^p(\mathbb{R}^n)$. Hence, we can define a modified Riesz transforms of f as

$$\tilde{R}_j f(x) = \lim_{\varepsilon \to 0} \tilde{R}_{j.\varepsilon} f(x), \quad j = 1, \ldots, n,$$

and

$$\tilde{R}_{j.\varepsilon} f(x) = c_n \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{(-y_j)(1 - \chi_{B(0,1)}(y))}{|y|^{n+1}} \right) f(y)\, dy.$$

We note that, if both $R_j f$ and $\tilde{R}_j f$ are well defined on $\mathbb{R}^n$, then $R_j f - \tilde{R}_j f$ is a constant function. More precisely,

$$R_j f(x) - \tilde{R}_j f(x) = c_n \int_{\mathbb{R}^n} \frac{(-y_j)(1 - \chi_{B(0,1)}(y))}{|y|^{n+1}} f(y)\, dy.$$

Theorem 11.3 ([57] (2019)). *Let $1 < p < \infty$. Assume that ϕ satisfies* (4.1) *and that there exists a positive constant A such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$r \int_r^\infty \frac{\phi(x,t)}{t^2}\, dt \le A\phi(x,r). \tag{11.6}$$

Assume also that there exists a growth function $\tilde{\phi}$ such that $\phi \le \tilde{\phi}$ and that $\tilde{\phi}$ satisfies (4.1), (4.2) *and* (9.5). *If $f \in \mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$, then, for each $j = 1, 2, \ldots, n$, $R_j f$ is well defined, $\sigma(R_j f) = \lim_{r\to\infty} (R_j f)_{B(0,r)} = 0$, and $\|R_j f\|_{\mathcal{L}^\natural_{p,\phi}} \le C \|f\|_{\mathcal{L}^\natural_{p,\phi}}$, where the constant C is independent of f.*

Remark 11.1. From Theorem 11.3 we conclude that, under the assumption, if $f \in \mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$, then $R_i R_j f$ is well defined and

$$\|R_i R_j f\|_{\mathcal{L}^\natural_{p,\phi}} \le C \|f\|_{\mathcal{L}^\natural_{p,\phi}}, \quad i, j = 1, \ldots, n.$$

12 Convolution with the heat kernel

Let

$$h_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \quad \text{for} \quad x \in \mathbb{R}^n,\ t \in (0,\infty). \tag{12.1}$$

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}. \tag{12.2}$$

Theorem 12.1 ([57] (2019)). *Let $1 \le p_2 \le p_1 < \infty$. Assume that ϕ satisfies (4.1) and (11.6). Then there exists a positive constant C such that, for all $t \in (0,\infty)$ and $f \in \mathcal{L}_{p_2,\phi}(\mathbb{R}^n)$,*

$$\|h_t * f\|_{\mathcal{L}_{p_1,\theta}} \le C(1+t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}_{p_2,\phi}},$$
$$\|(\nabla h_t) * f\|_{\mathcal{L}_{p_1,\theta}} \le Ct^{-1/2}(1+t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}_{p_2,\phi}},$$

where $\theta(x,r) = (1+r^{(1/p_2-1/p_1)n})\phi(x,r)$. Moreover, if there exists a positive constant C_ϕ such that, for all $x \in \mathbb{R}^n$, $\int_1^\infty \frac{\phi(x,t)}{t}\,dt \le C_\phi$, then there exists a positive constant C such that, for all $t \in (0,\infty)$ and $f \in \mathcal{L}^\natural_{p_2,\phi}(\mathbb{R}^n)$,

$$\|h_t * f\|_{\mathcal{L}^\natural_{p_1,\theta}} \le C(1+t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\phi}},$$
$$\|(\nabla h_t) * f\|_{\mathcal{L}^\natural_{p_1,\theta}} \le Ct^{-1/2}(1+t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\phi}}.$$

Further, if $\lim\limits_{r\to\infty} \sup\limits_{x\in\mathbb{R}^n} \int_r^\infty \frac{\phi(x,t)}{t}\,dt = 0$, then $\sigma(f) = 0$ implies $\sigma(h_t * f) = \sigma((\nabla h_t) * f) = 0$.

Theorem 12.2 ([57] (2019)). *Let $1 \le p_2 \le p_1 < \infty$. Assume that ψ satisfies (4.1) and (11.6) and that there exists a positive constant C_ψ such that, for all $x \in \mathbb{R}^n$, $\int_0^\infty \frac{\psi(x,t)}{t}\,dt \le C_\psi$. Assume also that $\lim\limits_{r\to\infty} \sup\limits_{x\in\mathbb{R}^n} \int_r^\infty \frac{\psi(x,t)}{t}\,dt = 0$. Let*

$$\phi(x,r) = \begin{cases} \psi(x,r) & r < 1, \\ \psi(x,r)^{p_2/p_1} & r \ge 1. \end{cases}$$

Then, for $f \in \mathcal{L}^\natural_{p_2,\psi}(\mathbb{R}^n)$ with $\sigma(f) = 0$, then $\sigma(h_t * f) = \sigma((\nabla h_t) * f) = 0$ and

$$\|h_t * f\|_{\mathcal{L}^\natural_{p_1,\phi}} \le C(1+t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\psi}},$$
$$\|(\nabla h_t) * f\|_{\mathcal{L}^\natural_{p_1,\phi}} \le Ct^{-1/2}(1+t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\psi}}.$$

13 An application: The Cauchy problem for the Navier-Stokes equation

The Navier-Stokes equation is expressed as

$$\begin{cases} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = 0 & \text{in } \mathbb{R}^n \times [0,T), \\ \nabla \cdot v = 0 & \text{in } \mathbb{R}^n \times [0,T), \\ v|_{t=0} = v_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{13.1}$$

where $v = (v_1, \dots, v_n)$ is a vector field representing velocity of the fluid, p is the pressure, and

$$\nabla \cdot v = \sum_{j=1}^{n} \partial_j v_j, \quad v \cdot \nabla = \sum_{j=1}^{n} v_j \partial_j, \quad \Delta = \sum_{j=1}^{n} \partial_j^2.$$

It is known that the pair of solutions (v, p) satisfies the relation

$$p = \sum_{i,j=1}^{n} R_i R_j (v_i v_j),$$

where the operators R_j $(j = 1, \dots, n)$ are the Riesz transforms (see [18, 24, 56] for example). Therefore, to estimate the solutions in some function space we need the properties of the Riesz transforms and pointwise multipliers (pointwise product operators) on the function space. Namely, we need the following norm boundedness:

$$\|fg\|_{\mathcal{L}^{\natural}_{q,\psi}} \le C \|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \|g\|_{\mathcal{L}^{\natural}_{p,\phi}}, \tag{13.2}$$

$$\|R_j f\|_{\mathcal{L}^{\natural}_{q,\psi}} \le C \|f\|_{\mathcal{L}^{\natural}_{q,\psi}}, \tag{13.3}$$

for Campanato spaces $\mathcal{L}^{\natural}_{p,\phi}$ and $\mathcal{L}^{\natural}_{q,\psi}$ with variable growth condition.

To solve (13.1) we consider the following equations:

$$u(t) = e^{t\Delta} u_0 + Gu(t),$$
$$Gu(t) = -\int_0^t \nabla e^{-(t-s)\Delta} P(u \otimes u)(s)\, ds,$$

where P is the Helmholtz projection; $P = (\delta_{jk} + R_j R_k)_{1 \le j,k \le n}$. Then we also need the estimate of the convolution with the heat kernel.

Using Theorems 11.3, 12.1, 12.2 and Corollary 10.4, we have the following theorem:

Theorem 13.1 ([57] (2019)). *Let* $\max(2, n) < p < \infty$*,* $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ *and*

$$\psi(x, r) = \begin{cases} \phi(x, r) & r < 1, \\ \phi(x, r)^2 & r \ge 1. \end{cases}$$

Assume that ϕ *and* ψ *satisfy* (4.1) *and* (11.6) *and that*

$$\int_0^\infty \frac{\phi(x, t)}{t}\, dt \le C_\phi \quad \text{for all } x \in \mathbb{R}^n,$$
$$\int_r^\infty \frac{\phi(x, t)}{t}\, dt \le C_\phi\, \phi(x, r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r \ge 1,$$
$$\lim_{r \to \infty} \sup_{x \in \mathbb{R}^n} \phi(x, r) = 0.$$

Assume also that there exists a growth function $\tilde{\psi}$ such that $\psi \le \tilde{\psi}$, that $\tilde{\psi}$ satisfies (4.1), (4.2) *and* (9.5). *Then, for all $u_0 \in (\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n))^n$ such that $\nabla \cdot u_0 = 0$ and $\sigma(u_0) = \lim_{r\to\infty}(u_0)_{B(0,r)} = 0$, there exist a positive constant T (depending only on the norm of initial data) and a unique solution $u \in C([0,T);(\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n))^n)$ to* (13.1).

For example, let $p > \max(2,n)$, $\alpha(\cdot) : \mathbb{R}^n \to (0,1)$, $\beta(\cdot) : \mathbb{R}^n \to [-n/p, 0)$, and let

$$\phi(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r \le 1, \\ r^{\beta(x)}, & r > 1, \end{cases} \qquad \psi(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r \le 1, \\ r^{2\beta(x)}, & r > 1, \end{cases}$$

and $\tilde{\psi}(x,r) = r^{2\beta_+}$, where $\alpha(\cdot)$, $\beta(\cdot)$ and β_+ satisfy

$$0 < \inf_{x\in\mathbb{R}^n} \alpha(x) \le \sup_{x\in\mathbb{R}^n} \alpha(x) < 1,$$
$$-n/p \le \inf_{x\in\mathbb{R}^n} \beta(x) \le \sup_{x\in\mathbb{R}^n} \beta(x) = \beta_+ < 0.$$

Then ϕ, ψ and $\tilde{\psi}$ satisfy the assumption in Theorem 13.1.

For other applications of generalized Campanato spaces, see [56, 57].

14 Generalized fractional integral operators on generalized Morrey spaces

In this section we state the boundedness of generalized fractional integral operators on generalized Morrey spaces. We also state on generalized fractional maximal operators.

In this and the next sections, we use the symbols $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ and $L^{(p,\varphi)}(\mathbb{R}^n)$ instead of $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$:

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$
$$\|f\|_{\mathcal{L}^{(p,\varphi)}} = \sup_B \left(\frac{1}{\varphi(B)} \fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$
$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\fint_B |f(y)|^p \, dy \right)^{1/p},$$
$$\|f\|_{L^{(p,\varphi)}} = \sup_B \left(\frac{1}{\varphi(B)} \fint_B |f(y)|^p \, dy \right)^{1/p}.$$

Note that $\mathcal{L}_{1,\varphi}(\mathbb{R}^n) = \mathcal{L}^{(1,\varphi)}(\mathbb{R}^n)$ and $L_{p,\varphi}(\mathbb{R}^n) = L^{(p,\varphi^p)}(\mathbb{R}^n)$.

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\theta(x,r) \le C\theta(x,s) \quad (\text{resp. } C\theta(x,r) \ge \theta(x,s)), \quad \text{if } r < s.$$

In this and the next sections we consider the following classes of φ:

Definition 14.1. Let $\mathcal{G}^{\text{dec}}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that φ is almost decreasing and that $r \mapsto \varphi(x,r)r^n$ is almost increasing. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$C\varphi(x,r) \geq \varphi(x,s), \quad \varphi(x,r)r^n \leq C\varphi(x,s)s^n, \quad \text{if } r < s.$$

Definition 14.2. Let $\mathcal{G}^{\text{inc}}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that φ is almost increasing and that $r \mapsto \varphi(x,r)/r$ is almost decreasing. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\varphi(x,r) \leq C\varphi(x,s), \quad C\varphi(x,r)/r \geq \varphi(x,s)/s, \quad \text{if } r < s.$$

If $\varphi \in \mathcal{G}^{\text{dec}}$ or $\varphi \in \mathcal{G}^{\text{inc}}$, then φ satisfies the doubling condition (4.1).

First we state the boundedness of the Hardy-Littlewood maximal operator M. It is defined by the following: For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let

$$Mf(x) = \sup_{B \ni x} ⨍_B |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls B containing x.

Theorem 14.1 ([45] (2014))**.** *Let $1 < p < \infty$ and $\varphi \in \mathcal{G}^{\text{dec}}$. Then the operator M is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to itself.*

For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we consider generalized fractional integral operators I_ρ defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(x,|x-y|)}{|x-y|^n} f(y)\, dy, \tag{14.1}$$

where we always assume that

$$\int_0^1 \frac{\rho(x,t)}{t}\, dt < \infty \quad \text{for each } x \in \mathbb{R}^n, \tag{14.2}$$

and that there exist positive constants C, K_1 and K_2 with $K_1 < K_2$ such that, for all $x \in \mathbb{R}^n$ and $r > 0$,

$$\sup_{r \leq t \leq 2r} \rho(x,t) \leq C \int_{K_1 r}^{K_2 r} \frac{\rho(x,t)}{t}\, dt. \tag{14.3}$$

The assumption (14.2) is needed so that $I_\rho f$ is well defined for all $f \in L^\infty_{\text{comp}}(\mathbb{R}^n)$. The condition (14.3) comes from [60, p. 664 (D)].

If $\rho(x,r)=r^{\alpha}$, then I_ρ is the usual fractional integral operator I_α. It is known as the Hardy-Littlewood-Sobolev theorem that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha\in(0,n)$, $p,q\in(1,\infty)$ and $-n/p+\alpha=-n/q$. If $\alpha(\cdot):\mathbb{R}^n\to(0,n)$ and $\rho(x,r)=r^{\alpha(x)}$, then I_ρ is a generalized fractional integral operator $I_{\alpha(x)}$ with variable order defined by

$$I_{\alpha(x)}f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha(x)}}\,dy.$$

The operator I_ρ was introduced in [36, 37] (2000, 2001) with $\phi:(0,\infty)\to(0,\infty)$ to extend the Hardy-Littlewood-Sobolev theorem to Orlicz spaces.

Theorem 14.2 ([45] (2014))**.** *Let $1<p<q<\infty$ and $\rho,\varphi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$. Assume that ρ satisfies* (14.2) *and* (14.3) *and that φ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies*

$$\lim_{r\to 0}\varphi(x,r)=\infty,\qquad \lim_{r\to\infty}\varphi(x,r)=0. \tag{14.4}$$

Assume also that there exists a positive constant C such that, for all $x\in\mathbb{R}^n$ and $r\in(0,\infty)$,

$$\int_0^r\frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/p}+\int_r^\infty\frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt\le C\varphi(x,r)^{1/q}. \tag{14.5}$$

Then I_ρ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

To prove the theorem above we use Hedberg's method in [20] and the boundedness of the Hardy-Littlewood maximal operator.

Next, we consider fractional maximal operators. For a function $\rho:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$, let

$$M_\rho f(x)=\sup_{B\ni x}\rho(B)\,⨍_B|f(y)|\,dy,\quad x\in\mathbb{R}^n, \tag{14.6}$$

where the supremum is taken over all balls B containing x. We do not postulate the condition (14.2) or (14.3) on the definition of M_ρ. The operator M_ρ was defined on Orlicz spaces in [38] (2001), and studied by Sawano, Sugano and Tanaka [62] (2011) on Morrey spaces in case of $\rho:(0,\infty)\to(0,\infty)$. If $\rho(B)=|B|^{\alpha/n}$, then M_ρ is the usual fractional maximal operator M_α. If $\rho\equiv 1$, then M_ρ is the Hardy-Littlewood maximal operator M.

If $\rho(x,r)/r^n\le C\rho(x,s)/s^n$ for $0<s<r<\infty$, then

$$M_\rho f(x)\le CI_\rho|f|(x),\quad x\in\mathbb{R}^n. \tag{14.7}$$

Hence, the boundedness of M_ρ follows from the boundedness of I_ρ. For example, the Hardy-Littlewood-Sobolev theorem yields that M_α is bounded from $L^p(\mathbb{R}^n)$ to

$L^q(\mathbb{R}^n)$, if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p+\alpha = -n/q$. However, for example, if

$$\rho(x,r) = \begin{cases} (\log(e+1/r))^{-\beta} & (0<r<1) \\ (\log(e+r)^{\gamma} & (r \ge 1), \end{cases} \qquad \beta > 1,\ \gamma > 0,$$

then it turns out that the boundedness of M_ρ is better than the boundedness of I_ρ by the following theorem. Actually, (14.5) cannot be replaced by (14.8), see [17, Theorem 1.1].

Theorem 14.3 ([1] (2018))**.** *Let $1 < p < q < \infty$ and $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies* (14.4)*. Assume also that there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\rho(x,r)\varphi(x,r)^{1/p} \le C_0 \varphi(x,r)^{1/q}. \tag{14.8}$$

Then M_ρ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

For the boundedness of I_ρ and M_ρ on Orlicz-Morrey spaces, see [40, 42]. For the boundedness of I_ρ on Campanato spaces, see [39, 16].

15 Commutators of integral operators with functions in Campanato spaces

It is known that any Calderón-Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. In 1976 Coifman, Rochberg and Weiss [10] proved that the commutator $[b,T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b,T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. For the fractional integral operator I_α, Chanillo [8] proved the boundedness of $[b, I_\alpha]$ in 1982. That is,

$$\|[b,I_\alpha]f\|_{L^q} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p+\alpha = -n/q$. These results were extended to Morrey spaces by Di Fazio and Ragusa [15] in 1991.

In this section we state the boundedness of the commutators $[b,T]$ and $[b,I_\rho]$ on generalized Morrey spaces with variable growth condition, where T is a Calderón-Zygmund operator, I_ρ is a generalized fractional integral operator and b is a function in generalized Campanato spaces with variable growth condition.

First we recall the definition of Calderón-Zygmund operators following [64]. Let Ω be the set of all nonnegative nondecreasing functions ω on $(0,\infty)$ such that $\int_0^1 \frac{\omega(t)}{t}dt < \infty$.

Definition 15.1 (standard kernel of type ω)**.** Let $\omega \in \Omega$. A continuous function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \in \mathbb{R}^{2n}\}$ is said to be a standard kernel of type ω if the following conditions are satisfied;

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \ne y, \tag{15.1}$$

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le \frac{C}{|x-y|^n}\,\omega\left(\frac{|y-z|}{|x-y|}\right) \qquad \text{for} \quad 2|y-z| \le |x-y|. \tag{15.2}$$

Definition 15.2 (Calderón-Zygmund operator)**.** Let $\omega \in \Omega$. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator of type ω, if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K of type ω such that, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy, \quad x \notin \operatorname{supp} f. \tag{15.3}$$

It is known by [64, Theorem 2.4] that any Calderón-Zygmund operator of type $\omega \in \Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

This result was extended to generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function φ by [34] as the following: Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and that there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\int_r^\infty \frac{\varphi(x,t)}{t}\,dt \le C\varphi(x,r). \tag{15.4}$$

For $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, $1 < p < \infty$, we define Tf on each ball B by

$$Tf(x) = T(f\chi_{2B})(x) + \int_{\mathbb{R}^n \setminus 2B} K(x,y) f(y)\,dy, \quad x \in B. \tag{15.5}$$

Then the first term in the right hand side is well defined, since $f\chi_{2B} \in L^p(\mathbb{R}^n)$, and the integral of the second term converges absolutely. Moreover, $Tf(x)$ is independent of the choice of the ball containing x. By this definition we can show that T is a bounded operator on $L^{(p,\varphi)}(\mathbb{R}^n)$. For the definition of Tf, see also [53, Section 5] and [61].

For functions f in Morrey spaces, we define $[b,T]f$ on each ball B by

$$[b,T]f(x) = [b,T](f\chi_{2B})(x) + \int_{\mathbb{R}^n \setminus 2B} (b(x) - b(y)) K(x,y) f(y)\,dy, \quad x \in B. \tag{15.6}$$

Then we have the following theorem.

Theorem 15.1 ([1] (2018))**.** *Let $1 < p \le q < \infty$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Let T be a Calderón-Zygmund operator of type $\omega \in \Omega$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

(i) *Assume that ψ satisfies (4.2), that φ satisfies (15.4), that $\int_0^1 \frac{\omega(t)\log(1/t)}{t}\,dt < \infty$ and that there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\psi(x,r)\varphi(x,r)^{1/p} \le C_0 \varphi(x,r)^{1/q}. \tag{15.7}$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then $[b,T]f$ in (15.6) is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|[b,T]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that φ satisfies (4.2) and that there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$C_0\psi(x,r)\varphi(x,r)^{1/p} \ge \varphi(x,r)^{1/q}. \tag{15.8}$$

If T is a convolution type such that

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)\,dy \tag{15.9}$$

with homogeneous kernel K satisfying $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, and if $[b,T]$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C\|[b,T]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}},$$

where $\|[b,T]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}}$ is the operator norm of $[b,T]$ from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

In the above theorem, if $\psi \equiv 1$ and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ and $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $p = q$. This is Coifman, Rochberg and Weiss's result in [10]. If $\psi(x,r) = r^\alpha$, $0 < \alpha \le 1$, and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{(q,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ with $-n/p + \alpha = -n/q$. That is,

$$\|[b,T]f\|_{L^q} \lesssim \|b\|_{\mathrm{Lip}_\alpha}\|f\|_{L^p}.$$

This is Janson's result in [22, Lemma 12].

Example 15.1 ([1] (2018))**.** Let $1 < p \le q < \infty$ and $\beta(\cdot), \lambda(\cdot) : \mathbb{R}^n \to (-\infty,\infty)$. Assume that

$$0 \le \inf_{x\in\mathbb{R}^n} \beta(x) \le \sup_{x\in\mathbb{R}^n} \beta(x) \le 1, \quad 0 \le \beta_* \le 1,$$
$$-n \le \inf_{x\in\mathbb{R}^n} \lambda(x) \le \sup_{x\in\mathbb{R}^n} \lambda(x) < 0, \quad -n \le \lambda_* < 0.$$

Let

$$\psi(x,r)=\begin{cases} r^{\beta(x)}, \\ r^{\beta_*}, \end{cases} \quad \varphi(x,r)=\begin{cases} r^{\lambda(x)}, & 0<r<1, \\ r^{\lambda_*}, & 1\le r<\infty. \end{cases}$$

Let T be a Calderón-Zygmund operator of type $\omega \in \Omega$ with $\int_0^1 \frac{\omega(t)\log(1/t)}{t}\,dt<\infty$. If $\beta(\cdot)$ is log-Hölder continuous and

$$\beta(x)+\lambda(x)/p \ge \lambda(x)/q, \quad \beta_*+\lambda_*/p \le \lambda_*/q,$$

then ψ and ϕ satisfy the assumption in Theorem 15.1 (i) and then the inequality

$$\|[b,T]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}$$

holds. Conversely, if $\lambda(\cdot)$ is log-Hölder continuous and

$$\beta(x)+\lambda(x)/p \le \lambda(x)/q, \quad \beta_*+\lambda_*/p \ge \lambda_*/q,$$

and if T is a convolution type with homogeneous kernel K satisfying $K(x)=|x|^{-n}K(x/|x|)$, $\int_{S^{n-1}}K=0$, $K\in C^\infty(S^{n-1})$ and $K\not\equiv 0$, then we have

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}.$$

We also consider the cases

$$\psi(x,r)=\begin{cases} r^{\beta(x)}(1/\log(e/r))^{\beta_1(x)}, & 0<r<1, \\ r^{\beta_*}(\log(er))^{\beta_{**}}, & 1\le r<\infty, \end{cases}$$

etc.

For the commutator $[b,I_\rho]$ we have the following theorem.

Theorem 15.2 ([1] (2018))**.** *Let $1<p<q<\infty$ and $\rho,\varphi,\psi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$. Assume that $\varphi\in\mathcal{G}^{\rm dec}$ and $\psi\in\mathcal{G}^{\rm inc}$. Assume also that ρ satisfies* (14.2) *and* (14.3)*. Let $\rho^*(x,r)=\int_0^r\frac{\rho(x,t)}{t}\,dt$ and $b\in L^1_{\rm loc}(\mathbb{R}^n)$.*

(i) *Assume that ρ, ρ^* and ψ satisfy* (4.2)*, that φ satisfies* (15.4) *and that there exist positive constants ϵ, C_ρ, C_0, C_1 and an exponent $\tilde{p}\in(p,q]$ such that, for all $x,y\in\mathbb{R}^n$ and $r,s\in(0,\infty)$,*

$$C_\rho\,\frac{\rho(x,r)}{r^{n-\epsilon}} \ge \frac{\rho(x,s)}{s^{n-\epsilon}}, \ \text{if } r<s, \tag{15.10}$$

$$\left|\frac{\rho(x,r)}{r^n}-\frac{\rho(y,s)}{s^n}\right| \le C_\rho\left(|r-s|+|x-y|\right)\frac{\rho^*(x,r)}{r^{n+1}}, \tag{15.11}$$

$$\text{if } \frac{1}{2}\le\frac{r}{s}\le 2 \text{ and } |x-y|<r/2,$$

$$\int_0^r\frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/p}+\int_r^\infty\frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \le C_0\varphi(x,r)^{1/\tilde{p}}, \tag{15.12}$$

$$\psi(x,r)\varphi(x,r)^{1/\tilde{p}} \le C_1\varphi(x,r)^{1/q}. \tag{15.13}$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then $[b, I_\rho]f$ is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that φ satisfies (4.2), that $\rho(x,r) = r^\alpha$, $0 < \alpha < n$, and that*

$$C_0\psi(x,r)r^\alpha\,\varphi(x,r)^{1/p} \ge \varphi(x,r)^{1/q}. \tag{15.14}$$

If $[b, I_\alpha]$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C\|[b, I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}},$$

where $\|[b, I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ is the operator norm of $[b, I_\alpha]$ from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

In the above theorem, if $\rho(x,r) = r^\alpha$, $0 < \alpha < n$, $\psi \equiv 1$ and $\varphi(x,r) = r^{-n}$, then $I_\rho = I_\alpha$, $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$, $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{(q,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$. This is Chanillo's result in [8]. See also [25].

Example 15.2 ([1] (2018))**.** Let $1 < p < \tilde{p} \le q < \infty$ and $\alpha(\cdot), \beta(\cdot), \lambda(\cdot) : \mathbb{R}^n \to (-\infty, \infty)$. Assume that

$$\begin{aligned}
0 &< \inf_{x\in\mathbb{R}^n} \alpha(x) \le \sup_{x\in\mathbb{R}^n} \alpha(x) < n, \quad 0 < \alpha_* < n,\\
0 &\le \inf_{x\in\mathbb{R}^n} \beta(x) \le \sup_{x\in\mathbb{R}^n} \beta(x) \le 1, \quad 0 \le \beta_* \le 1,\\
-n &\le \inf_{x\in\mathbb{R}^n} \lambda(x) \le \sup_{x\in\mathbb{R}^n} \lambda(x) < 0, \quad -n \le \lambda_* < 0.
\end{aligned}$$

Let

$$\rho(x,r) = \begin{cases} r^{\alpha(x)}, \\ r^{\alpha_*}, \end{cases} \quad \psi(x,r) = \begin{cases} r^{\beta(x)}, \\ r^{\beta_*}, \end{cases} \quad \varphi(x,r) = \begin{cases} r^{\lambda(x)}, & 0 < r < 1, \\ r^{\lambda_*}, & 1 \le r < \infty. \end{cases}$$

If $\alpha(\cdot)$ is Lipschitz continuous, $\beta(\cdot)$ is log-Hölder continuous and

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} (\alpha(x) + \lambda(x)/p) < 0,\\
&\alpha(x) + \lambda(x)/p \ge \lambda(x)/\tilde{p}, \quad \alpha_* + \lambda_*/p \le \lambda_*/\tilde{p},\\
&\beta(x) + \lambda(x)/\tilde{p} \ge \lambda(x)/q, \quad \beta_* + \lambda_*/\tilde{p} \le \lambda_*/q,
\end{aligned}$$

then

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Conversely, if $\lambda(\cdot)$ is log-Hölder continuous, α is constant and

$$\alpha+\beta(x)+\lambda(x)/p\le\lambda(x)/q,\quad \alpha+\beta_*+\lambda_*/p\ge\lambda_*/q,$$

then

$$\|b\|_{\mathcal{L}^{(1,\psi)}}\le C\|[b,I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}.$$

We also take the cases

$$\rho(x,r)=\begin{cases} r^{\alpha(x)}(1/\log(e/r))^{\alpha_1(x)}, & 0<r<1,\\ r^{\alpha_*}(\log(er))^{\alpha_{**}}, & 1\le r<\infty,\end{cases}$$

$$\rho(x,r)=\begin{cases} r^{\alpha(x)}, & 0<r<1,\\ e^{-(r-1)}, & 1\le r<\infty,\end{cases}$$

etc.

To prove Theorems above we use the following three propositions and a corollary. For $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$M^\sharp f(x)=\sup_{B\ni x}\fint_B|f(y)-f_B|\,dy,\quad x\in\mathbb{R}^n, \tag{15.15}$$

where the supremum is taken over all balls B containing x.

Proposition 15.3 ([1] (2018)). *Let $p,\eta\in(1,\infty)$, $\varphi\in\mathcal{G}^{\mathrm{dec}}$ and $\psi\in\mathcal{G}^{\mathrm{inc}}$. Let T be a Calderón-Zygmund operator of type ω. Assume that ψ satisfies (4.2), that φ satisfies (15.4), that $\int_0^1\frac{\omega(t)\log(1/t)}{t}\,dt<\infty$ and that $\int_r^\infty\frac{\psi(x,t)\varphi(x,t)^{1/p}}{t}\,dt<\infty$ for each $x\in\mathbb{R}^n$ and $r>0$. Then there exists a positive constant C such that, for all $b\in\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, $f\in L^{(p,\varphi)}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,*

$$M^\sharp[b,T]f(x)\le C\|b\|_{\mathcal{L}^{(1,\psi)}}\left(\big(M_{\psi^\eta}(|Tf|^\eta)(x)\big)^{1/\eta}+\big(M_{\psi^\eta}(|f|^\eta)(x)\big)^{1/\eta}\right). \tag{15.16}$$

Proposition 15.4 ([1] (2018)). *Let $p,\eta\in(1,\infty)$, $\varphi\in\mathcal{G}^{\mathrm{dec}}$ and $\psi\in\mathcal{G}^{\mathrm{inc}}$. Assume that $\rho:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ satisfies (14.2) and (14.3). Let $\rho^*(x,r)=\int_0^r\frac{\rho(x,t)}{t}\,dt$. Assume that ρ, ρ^* and ψ satisfy (4.2), that φ satisfies (15.4) and that there exist positive constants ϵ, C_ρ such that (15.10) and (15.11) hold. Assume also that*

$$\int_r^\infty\frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt<\infty,\quad \int_r^\infty\frac{\psi(x,t)}{t}\left(\int_t^\infty\frac{\rho(x,u)\varphi(x,u)^{1/p}}{u}\,du\right)dt<\infty, \tag{15.17}$$

for each $x\in\mathbb{R}^n$ and $r>0$. Then there exists a positive constant C such that, for all $b\in\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, $f\in L^{(p,\varphi)}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$M^\sharp([b,I_\rho]f)(x)\le C\|b\|_{\mathcal{L}^{(1,\psi)}}\left(\big(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\big)^{1/\eta}+\big(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\big)^{1/\eta}\right). \tag{15.18}$$

Proposition 15.5 ([1] (2018))**.** *Let* $1 \le p < \infty$ *and* $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$*. If* φ *satisfies the doubling condition* (4.1)*, then, for* $f \in L^1_{\rm loc}(\mathbb{R}^n)$*,*

$$\|f\|_{\mathcal{L}^{(p,\varphi)}} \le C\|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{15.19}$$

where C *is a positive constant independent of* f*.*

By Theorem 9.3 we have the following corollary.

Corollary 15.6 ([1] (2018))**.** *Let* $1 \le p < \infty$ *and* $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$*. Assume that* $\varphi \in \mathcal{G}^{\rm dec}$ *and that* φ *satisfies* (15.4)*. For* $f \in L^1_{\rm loc}(\mathbb{R}^n)$*, if* $\lim_{r\to\infty} f_{B(0,r)} = 0$*, then*

$$\|f\|_{L^{(p,\varphi)}} \le C\|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{15.20}$$

where C *is a positive constant independent of* f*.*

Then, using Propositions 15.3 and 15.4, Corollary 15.6 and the boundedness of T, I_ρ and M_ρ (Theorems 11.2, 14.2 and 14.3, respectively), we have

$$\begin{aligned} \|[b,T]f\|_{L^{(q,\varphi)}} &\lesssim \|M^\sharp([b,T]f)\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}, \\ \|[b,I_\rho]f\|_{L^{(q,\varphi)}} &\lesssim \|M^\sharp([b,I_\rho]f)\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}. \end{aligned}$$

These shows Theorem 15.1 (i) and Theorem 15.2 (i). The parts (ii) in Theorems 15.1 and 15.2 are proved by Janson's method in [22].

We also have the compactness of $[b,T]$ and $[b,I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$, see [2, 3].

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