Hydrodynamic boundary value problem of mean field equations on annular domains

金沢大学・理工研究域数物科学系　大塚　浩史

Hiroshi Ohtsuka*
Faculty of Mathematics and Physics,
Institute of Science and Engineering,
Kanazawa University†

1 Introduction

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain whose boundary is the disjoint union of a finite number of smooth Jordan curves Γ_i $(i = 0, 1, \cdots, K)$, that is, $\partial\Omega = \cup_{i=0}^{K}\Gamma_k$. We assume that Ω is the bounded region with boundary Γ_0 with the regions bounded by Γ_i $(i = 1, \cdots, K)$ removed. In this domain, we consider the so-called *mean field* equation

$$-\Delta u = \sigma \frac{e^u}{\int_\Omega e^{u(x)}dx} \quad \text{in } \Omega \tag{1.1}$$

for a parameter $\sigma \in \mathbf{R}$ with the following boundary conditions:

$$u(x) \equiv \mu_i \quad \text{on } \Gamma_i,$$

where $\mu_0 = 0$ and $\mu_i \in \mathbf{R}$ $(i = 1, \cdots, K)$. In this note, we refer to this problem as *hydrodynamic boundary value problems.*

* ohtsuka@se.kanazawa-u.ac.jp
† Kakuma-machi, Kanazawa-shi, Ishikawa, 920-1192, JAPAN

We note that a solution of the problem corresponds to *the stream function* for an incompressible fluid motion in Ω that is tangential at the boundary $\partial\Omega$. Indeed,

$$\omega := \sigma \frac{e^u}{\int_\Omega e^{u(x)} dx}$$

is assumed to be a vorticity field of such a motion. More precisely, the vector field $\boldsymbol{v} =^t (v_1, v_2)$ determined by

$$\boldsymbol{v} := \nabla^\perp u = J\nabla u \quad \text{for} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

satisfies

$$\nabla \cdot \boldsymbol{v} := \frac{\partial}{\partial x_1} v_1 + \frac{\partial}{\partial x_2} v_2 = 0 \quad \text{in } \Omega, \qquad \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega,$$

where $x = (x_1, x_2)$ and $\boldsymbol{\nu}$ is the outer normal vector field to $\partial\Omega$. We note that when $K = 0$, that is, Ω is simply connected, the problem reduces to the usual Dirichlet boundary value problem. For $K \geq 1$ cases, however, the previous studies for the mean field equations seem to be concerned with the case $\mu_1 = \cdots = \mu_K = 0$ only. It is true that the mean field equations appear in the various fields of mathematical science, but we note that the origin of the name, *mean field*, comes from the *mean field* limit of the equilibrium vortices in the context of hydrodynamic motions. Therefore it seems natural to consider the general *hydrodynamic boundary value problems* .

In this note, we summarize several facts for the simplest annular case $\Omega = \Lambda_a := \{x \in \mathbf{R}^2 \mid a < |x| < 1\}$ for $a \in (0, 1)$, that is,

$$-\Delta u = \sigma \frac{e^u}{\int_{\Lambda_a} e^{u(x)} dx} \quad \text{in } \Lambda_a \tag{1.2}$$

for a parameter $\sigma \in \mathbf{R}$ with the following boundary conditions:

$$u(x) \equiv \mu \quad \text{on } \partial B_a(0) \text{ and } u(x) \equiv 0 \quad \text{on } \partial B_1(0).$$

Especally, we study the radially symmetric solutions for this problem.

We note that this problem is referred to in [1, p.514] and the following fact are asserted without proof:

Theorem 1.1 ([1])**.** For every $\mu \in \mathbf{R}$ and $\sigma \in \mathbf{R}$, there exists a unique radial solution to the problem (1.2).

Remark 1.2. We note that it is also asserted without proof that "in order to recover the solution in disk from the solution in the annulus in the limit $a \longrightarrow 0$, we need to prescribe a large value of μ when σ is close to 8π".[*1]

In this note, we start with the proof of Theorem 1.1 and give an interpritation of the assertion mentioned in Remark 1.2.

2 The ODE corresponding to the problem (1.2)

To simplify the presentation, we set $\Lambda := B_1(0)$. In [1, section 5], the problems (1.1) for $\Omega = \Lambda$ and (1.2) for $\mu = 0$ are studied, which we will review here.

For both cases, the problem reduces to the following ordinary differential equation:

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}u\right) = \frac{\sigma}{Z}e^u. \tag{2.1}$$

where $u = u(|x|)$ and

$$Z = \int_\Omega e^{u(|x|)}dx = 2\pi\int_a^1 e^{u(r)}rdr. \tag{2.2}$$

From the transformation $t = \log r$ attributed to R. Emden, the problem (2.1) reduces to

$$\frac{d^2H}{dt^2} + \frac{\sigma}{Z}e^H = 0, \tag{2.3}$$

where

$$H(t) = u\left(e^t\right) + 2t.$$

Therefore we get

$$\frac{1}{2}\left(\frac{dH}{dt}\right)^2 + \frac{\sigma}{Z}e^H = E \tag{2.4}$$

for some constant $E \in \mathbf{R}$ from integration of (2.3).

The equation (2.4) can be solved easily. Indeed, from (2.3) and (2.4), it holds that

$$\frac{1}{2}\left(\frac{dH}{dt}\right)^2 - \frac{d^2H}{dt^2} = E,$$

[*1] In the original terminology, our u, σ, and μ are $-\beta\psi$, $-\beta$, and $-\beta\alpha$, respectively, that is, the original statement is as follows: we need to prescribe a large value of α when β is close to -8π.

that is, $\xi(t) := \frac{dH}{dt}$ satisfies the following separable differential equation:

$$\frac{d\xi}{\xi^2 - 2E} = \frac{dt}{2} \quad \text{for } t \in [\log a, 0]. \tag{2.5}$$

Then the equation (2.5) is easily solved as *the final value problem*:

$$\xi(t) = \begin{cases} \sqrt{2E}\dfrac{1 + Ae^{t\sqrt{2E}}}{1 - Ae^{t\sqrt{2E}}}, & \text{for } E > 0, \\ -\dfrac{2}{t+A}, & \text{for } E = 0, \\ \sqrt{-2E}\tan\left(\frac{t\sqrt{-2E}}{2} + A\right), & \text{for } E < 0 \end{cases}$$

for some constant $A \in \mathbf{R}$ determined later. Therefore it holds that

$$\frac{\sigma}{Z}e^H = E - \frac{\xi^2}{2} = \begin{cases} -\dfrac{4EAe^{t\sqrt{2E}}}{\left(1 - Ae^{t\sqrt{2E}}\right)^2}, & \text{for } E > 0, \\ -\dfrac{2}{(t+A)^2}, & \text{for } E = 0, \\ \dfrac{E}{\cos^2\left(\frac{t\sqrt{-2E}}{2} + A\right)}, & \text{for } E < 0. \end{cases} \tag{2.6}$$

From the boundary condition $H(0) = u(1) = 0$, it holds that

$$\frac{\sigma}{Z} = \begin{cases} -\dfrac{4EA}{(1-A)^2}, & \text{for } E > 0, \\ -\dfrac{2}{A^2}, & \text{for } E = 0, \\ \dfrac{E}{\cos^2 A}, & \text{for } E < 0. \end{cases} \tag{2.7}$$

Since $Z > 0$, the constraint (2.7) implies the following:

$$\begin{array}{lcl} \sigma > 0 & \Rightarrow & E > 0 \text{ and } A < 0, \\ \sigma = 0 & \Rightarrow & E > 0 \text{ and } A = 0, \\ \sigma < 0 & \Rightarrow & E > 0 \text{ and } A > 0, \text{ or } E \leq 0. \end{array} \tag{2.8}$$

In combination with (2.6) and (2.7), the following holds around $r = 1$:

$$e^{u(r)} = \frac{e^{H(\log r)}}{r^2} = \begin{cases} \dfrac{(1-A)^2 r^{\sqrt{2E}}}{r^2\left(1 - Ar^{\sqrt{2E}}\right)^2}, & \text{for } E > 0, \\ \dfrac{A^2}{r^2(A + \log r)^2}, & \text{for } E = 0, \\ \dfrac{\cos^2 A}{r^2\cos^2\left(A + \frac{\sqrt{-2E}}{2}\log r\right)}, & \text{for } E < 0. \end{cases} \tag{2.9}$$

We note that these functions must be defined in the interval $(a, 1)$. This is established for every $\sigma \geq 0$ and $a \in (0,1)$ because $E > 0$ and $A \leq 0$, see (2.8). For the case $\sigma < 0$, we have to choose A carefully as follows to avoid dividing by zero during the interval $(a, 1)$:

$$\begin{array}{lll} E > 0 & \Rightarrow & (1 - Aa^{\sqrt{2E}})(1-A) > 0, \\ E = 0 & \Rightarrow & A(A + \log a) > 0, \\ E < 0 & \Rightarrow & -\frac{\pi}{2} < A + \frac{\sqrt{-2E}}{2}\log a < A < \frac{\pi}{2}. \end{array} \tag{2.10}$$

We asume these conditions from now on when $\sigma < 0$. We note that the $E < 0$ must be chosen satsfying

$$-\pi < \frac{\sqrt{-2E}}{2}\log a < 0 \tag{2.11}$$

for a given $a \in (0,1)$.

From the formula (2.9), we are able to calculate the value of $Z = \int_{\Lambda_a} e^{u(|x|)}dx$ as follows:

$$Z = \begin{cases} \dfrac{2\pi(1-A)(1-a^{\sqrt{2E}})}{\sqrt{2E}\left(1 - Aa^{\sqrt{2E}}\right)}, & \text{for } E > 0, \\ -\dfrac{2\pi A \log a}{A + \log a}, & \text{for } E = 0, \\ \frac{4\pi}{\sqrt{-2E}}\left(\cos^2 A\right)\left[\tan A - \tan\left(A + \frac{\sqrt{-2E}}{2}\log a\right)\right], & \text{for } E < 0. \end{cases} \tag{2.12}$$

This determines the value σ from the constraint (2.7):

$$\sigma = \begin{cases} -\dfrac{4\pi A\sqrt{2E}(1-a^{\sqrt{2E}})}{(1-A)\left(1 - Aa^{\sqrt{2E}}\right)}, & \text{for } E > 0, \\ \dfrac{4\pi \log a}{A(A + \log a)}, & \text{for } E = 0, \\ -2\pi\sqrt{-2E}\left[\tan A - \tan\left(A + \frac{\sqrt{-2E}}{2}\log a\right)\right], & \text{for } E < 0. \end{cases} \tag{2.13}$$

Finally we assume the boundary condition $u(a) = \mu$ on the *inner* boundary $\partial B_a(0)$:

$$e^{\mu} = \begin{cases} \dfrac{(1-A)^2 a^{\sqrt{2E}}}{a^2\left(1 - Aa^{\sqrt{2E}}\right)^2}, & \text{for } E > 0, \\ \dfrac{A^2}{a^2(A + \log a)^2}, & \text{for } E = 0, \\ \dfrac{\cos^2 A}{a^2\cos^2\left(A + \frac{\sqrt{-2E}}{2}\log a\right)}, & \text{for } E < 0. \end{cases} \tag{2.14}$$

In these settings, what we have to do becomes to find appropriate $E \in \mathbf{R}$ and $A \in \mathbf{R}$ satisfying (2.8), (2.10), (2.13), and (2.14) for given $a \in (0,1)$, $\sigma \in \mathbf{R}$, and $\mu \in \mathbf{R}$.

3 When $\Omega = \Lambda (= B_1(0))$.

In this case, the righthand side of (2.9) must be regular at $r = 0$, which is established only by the case $E = 2$. We are able to use (2.12) here with $a = 0$ and we do not use the inner boundary condition (2.14) in this case. Then the solution becomes

$$e^{u(r)} = \frac{(1-A)^2}{(1-Ar^2)^2}, \quad \sigma = -\frac{8\pi A}{1-A}. \tag{3.1}$$

The condition $Z = \pi(1-A) > 0$ implies that $A < 1$, which satisfies (2.8) and (2.10). Then the solution exists if and only if

$$\sigma \in (-\infty, 8\pi).$$

For the case $\Omega = \Lambda$, the conclusion is summarized as follows:

Proposition 3.1 ([1, p.513])**.** The regular solution of (2.1) exists if and only if $\sigma \in (-\infty, 8\pi)$ and is given as (3.1).

4 When $\Omega = \Lambda_a$ for $a \in (0,1)$: The proof of Theorem 1.1.

First we note that

$$e^{\frac{\mu}{2}} = \begin{cases} \dfrac{(1-A)a^{\frac{\sqrt{2E}}{2}-1}}{1-Aa^{\sqrt{2E}}}, & \text{for } E > 0, \\ \dfrac{A}{a(A+\log a)}, & \text{for } E = 0, \\ \dfrac{\cos A}{a\cos\left(A + \frac{\sqrt{-2E}}{2}\log a\right)}, & \text{for } E < 0. \end{cases} \tag{4.1}$$

holds because of (2.10). Then we are able get the formula representing A as follows:

$$A = \begin{cases} \dfrac{1-e^{\frac{\mu}{2}}a^{1-\frac{\sqrt{2E}}{2}}}{1-e^{\frac{\mu}{2}}a^{1+\frac{\sqrt{2E}}{2}}}, & \text{for } E > 0, \\ \dfrac{e^{\frac{\mu}{2}}a\log a}{1-e^{\frac{\mu}{2}}a}, & \text{for } E = 0, \end{cases} \tag{4.2}$$

and

$$\tan A = \frac{e^{\frac{\mu}{2}} a \cos\left(\frac{\sqrt{-2E}}{2}\log a\right) - 1}{e^{\frac{\mu}{2}} a \sin\left(\frac{\sqrt{-2E}}{2}\log a\right)}, \quad \text{for } E < 0. \tag{4.3}$$

We note that, when $E > 0$, it holds that

$$1 - e^{\frac{\mu}{2}} a^{1+\frac{\sqrt{2E}}{2}} \neq 0.$$

Indeed, if it does not holds, then $1 - e^{\frac{\mu}{2}} a^{1-\frac{\sqrt{2E}}{2}} = 0$ must be hold, which implies

$$a^{1-\frac{\sqrt{2E}}{2}} - a^{1+\frac{\sqrt{2E}}{2}} = 0 \quad \Leftrightarrow \quad a^{\frac{\sqrt{2E}}{2}} = 1.$$

This does not occur when $E > 0$ and $a \in (0, 1)$. Similarly, when $E < 0$, it holds that

$$\sin\left(\frac{\sqrt{-2E}}{2}\log a\right) \neq 0$$

because of (2.10). When $E = 0$, however, there are no solution if $1 - e^{\frac{\mu}{2}} a = 0$.

Consequently we get the following relation between σ and E:

$$\sigma = \begin{cases} 2\pi\sqrt{2E}\dfrac{e^{\mu}a^2 - 2e^{\frac{\mu}{2}} a \cosh\left(\frac{\sqrt{2E}}{2}\log a\right) + 1}{e^{\frac{\mu}{2}} a \sinh\left(\frac{\sqrt{2E}}{2}\log a\right)}, & \text{for } E > 0, \\ \dfrac{4\pi\left(e^{\frac{\mu}{2}} a - 1\right)^2}{e^{\frac{\mu}{2}} a \log a}, & \text{for } E = 0, \\ 2\pi\sqrt{-2E}\dfrac{e^{\mu}a^2 - 2e^{\frac{\mu}{2}} a \cos\left(\frac{\sqrt{-2E}}{2}\log a\right) + 1}{e^{\frac{\mu}{2}} a \sin\left(\frac{\sqrt{-2E}}{2}\log a\right)}, & \text{for } E < 0. \end{cases} \tag{4.4}$$

We note that σ is a continuous function of E at $E = 0$ for fixed $\mu \in \mathbf{R}$ and $a \in (0, 1)$.

Now we introduce $B := e^{\frac{\mu}{2}} a > 0$ instead of μ in order to simplifies the presentation. We also introduce following new variable instead of E:

$$\begin{aligned} \xi &:= \tanh\left(-\frac{\sqrt{2E}}{4}\log a\right) > 0 \qquad \text{for } E > 0, \\ \xi &:= \tan\left(\frac{\sqrt{-2E}}{4}\log a\right) < 0 \qquad \text{for } E < 0. \end{aligned} \tag{4.5}$$

Then (4.4) is translated to as follows:

$$\sigma = \begin{cases} \frac{4\pi}{B\log a} \cdot \left[(B-1)^2 - (B+1)^2\xi^2\right] \cdot \frac{\tanh^{-1}\xi}{\xi}, & \text{for } 0 < \xi, \\ \frac{4\pi}{B\log a} \cdot (B-1)^2, & \text{for } \xi = 0, \\ \frac{4\pi}{B\log a} \cdot \left[(B-1)^2 + (B+1)^2\xi^2\right] \cdot \frac{\tan^{-1}\xi}{\xi}, & \text{for } \xi < 0. \end{cases} \tag{4.6}$$

It is easy to see that σ is an monotone increasing function of $\xi \in \mathbf{R}$, that has $\mathbf{R}$ as its range. This means the unique existence of $\xi \in \mathbf{R}$ satisfying (4.6) for each $\sigma \in \mathbf{R}$, $a \in (0,1)$, and $B = e^{\frac{\mu}{2}} a > 0$. Then we get the unique $E \in \mathbf{R}$ from (4.5) and $A \in \mathbf{R}$ from (4.2), that is, we get the unique solution (2.9).

5 On Remark 1.2: A behavior of solution as $a \searrow 0$.

From the above calculations, we are able to interpret the sentence in [1] referred in Remark 1.2. It is obvious that the solution in disk restricted to the annulus Λ_a will be a solution in Λ_a. However the parameter σ of the restricted solution, which we write σ_a, will differs from the original σ. We calculate this σ_a and observe the behavior of σ_a as $a \searrow 0$.

Proposition 5.1. Suppose $\sigma_0 \in (-\infty, 8\pi)$ and u_{σ_0} is the solution of (1.1) for $\sigma = \sigma_0$ on $\Omega = \Lambda(= B_1(0))$. Then $u_\sigma|_{\Lambda_a}$ is the solution of (1.2) for

$$\mu = \mu_a := u_\sigma(a) = 2\log\frac{8\pi}{8\pi - \sigma + \sigma a^2}, \quad \sigma = \sigma_a := \frac{\sigma_0}{1 + \frac{8\pi}{8\pi-\sigma}\cdot\frac{a^2}{1-a^2}}.$$

Remark 5.2. We note that

$$\mu_a \longrightarrow 2\log\frac{8\pi}{8\pi - \sigma}, \quad \sigma_a \nearrow \sigma_0$$

as $a \searrow 0$, that is, for every $\sigma \in (-\infty, 8\pi)$, there is at least one way "to recover the solution in disk from the solution in the annulus in the limit $a \longrightarrow 0$". It is obvious that

$$2\log\frac{8\pi}{8\pi - \sigma} \nearrow +\infty$$

as $\sigma \nearrow 8\pi$. From this fact, we may say that "we need to prescribe a large value of μ when σ is close to 8π".

Proof of Proposition 5.1. u_σ is given by (3.1) for $\sigma = \sigma_a$, which gives μ_a. Using the value of the parameter A given in (3.1), we are able to get the value of the parameter Z from (2.12). We note that u_σ corresponds to the case $E = 2$. Then

$$\sigma_a = \sigma_0\frac{Z}{\pi(1-A)}.$$

□

References

[1] Caglioti, E., Lions, P.L., Marchioro, C., and Pulvirenti, M.: A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description Comm. Math. Phys. **143**, 501–525 (1992)