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Extension of the HKLL bulk reconstruction for small Δ

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Abstract: We re-analyse the bulk reconstruction for a scalar field in Lorentzian AdS spacetime, both for the case of even and odd dimensions, for an extended range of conformal dimensions where the original HKLL reconstruction has to be modified. We also discuss the use of space-like Green's functions in the bulk reconstruction. We demonstrate that in the extended range also the singular part of the Green's function, omitted in the original papers, has be included. The results are particularly simple and physically interesting for integer conformal dimensions below the range considered in the original HKLL papers.

Keywords: AdS-CFT Correspondence, Conformal Field Models in String Theory, Field Theories in Higher Dimensions

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1 Introduction

The AdS/CFT correspondence [2, 3] relates a theory of gravity in AdS space to a conformal field theory on the boundary. One consequence of the correspondence is that bulk quantum fields can be expressed as CFT operators. In the large N limit the bulk fields are free and can be written as smeared CFT operators. The explicit construction, called the HKLL (Hamilton, Kabat, Lifschytz, and Lowe) bulk reconstruction, was accomplished in a series of papers [4–6]. In the simplest case a massive free scalar field operator $\Phi(Y)$ is considered in AdS_{d+1}. The HKLL bulk reconstruction represents it in terms of the boundary CFT primary of weight Δ, $O(x)$, as

$$\Phi(Y) = \int_{\Sigma_Y} \mathrm{d}x\, K(Y, x) O(x), \tag{1.1}$$

where $K(Y, x)$ is a smearing function, and the integration at the boundary should be performed in a region Σ_Y space-like separated from the bulk point Y. We refer to [7–9] for recent reviews. See also [10] for an alternative derivation based on Gel'fand-Graev-Radon transforms. Later the reconstruction has been extended to higher spins as well [6, 11–15]. After having constructed the free case the next step is to study bulk interactions [16]. An

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elegant way to introduce interactions as well as to reproduce the bulk reconstruction for free fields is the method based on space-like Green's functions [5, 17].

In the original papers it was not explicitly stated that (1.1) holds only for $\Delta > d-1$, due to the convergence for the integral. This restriction is not essential for applications of the AdS/CFT correspondence in the case of supersymmetric gauge theories, in particular in the prime example of the $\mathcal{N}=4$ SUSY U(N) gauge theory in $d=4$ dimensions, since the conformal dimensions of physically relevant operators are typically (much) larger than this lower bound $\Delta > d-1$. See however [18] for some explicit examples for small Δ primaries in the AdS/CFT context. More importantly there exists an other family of models often used in the AdS/CFT context, namely O(N) vector models and their holographic duals, higher spin theories in the bulk [19, 20]. In the most interesting $d=3$ case, the simplest singlet operator has $\Delta = 1$ $(d-2)$. Furthermore, its square, an operator which can be used as a relevant deformation, has $\Delta = 2$ $(d-1)$. The HKLL formula (1.1) can not be used to relate these singlet scalar operators in the free $O(N)$ vector model to bulk operators.

It turned out [6] that for the special case $\Delta = d-1$ the smearing function in Poincare coordinates is supported on the intersection of the light-cone of the bulk point and the boundary. In [21] the range of allowed Δ values was extended down to $\Delta > d/2$ by analytic continuation. While the bulk-boundary relation remains linear, the smearing kernel in (1.1) is replaced by a suitable distribution.

In [1] we found, in some special cases mainly concentrating on the (simpler) case of even AdS spaces, a generalized HKLL formula for Δ values below the original lower bound $d-1$ by a direct derivation, without using analytic continuation. When we explicitly evaluated the results of [21], we found that they precisely agree with the results of the direct calculation in the range where they overlap. We also discussed the interesting special cases $\Delta = d-s$, where s is a positive integer only limited by the requirement that the conformal weight satisfies the unitarity bound $\Delta \geq (d-2)/2$ (equality holds for the free scalar theory). In these integer Δ cases the bulk operator $\Phi(Y)$ is expressed in terms of CFT operators living on $\Sigma_Y^{(0)}$ (boundary points light-like separated from Y).

In this paper we carefully re-analyse the HKLL bulk reconstruction, both for the case of even and odd AdS spaces (odd and even boundary manifolds), paying special attention to the range of conformal dimensions where the construction is valid (not emphasized in the original HKLL papers).

After a setup for the HKLL bulk reconstruction in section 2, we consider the case of even and odd AdS_{d+1} (odd and even d) in sections 3 and 4, respectively and first recall the very well-known HKLL bulk reconstruction [4, 5] for a massive free scalar boson field with conformal weight $\Delta > d-1$ in each section. The purpose of this review is to introduce our notations and conventions, which will be needed later in the paper when we extend the validity of the construction to smaller values of Δ. We then recall some pertinent results from [1] in both sections 3 and 4 before discussing explicit reconstruction formulas for the regions $d-1 > \Delta > d-2$ and $d-2 > \Delta > d-3$. Some detailed calculations and necessary properties are summarized in several appendices. In addition, in appendices G and H, we discuss the use of space-like Green's functions in the bulk reconstruction. We demonstrate that this alternative method correctly reproduces the same results but in the extended range also the singular part of the Green's function, omitted in the original paper [5], has to be included.

2 Setup for the HKLL bulk reconstruction in AdS_{d+1}

The HKLL bulk reconstruction [4, 5] starts with a free scalar operator $\Phi(t,\rho,n)$ on the $d+1$ dimensional global AdS spacetime, whose metric is given by

$$ds^2 = R^2 d\rho^2 - R^2\cosh^2\rho\, dt^2 + R^2\sinh^2\rho\, dn^i dn^i, \tag{2.1}$$

where R is the AdS radius and $Y = (t,\rho,n^i)$ with $n\cdot n = 1$ (or $Y = (t,\rho,\Omega)$) are the standard global coordinates of AdS_{d+1}.

The value of Φ at the middle of the AdS, $Y_o = (0,0,n)$, is expressed as (see appendix A)

$$\Phi(Y_o) = D(1) + D_1(1), \quad D(z) = \sum_{n=0}^{\infty} d_n z^n, \qquad D_1(z) = \sum_{n=0}^{\infty} d_n^\dagger z^n, \tag{2.2}$$

where d_n and $d_n^\dagger$ are (rescaled) annihilation and creation operators, and Δ is related to the mass of the free scalar m as $m^2R^2 = \Delta(\Delta-d)$. On the other hand, using the BDHM relation

$$O(x) = \lim_{\rho\to\infty}(\sinh\rho)^{\Delta}\Phi(\tilde{t},\rho,\tilde{n}), \tag{2.3}$$

where $O(x)$ is a CFT operator with conformal dimension Δ at the AdS boundary $x = (\tilde{t},\tilde{n})$ with $\tilde{n}\cdot\tilde{n} = 1$, we have

$$\mathcal{C}(\tilde{t}) := \int \mathrm{d}\tilde{\Omega}\, O(\tilde{t},\tilde{\Omega}) = e^{-i\Delta\tilde{t}}B(-e^{-2i\tilde{t}}) + e^{i\Delta\tilde{t}}B_1(-e^{2i\tilde{t}}), \tag{2.4}$$

$$B(z) := \sum_{n=0}^{\infty} b_n z^n, \qquad B_1(z) := \sum_{n=0}^{\infty} b_n^\dagger z^n, \tag{2.5}$$

and $b_n, b_n^\dagger$ are related to $d_n, d_n^\dagger$ as

$$b_n := \Omega_d \frac{P_n(1+\alpha)}{P_n(d/2)} d_n, \quad b_n^\dagger := \Omega_d \frac{P_n(1+\alpha)}{P_n(d/2)} d_n^\dagger, \tag{2.6}$$

where $\alpha := \Delta - d/2$, $P_n(z) := \Gamma(z+n)/\Gamma(z)$ is the Pochhammer symbol, and $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of the d dimensional unit sphere.

The HKLL bulk reconstruction goes as follows. First a relation between $\Phi(Y_o)$ and $O(x)$ is derived, then $\Phi(Y)$ is obtained using the AdS isometry g and associated unitary operator $\mathrm{U}(g)$ as $\Phi(Y) = U^\dagger(g)\Phi(Y_o)\mathrm{U}(g)$, where $Y = g^{-1}Y_o$ is a generic point in the AdS space.

3 Bulk reconstruction for odd d

We first consider the bulk reconstruction for the odd d case.

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3.1 Results of the original HKLL bulk reconstruction

In papers by HKLL [4, 5], a relation between $\Phi(Y_o)$ and $O(x)$ has been derived for odd d (see also appendix B)

$$\Phi(Y_o) = \xi \int \mathcal{D}x\, k_0(\tilde{t})\Theta(\pi/2 - \tilde{t})\Theta(\tilde{t} + \pi/2)O(x), \tag{3.1}$$

where

$$\xi := \frac{1}{\pi\Omega_d}\frac{\Gamma(1-d/2)\Gamma(1+\alpha)}{\Gamma(\nu+1)}, \qquad k_0(u) := (2\cos u)^\nu, \qquad \nu := \Delta - d, \tag{3.2}$$

The convergence of the $\tilde{t}$ integral near $\tilde{t} = \pm\pi/2$ implies $\nu > -1$. Thus the condition $\Delta > d-1$ is (implicitly) assumed for the original HKLL bulk reconstruction.

A relation at a generic point $Y = (t, \rho, n)$ was given as

$$\Phi(Y) = \xi \int \mathcal{D}x\, I^\nu(Y,x)T(Y,x)O(x), \tag{3.3}$$

where

$$I(Y,x) = 2[\cosh(\rho)\cos(t-\tilde{t}) - (\sinh\rho)n^i\tilde{n}^i] \qquad T(Y,x) = \Theta(T_2 - \tilde{t})\Theta(\tilde{t} - T_1), \tag{3.4}$$

where $T_1 = t - \omega, T_2 = t + \omega$ with $\omega = \arccos[(\tanh\rho)n \cdot \tilde{n}]$ and $0 < \omega < \pi$. Note that

$$\lim_{\rho,t\to 0} I^\nu(Y,x) = k_0(\tilde{t}), \qquad \lim_{\rho,t\to 0} T(Y,x) = \Theta(\pi/2 - \tilde{t})\Theta(\tilde{t} + \pi/2). \tag{3.5}$$

3.2 Results of ref. [1] for smaller Δ

In our previous paper [1] we have derived (see appendix C)

$$\Phi(Y_o) = \frac{\eta}{2\Omega_d}\left[\mathcal{C}(\pi/2) + \mathcal{C}(-\pi/2)\right] + \xi \int_{(\text{sub})} \mathrm{d}\tilde{t}\, k_0(\tilde{t})\mathcal{C}(\tilde{t}), \tag{3.6}$$

where

$$\eta = \frac{\Gamma(1-d/2)\Gamma(1+\alpha)}{\Gamma^2(1+\nu/2)}, \tag{3.7}$$

and the subtracted integral is defined by

$$\int_{(\text{sub})} \mathrm{d}\tilde{t}\, K(\tilde{t})f(\tilde{t}) = \int_{-\pi/2}^{0} \mathrm{d}\tilde{t}K(\tilde{t})\left[f(\tilde{t}) - f(-\pi/2)\right] + \int_0^{\pi/2} \mathrm{d}\tilde{t}K(\tilde{t})\left[f(\tilde{t}) - f(\pi/2)\right], \tag{3.8}$$

which converges for $\nu > -2$ thanks to subtractions. Thus (3.6) is valid for $\Delta > d-2$ and it reduces to (3.1) for $\Delta > d-1$.

For $\Delta = d - s$ with an integer s, simple relations without integral have been given [1]:

$$\Phi(Y_o) = \xi_o \frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2} + (2k-1)^2\right\}}{\prod_{k=1}^{2\ell}(d-2k)} C_+(t)\Big|_{t=0}, \qquad \xi_o := \frac{(-1)^{\frac{d-1}{2}}}{2\Omega_d} \tag{3.9}$$

for $\Delta = d - (2\ell+1)$, where $C_+(t) = \mathcal{C}(t+\frac{\pi}{2}) + \mathcal{C}(t-\frac{\pi}{2})$, and

$$\Phi(Y_o) = \xi_o \frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2} + 4k^2\right\}}{\prod_{k=1}^{2\ell+1}(d-2k)} \frac{\partial}{\partial t} C_-(t)\Big|_{t=0} \tag{3.10}$$

for $\Delta = d - 2(\ell+1)$, where $C_-(0) = \mathcal{C}(t+\frac{\pi}{2}) - \mathcal{C}(t-\frac{\pi}{2})$.

In the previous paper we have not derived a formula for $\Phi(Y)$ at a generic point Y for the whole extended range. Results at the special points $\Delta = d-1$ and $d-2$ only were given, which are as follows.

$$\Phi(Y) = \xi_o \int \mathrm{d}\tilde{\Omega}\, \frac{1}{\mathcal{R}(Y,x)} \left[O(T_1,\tilde{\Omega}) + O(T_2,\tilde{\Omega})\right] \tag{3.11}$$

for $\Delta = d-1$, where $\mathcal{R}(Y,x) = \sqrt{\cosh^2\rho - (n\cdot\tilde{n})^2 \sinh^2\rho}$, and

$$\Phi(Y) = \tilde{\xi}_o \int \frac{d\tilde{\Omega}}{\mathcal{R}^2(Y,x)} \left[\dot{O}(T_2,\tilde{\Omega}) - \dot{O}(T_1,\tilde{\Omega}) - \cot\omega\{O(T_2,\tilde{\Omega}) + O(T_1,\tilde{\Omega})\}\right] \tag{3.12}$$

for $\Delta = d-2$, where $\tilde{\xi}_o := \frac{(-1)^{\frac{d-1}{2}}}{2(d-2)\Omega_d}$, and $\dot{O}(x) := \partial_{\tilde{t}} O(\tilde{t},\tilde{n})$.

3.3 Bulk reconstruction for the extended range $\Delta > d-3$ for odd d

In this subsection we derive a bulk reconstruction formula for a generic bulk point for $d-1 \geq \Delta > d-3$.

3.3.1 Formula at a generic point by partial integration

Since it is not easy to transform (3.6) to a generic bulk point by the AdS isometry, we take a different strategy and we start from (3.3), which is first rewritten by partial integration as

$$\begin{aligned}\Phi(Y) = \eta_o (2\cosh\rho)^\nu \int \mathrm{d}\tilde{\Omega}\ \Bigg\{ &-\frac{1}{\Gamma(\nu+1)} \int_{-\omega}^{\omega} \mathrm{d}\tilde{t}\, \phi_1(\tilde{t}) \dot{O}(t+\tilde{t},\tilde{\Omega}) \\ &+ K_1(\nu,\omega) \left[O(T_1,\tilde{\Omega}) + O(T_2,\tilde{\Omega})\right] \Bigg\},\end{aligned} \tag{3.13}$$

where $\eta_o := \Gamma(\nu+1)\xi = \frac{\Gamma(1-d/2)}{\pi\Omega_d}\Gamma(1+\alpha)$,

$$\phi_1(u) = \int_0^u \mathrm{d}v\, \phi_0(v),\ \ \phi_0(u) := (\cos u - \cos\omega)^\nu,\ \ K_1(\nu,\omega) = \frac{\phi_1(\omega)}{\Gamma(\nu+1)}. \tag{3.14}$$

Since $\phi_1(u) \sim (\omega - |u|)^{\nu+1}$ at $u \sim \pm\omega$, the $\tilde{t}$ integral is convergent for $\nu > -2$.

Performing a second integration by parts, (3.13) becomes

$$\begin{aligned}\Phi(Y) = \eta_o (2\cosh\rho)^\nu \int \mathrm{d}\tilde{\Omega} \Bigg\{ &\frac{1}{\Gamma(\nu+1)} \int_{-\omega}^{\omega} \mathrm{d}\tilde{t}\, \phi_2(\tilde{t}) \ddot{O}(t+\tilde{t},\tilde{\Omega}) \\ &+ K_2(\nu,\omega) \left[\dot{O}(T_1,\tilde{\Omega}) - \dot{O}(T_2,\tilde{\Omega})\right] + K_1(\nu,\omega) \left[O(T_1,\tilde{\Omega}) + O(T_2,\tilde{\Omega})\right] \Bigg\},\end{aligned} \tag{3.15}$$

where

$$\phi_2(u) = \int_0^u \mathrm{d}v (u-v)\phi_0(v),\ \ \phi_2'(u) = \phi_1(u), \qquad K_2(\nu,\omega) = \frac{\phi_2(\omega)}{\Gamma(\nu+1)}. \tag{3.16}$$

The $\tilde{t}$ integral in this expression is convergent for $\nu > -3$.

Although we do not need to go further for later analysis, we can repeat the procedure to obtain

$$\begin{aligned}\Phi(Y) = \eta_o(2\cosh\rho)^\nu \int \mathrm{d}\tilde{\Omega} \Bigg\{ \frac{(-1)^k}{\Gamma(\nu+1)} \int_{-\omega}^{\omega} \mathrm{d}\tilde{t}\, \phi_k(\tilde{t}) O^{(k)}(t+\tilde{t},\tilde{\Omega}) \\ + \sum_{\ell=1}^{k} \frac{\phi_\ell(\omega)}{\Gamma(\nu+1)} \left[O^{(\ell-1)}(T_1,\tilde{\Omega}) + (-1)^{\ell-1} O^{(\ell-1)}(T_2,\tilde{\Omega}) \right] \Bigg\}\end{aligned} \tag{3.17}$$

for an arbitrary positive integer k, where

$$\phi_\ell(u) := \frac{1}{(\ell-1)!} \int_0^u \mathrm{d}v\, (u-v)^{\ell-1} \phi_0(v), \qquad O^{(\ell)}(u,\tilde{\Omega}) := \frac{\partial^\ell}{\partial u^\ell} O(u,\tilde{\Omega}). \tag{3.18}$$

The $\tilde{t}$ integral is convergent for $\nu > -(k+1)$.

3.3.2 Analytic continuation of $K_1(\nu,\omega)$

While the $\tilde{t}$ integral in (3.13) is convergent for $\nu > -2$, we must show that $K_1(\nu,\omega)$ is convergent for $\nu > -2$. Using a limiting case of (3.663-1) in the table of integrals by Gradshteyn and Ryzhik, $K_1(\nu,\omega)$ can be evaluated for $\nu > -1$ as

$$K_1(\nu,\omega) = \sqrt{\pi} 2^\nu \left(\sin\frac{\omega}{2}\right)^{2\nu+1} \frac{1}{\Gamma(\nu+3/2)} {}_2F_1\left(\frac{1}{2},\frac{1}{2};\nu+\frac{3}{2};\sin^2\frac{\omega}{2}\right). \tag{3.19}$$

Since the Gamma function in the denominator of (3.19) regularizes the hypergeometric function, (3.19) can be analytically continued to all ν. The integral part is convergent for $\nu > -2$ and therefore (3.13) provides the analytic extension of the bulk reconstruction to the range $\nu > -2$.

3.3.3 Analytic continuation of $K_2(\nu,\omega)$

Since the $\tilde{t}$ integral in (3.15) is convergent for $\nu > -3$ and we have already seen that $K_1(\nu,\omega)$ is analytic for all ν, we now concentrate on the integral

$$K_2(\nu,\omega) = \omega K_1(\nu,\omega) - J_1(\nu,\omega), \tag{3.20}$$

where

$$J_1(\nu,\omega) := \frac{1}{\Gamma(\nu+1)} \int_0^\omega \mathrm{d}u\, u(\cos u - \cos\omega)^\nu, \tag{3.21}$$

which, unfortunately, can not be found in integral tables.

In the absence of an explicit formula for (3.21) we have derived (see appendix F) a recursion relation for $J_1(\nu,\omega)$, which can also be used for analytic continuation:

$$J_1(\nu,\omega) = \frac{1}{\sin^2\omega} \left\{ (\nu+2)^2 J_1(\nu+2,\omega) + (2\nu+3)\cos\omega\, J_1(\nu+1,\omega) + \frac{(1-\cos\omega)^{\nu+2}}{\Gamma(\nu+3)} \right\}. \tag{3.22}$$

The integrals related to the right hand side of (3.22) are convergent for $\nu > -2$ and so this relation can be used to extend the left hand side to $\nu > -2$ too. After this extension the right hand side will be defined to $\nu > -3$ and it defines the left hand side also to

$\nu > -3$. In this way we can extend, step by step, $J_1(\nu,\omega)$ for all ν. Therefore, (3.20) implies that $K_2(\nu,\omega)$ is analytic for all ν. Since the $\tilde{t}$ integral is convergent for $\nu > -3$, and $K_{1,2}(\nu,\omega)$ are analytic for all ν, $\Phi(Y)$ in (3.15) provides the analytic extension of the bulk reconstruction to the range $\nu > -3$ ($\Delta > d-3$), as promised.

3.3.4 Comparisons with previous results

Since the starting formula in (3.3) is valid only for $\nu > -1$, results (3.13) for $\nu > -2$ and (3.15) for $\nu > -3$ are not regarded as direct derivations but should be considered as analytic continuations to $\nu > -2$ and $\nu > -3$. Therefore, it is useful to compare (3.13) and (3.15) for special cases with previous results obtained without using analytic continuation.

For this purpose, using the hypergeometric identities ${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1(a,c-b;c;z/(z-1))$, we rewrite (3.19) as

$$K_1(\nu,\omega) = \sqrt{\pi}2^\nu \frac{(\sin\frac{\omega}{2})^{2\nu+1}}{\cos\frac{\omega}{2}} \frac{1}{\Gamma(\nu+3/2)} {}_2F_1\left(\frac{1}{2},\nu+1;\nu+\frac{3}{2};-\tan^2\frac{\omega}{2}\right), \tag{3.23}$$

which gives

$$K_1(-1,\omega) = \frac{1}{\sin\omega}, \quad K_1(-2,\omega) = -\frac{\cos\omega}{\sin^3\omega}. \tag{3.24}$$

To start the recursion of $J_1(\nu,\omega)$ for a negative integer ν, we calculate

$$J_1(0,\omega) = \frac{\omega^2}{2}, \qquad J_1(1,\omega) = \int_0^\omega \mathrm{d}u\, u(\cos u - \cos\omega) = -\frac{\omega^2}{2}\cos\omega + \cos\omega + \omega\sin\omega - 1, \tag{3.25}$$

which, through the recursion (3.22), lead to

$$J_1(-1,\omega) = \frac{\omega}{\sin\omega}, \qquad J_1(-2,\omega) = \frac{1}{\sin^2\omega} - \frac{\omega\cos\omega}{\sin^3\omega}. \tag{3.26}$$

Thus, (3.20) leads to

$$K_2(-1,\omega) = 0, \quad K_2(-2,\omega) = -\frac{1}{\sin^2\omega}. \tag{3.27}$$

Using these, (3.13) for $\nu = -1$ and (3.15) for $\nu = -2$ reduce to

$$\Phi(Y) = \xi_o \int \frac{d\tilde{\Omega}}{\mathcal{R}(Y,x)} \left[O(T_1,\tilde{\Omega}) + O(T_2,\tilde{\Omega})\right], \tag{3.28}$$

$$\Phi(Y) = \tilde{\xi}_o \int \frac{\mathrm{d}\Omega}{\mathcal{R}^2(Y,x)} \Big\{[\dot{O}(T_2,\Omega) - \dot{O}(T_1,\Omega)] - \cot\omega[O(T_2,\Omega) + O(T_1,\Omega)]\Big\}, \tag{3.29}$$

which reproduce our previous results from the direct evaluation, (3.11) and (3.12).

By applying a (backward) partial integration to the integral in (3.13), we obtain

$$\Phi(Y) = \eta_o(2\cosh\rho)^\nu \int \mathrm{d}\Omega\Big\{\frac{1}{\Gamma(\nu+1)}\int_0^\omega \mathrm{d}u\,\phi(u)[O(t+u,\Omega) - O(T_2,\Omega)] + \frac{1}{\Gamma(\nu+1)} \\ \times \int_{-\omega}^0 \mathrm{d}u\,\phi(u)[O(t+u,\Omega) - O(T_1,\Omega)] + K_1(\nu,\omega)[O(T_2,\Omega) + O(T_1,\Omega)]\Big\}. \tag{3.30}$$

For the middle point Y_o this reduces to (3.6), which was obtained by direct calculation for $\Delta > d - 2$.

The above results show that the analytic continuation and the direct calculation without analytic continuation lead to the same formula (at least in these special cases) for the extended range of Δ.

4 Bulk reconstruction for even d

We now consider a more difficult task, the extension of the bulk reconstruction for even d to smaller Δ.

4.1 Results of the original HKLL bulk reconstruction

For the even d case, the result at the middle point has been obtained by HKLL [4, 5]:

$$\Phi(Y_o) = \tilde{\xi} \int \mathcal{D}x\, T(Y_o, x) I^{\nu}(Y_o, x) \ln[I(Y_o, x)] O(x), \quad \tilde{\xi} := \left(-\frac{1}{\pi}\right)^{d/2+1} \frac{\Gamma(1+\alpha)}{\Gamma(\nu+1)}. \tag{4.1}$$

Using transformation properties under the AdS isometry g

$$U^{\dagger}(g)\Phi(Y_o)\mathrm{U}(g) = \Phi(g^{-1}Y_o), \quad U^{\dagger}(g)O(x)\mathrm{U}(g) = H^{\Delta}(g^{-1}, x)O(g^{-1}x), \tag{4.2}$$

Φ at a generic bulk point $Y = g^{-1}Y_o$ becomes

$$\begin{aligned}\Phi(Y) &= \tilde{\xi} \int \mathcal{D}y\, T(Y_o, y) I^{\nu}(Y_o, y) \ln[I(Y_o, y)] H^{\Delta}(g^{-1}, y) O(g^{-1}y)\\ &= \tilde{\xi} \int \mathcal{D}x\, T(g^{-1}Y_o, x) I^{\nu}(g^{-1}Y_o, x) \ln[I(g^{-1}Y_o, x) H(g, x)] O(x)\\ &= \Phi^{\mathrm{HKLL}}(Y) + \tilde{\xi}\hat{\Phi}(g),\end{aligned} \tag{4.3}$$

where

$$\Phi^{\mathrm{HKLL}}(Y) = \tilde{\xi} \int \mathcal{D}x\, T(Y, x) I^{\nu}(Y, x) \ln[I(Y, x)] O(x), \tag{4.4}$$

$$\hat{\Phi}(g) = \int \mathcal{D}x\, T(g^{-1}Y_o, x) I^{\nu}(g^{-1}Y_o, x) \ln[H(g, x)] O(x). \tag{4.5}$$

In the above derivation we used results in appendix D ((D.7), (D.9), (D.14)) in the form

$$I(Y_o, y) H(g^{-1}, y) = I(g^{-1}Y_o, g^{-1}y), \quad T(Y_o, y) = T(g^{-1}Y_o, g^{-1}y), \tag{4.6}$$

and

$$\mathcal{D}(gx) H^d(g^{-1}, gx) = \mathcal{D}x, \quad H(g, x) = \frac{1}{H(g^{-1}, gx)}. \tag{4.7}$$

It has been claimed in the HKLL papers [4, 5] that the bulk field at a generic point is given by $\Phi^{\mathrm{HKLL}}(Y)$, which is true if

$$\hat{\Phi}(g) \equiv 0 \tag{4.8}$$

for all group elements g. Note that from the derivation it follows that $\hat{\Phi}(g)$ only depends on $Y = g^{-1}Y_o$. Although this was already discussed in appendix B of [5], an elementary proof of (4.8) is presented in appendix E for the sake of completeness.

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4.2 Previous results for integer $\Delta = d - s$

In [1] we have derived results at the middle point for $\Delta = d - s$ with a positive integer s, which are summarized as

$$\Phi(Y_o) = \frac{(-1)^{d/2}}{\pi\Omega_d} \frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2} + (2k-1)^2\right\}}{\prod_{k=1}^{2\ell}(d-2k)} \frac{\partial}{\partial\Delta} C_+(t)\Bigg|_{t=0,\Delta=d-(2\ell+1)}, \tag{4.9}$$

$$\Phi(Y_o) = \frac{(-1)^{d/2}}{\pi\Omega_d} \frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2} + 4k^2\right\}}{\prod_{k=1}^{2\ell+1}(d-2k)} \frac{\partial}{\partial t}\frac{\partial}{\partial\Delta} C_-(t)\Bigg|_{t=0,\Delta=d-2(\ell+1)}. \tag{4.10}$$

4.3 New results for smaller Δ

4.3.1 Bulk reconstruction at the middle point

For even d, the bulk reconstruction at the middle point Y_o is given by

$$\Phi(Y_o) = \tilde{\xi} \int_{(\text{sub})} \mathrm{d}\tilde{t}\, k_1(\tilde{t})\mathcal{C}(\tilde{t}) + \tilde{\xi}\, g'(\nu)[\mathcal{C}(\pi/2) + \mathcal{C}(-\pi/2)]. \tag{4.11}$$

where

$$k_1(u) = (2\cos u)^\nu \ln(2\cos u), \quad g(\nu) = \int_0^{\pi/2} du\,(2\cos u)^\nu = \frac{\pi}{2}\frac{\Gamma(\nu+1)}{\Gamma^2(\nu/2+1)}. \tag{4.12}$$

Details of the derivation are presented in appendix C.

The derivative of the identity (C.30) with respect to ν gives

$$\int_{(\text{sub})} \mathrm{d}\tilde{t}\, k_1(\tilde{t})\mathcal{C}(\tilde{t}) + \int_{(\text{sub})} \mathrm{d}\tilde{t}\, k_0(\tilde{t})\mathcal{C}_\Delta(\tilde{t})$$
$$+ g'(\nu)[\mathcal{C}(\pi/2) + \mathcal{C}(-\pi/2)] + g(\nu)[\mathcal{C}_\Delta(\pi/2) + \mathcal{C}_\Delta(-\pi/2)] = 0, \tag{4.13}$$

which leads to an alternative form of the bulk reconstruction as

$$\Phi(Y_o) = -\tilde{\xi} \int_{(\text{sub})} \mathrm{d}\tilde{t}\, k_0(\tilde{t})\mathcal{C}_\Delta(\tilde{t}) - \tilde{\xi} g(\nu)[\mathcal{C}_\Delta(\pi/2) + \mathcal{C}_\Delta(-\pi/2)], \tag{4.14}$$

where we define

$$\mathcal{C}_\Delta(t) := \frac{\partial}{\partial\Delta}\mathcal{C}(t) \tag{4.15}$$

4.3.2 Analytic continuation

As in the odd d case, analytic continuation is employed to obtain the bulk reconstruction at a generic point Y also for even d. We start with the formula $\Phi(Y) = \Phi^{\text{HKLL}}(Y)$ in (4.4) rewritten as

$$\Phi(Y) = \tilde{\eta}(2\cosh\rho)^\nu \int \mathrm{d}\tilde{\Omega}\, H(t, \omega, \tilde{\Omega}), \qquad \tilde{\eta} := \left(-\frac{1}{\pi}\right)^{d/2+1}\Gamma(1+\alpha), \tag{4.16}$$

where

$$H(t, \omega, \tilde{\Omega}) := \frac{1}{\Gamma(\nu+1)} \int_{-\omega}^{\omega} \mathrm{d}\tilde{t}\, g(\tilde{t}, \omega) O(\tilde{t} + t, \tilde{\Omega}), \tag{4.17}$$

with $g(u,\omega) := (\cos u - \cos\omega)^\nu \ln(\cos u - \cos\omega)$. A partial integration gives

$$H(t,\omega,\tilde{\Omega}) = -\frac{1}{\Gamma(\nu+1)}\int_{-\omega}^{\omega} \mathrm{d}\tilde{t}\, g_1(\tilde{t},\omega)\dot{O}(\tilde{t}+t,\tilde{\Omega}) + P_1(\nu,\omega)\left[O(T_1,\tilde{\Omega}) + O(T_2,\tilde{\Omega})\right], \tag{4.18}$$

where

$$g_1(u,\omega) := \int_0^u dv\, g(v,\omega), \quad P_1(\nu,\omega) := \frac{g_1(\omega,\omega)}{\Gamma(\nu+1)}, \tag{4.19}$$

The $\tilde{t}$ integral in (4.18) is convergent for $\nu > -2$. Although the integral defining $P_1(\nu,\omega)$ is convergent only for $\nu > -1$, it can be analytically continued using the recursion relation

$$\begin{aligned} P_1(\nu,\omega) &= \frac{1}{\sin^2\omega}\Big\{(2\nu+3)\cos\omega P_1(\nu+1,\omega) + (\nu+2)^2 P_1(\nu+2,\omega) \\ &\quad +2\cos\omega K_1(\nu+1,\omega) + (\nu+2)K_1(\nu+2,\omega)\Big\} - \frac{K_1(\nu,\omega)}{\nu+1}, \end{aligned} \tag{4.20}$$

which is derived in appendix F. The right hand side of the recursion relation is defined for $\nu > -2$ except a pole at $\nu = -1$, whose residue is given by

$$-K_1(-1,\omega) = -\frac{1}{\sin\omega}. \tag{4.21}$$

The recursion relation allows to extend $P_1(\nu,\omega)$ step by step to all ν but there will be poles for all negative integer values of ν.

4.3.3 Comparison at the middle point for $\Delta = d - 1$

While the presence of poles at negative integer ν prevents us from extending $\Phi(Y)$ to $\nu = -1$ by analytic continuation, we can proceed for the special case $Y = Y_o$ (the middle point) starting from

$$\Phi(Y_o) = 2^\nu \tilde{\eta}\left\{-\frac{1}{\Gamma(\nu+1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tilde{t}\, g_1\left(\tilde{t},\frac{\pi}{2}\right)\dot{\mathcal{C}}(\tilde{t}) + P_1\left(\nu,\frac{\pi}{2}\right)\left[\mathcal{C}\left(\frac{\pi}{2}\right) + \mathcal{C}\left(-\frac{\pi}{2}\right)\right]\right\}. \tag{4.22}$$

We then explicitly evaluate the integral (convergent for $\nu > -1$)

$$g_1\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \frac{\partial}{\partial\nu}[2^{-\nu}g(\nu)] = \frac{\pi}{2^{\nu+1}}\frac{\Gamma(\nu+1)}{\Gamma^2(\nu/2+1)}\left\{-\ln 2 + \psi(\nu+1) - \psi(\nu/2+1)\right\}, \tag{4.23}$$

giving

$$P_1\left(\nu,\frac{\pi}{2}\right) = \frac{\pi}{2^{\nu+1}}\frac{1}{\Gamma^2(\nu/2+1)}\left\{-\ln 2 + \psi(\nu+1) - \psi(\nu/2+1)\right\}. \tag{4.24}$$

Since the digamma function in (4.24) has a pole at $\nu = -1$ we see that

$$P_1\left(\nu,\frac{\pi}{2}\right) = -\frac{1}{\nu+1} + O(1). \tag{4.25}$$

On the other hand, since $\mathcal{C}(\tilde{t})$ in (2.4) is anti-periodic in $\tilde{t}$ with a period π for an odd integer Δ, we have

$$\mathcal{C}\left(t+\frac{\pi}{2}\right) + \mathcal{C}\left(t-\frac{\pi}{2}\right) = (\nu+1)\left[\mathcal{C}_\Delta\left(t+\frac{\pi}{2}\right) + \mathcal{C}_\Delta\left(t-\frac{\pi}{2}\right)\right] + O\left((\nu+1)^2\right) \tag{4.26}$$

near $\Delta = d+1$ for even d. Thus we conclude that $\Phi(Y_o)$ is finite in the $\nu \to -1$ limit as

$$\Phi(Y_o) = \frac{(-1)^{d/2}}{\pi \Omega_d} \left[\mathcal{C}_\Delta \left(\frac{\pi}{2} \right) + \mathcal{C}_\Delta \left(-\frac{\pi}{2} \right) \right], \tag{4.27}$$

which reproduce the previous result in (4.9) for $\ell = 0$, obtained by a different method without analytic continuation.

4.3.4 An alternative expression at a generic point

We can avoid difficulties arising from poles of $P_1(\nu, \omega)$ by considering an alternative expression of Φ.

We start from the fact that for even d and $\nu > -1$ (see (B.14))

$$\int \mathcal{D}x \, I^\nu(Y_0, x) T(Y_o, x) O(x) = 0, \tag{4.28}$$

which can be transformed to a generic point by the isometry transformation as

$$\int \mathcal{D}x \, I^\nu(Y, x) T(Y, x) O(x) = 0. \tag{4.29}$$

By taking its derivative with respect to ν and comparing to (4.4), we arrive at an alternative expression (valid for $\nu > -1$):

$$\Phi(Y) = -\tilde{\xi} \int \mathcal{D}x \, I^\nu(Y, x) T(Y, x) O_\Delta(x), \quad O_\Delta(x) := \frac{\partial}{\partial \Delta} O(x). \tag{4.30}$$

Applying the partial integration to this alternative expression, H in (4.16) becomes

$$H(t, \omega, \tilde{\Omega}) = \frac{1}{\Gamma(\nu+1)} \int_{-\omega}^{\omega} \mathrm{d}\tilde{t} \, \phi_1(\tilde{t}) \dot{O}_\Delta(\tilde{t} + t, \tilde{\Omega}) - K_1(\nu, \omega) \left[O_\Delta(T_1, \tilde{\Omega}) + O_\Delta(T_2, \tilde{\Omega}) \right]. \tag{4.31}$$

Since the $\tilde{t}$ integral is convergent for $\nu > -2$ and $K_1(\nu, \omega)$ is analytic for all ν, $\Phi(Y)$ can be analytically continued to $\nu > -2$ as

$$\Phi(Y) = \tilde{\eta} (2 \cos \rho)^\nu \int \mathrm{d}\tilde{\Omega} \, H(t, \omega, \tilde{\Omega}) \tag{4.32}$$

with $H(t, \omega, \tilde{\Omega})$ in (4.31). In particular for $\nu = -1$, we have

$$\Phi(Y) = \frac{(-1)^{d/2}}{\pi \Omega_d} \int \mathrm{d}\tilde{\Omega} \frac{1}{\mathcal{R}(Y, x)} \left[O_\Delta(T_1, \tilde{\Omega}) + O_\Delta(T_2, \tilde{\Omega}) \right]. \tag{4.33}$$

For Y_o this agrees with (4.27), and thus with (4.9) for $\ell = 0$.

For Y_o, but for generic $\nu > -2$ (4.32) reduces to

$$\begin{aligned} \Phi(Y_o) &= -2^\nu \tilde{\xi} \int_{(\mathrm{sub})} \mathrm{d}\tilde{t} \, (\cos(\tilde{t}))^\nu \mathcal{C}_\Delta(\tilde{t}) - 2^\nu \tilde{\eta} K_1(\nu, \pi/2) \left[\mathcal{C}_\Delta(\pi/2) + \mathcal{C}_\Delta(-\pi/2) \right] \\ &= -\tilde{\xi} \int_{(\mathrm{sub})} \mathrm{d}\tilde{t} \, k_0(\tilde{t}) \mathcal{C}_\Delta(\tilde{t}) - \tilde{\xi} g(\nu) \left[\mathcal{C}_\Delta(\pi/2) + \mathcal{C}_\Delta(-\pi/2) \right], \end{aligned} \tag{4.34}$$

reproducing (4.14), which has been obtained without analytic continuation.

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A Fock space representation of the bulk and boundary fields

The construction of a massive bulk scalar field and the corresponding boundary field with conformal weight $\Delta > d/2 - 1$ was reviewed in [1]. Here we recall some formulas which will be used in this paper.

A free bulk scalar field Φ in terms of canonical creation and annihilation operators $\mathcal{A}^{+}_{n\ell\underline{m}}$ and $\mathcal{A}_{n\ell\underline{m}}$ is represented as

$$\Phi(t,\rho,\Omega) = \sum_{n\ell\underline{m}} \sqrt{\frac{\mathcal{N}R}{2\nu_{n\ell}}}\Big\{u_{n\ell}(\rho)Y_{\ell\underline{m}}(\Omega)\mathcal{A}_{n\ell\underline{m}}\,\mathrm{e}^{-i\nu_{n\ell}t} + u_{n\ell}(\rho)Y_{\ell\underline{m}}(\Omega)\mathcal{A}^{\dagger}_{n\ell\underline{m}}\,\mathrm{e}^{i\nu_{n\ell}t}\Big\}, \tag{A.1}$$

where $\mathcal{N}$ is a normalization constant related to the free Lagrangian, R is the AdS radius, $\nu_{n\ell} = \Delta + \ell + 2n$ is the eigenfrequency, $u_{n\ell}(\rho)$ is the radial wave function, and $Y_{\ell\underline{m}}(\Omega)$ are hyper-spherical harmonics for the $d-1$ dimensional sphere parametrized alternatively by the angular variables Ω or by the d dimensional unit vector n^i. Note that for simplicity we are using real hyper-spherical harmonics.

The field at the middle point, (2.2), can be expressed with the rescaled Fock space operator,

$$d_n = \sqrt{\frac{\mathcal{N}R}{2\nu_{n0}}}\,(-1)^n \frac{P_n(d/2)}{n!}\mathcal{N}_{n0}\frac{1}{\sqrt{\Omega_d}}\mathcal{A}_{n0\underline{0}}. \tag{A.2}$$

The explicit value of the normalization constant $\mathcal{N}_{n0}$ is not needed in our calculation.

The BDHM relation [22] gives the boundary field $O(\tilde{t},\tilde{\Omega})$ of conformal weight Δ as

$$O(\tilde{t},\tilde{\Omega}) := \lim_{\rho\to\infty}(\sinh\rho)^{\Delta}\,\Phi(\tilde{t},\rho,\tilde{\Omega}) = \sum_{n\ell\underline{m}}\sqrt{\frac{\mathcal{N}R}{2\nu_{n\ell}}}\Big\{\mathrm{e}^{-i\nu_{n\ell}\tilde{t}}\,\frac{P_n(1+\alpha)}{n!}\mathcal{N}_{n\ell}Y_{\ell\underline{m}}(\tilde{\Omega})\mathcal{A}_{n\ell\underline{m}} + \mathrm{e}^{i\nu_{n\ell}\tilde{t}}\,\frac{P_n(1+\alpha)}{n!}\mathcal{N}_{n\ell}Y_{\ell\underline{m}}(\tilde{\Omega})\mathcal{A}^{\dagger}_{n\ell\underline{m}}\Big\}. \tag{A.3}$$

The integrated boundary field (2.4) is given in terms of b_n, Fock space operators rescaled differently from d_n:

$$b_n = \sqrt{\frac{\mathcal{N}R}{2\nu_{n0}}}\,(-1)^n\frac{P_n(1+\alpha)}{n!}\mathcal{N}_{n0}\sqrt{\Omega_d}\mathcal{A}_{n0\underline{0}} = \Omega_d\,\frac{P_n(1+\alpha)}{P_n(d/2)}d_n. \tag{A.4}$$

B Reconstruction of the bulk field at the middle point

The relation (A.4) leads to the basic formula for bulk reconstruction:

$$
\begin{aligned}
D(w) &= \frac{1}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z} B(z) \sum_{n=0}^{\infty} \frac{P_n(d/2)}{P_n(1+\alpha)} \left(\frac{w}{z}\right)^n = \frac{1}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z} B(z) {}_2F_1(1, d/2; 1+\alpha; w/z) \\
&= \frac{1}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z} B(wz) {}_2F_1(1, d/2; 1+\alpha; 1/z),
\end{aligned}
\tag{B.1}
$$

which is valid for an arbitrary physical Δ (except at $1+\alpha=0$). Starting with (B.1), calculations go differently for odd and even d (even and odd AdS).

B.1 Odd d

For odd d we can evaluate the integral (B.1) using the hypergeometric function identity (valid for odd d)

$$
{}_2F_1\left(1, \frac{d}{2}; 1+\alpha; \frac{1}{z}\right) = \frac{2\alpha z}{2-d}\, {}_2F_1\left(1, 1-\alpha; 2-\frac{d}{2}; z\right) + \frac{\Gamma(1-\frac{d}{2})\Gamma(1+\alpha)}{\Gamma(\nu+1)} \left(-\frac{1}{z}\right)^{-\frac{d}{2}} (1-z)^{\nu},
\tag{B.2}
$$

where the first term is regular except for a cut starting at $z=1$. Around the branch point $z=1$, its behavior is

$$
\text{regular} \;+\; \text{const.}\, (1-z)^{\nu}.
\tag{B.3}
$$

When calculating the integral of this first term in (B.1), we can shrink our contour so that it becomes a very small circle around the branch point $z=1$, and then, its contribution vanishes for $\nu > -1$ (i.e. $\Delta > d-1$), because the value of the integral gets smaller and smaller as our integral contour gets smaller and smaller.

The second term has a cut starting already at $z=0$. The contour can be shrunken so that it becomes just the unit circle, since the singularity around the second branch point $z=1$ is an integrable one for $\nu > -1$. After a change of integration variable $z = -\mathrm{e}^{-2iu}$, the integral along the unit circle becomes

$$
D(w) = \frac{1}{\pi \Omega_d} \frac{\Gamma(1-d/2)\Gamma(1+\alpha)}{\Gamma(\nu+1)} \int_{-\pi/2}^{\pi/2} \mathrm{d}u B\left(-w\mathrm{e}^{-2iu}\right) \mathrm{e}^{-i\Delta u} (2\cos u)^{\nu}.
\tag{B.4}
$$

Thus we obtain

$$
D(1) = \xi \int_{-\pi/2}^{\pi/2} \mathrm{d}u\, \mathrm{e}^{-i\Delta u} B\left(-\mathrm{e}^{-2iu}\right) [2\cos(u)]^{\nu},
\tag{B.5}
$$

where ξ is given by (3.2). If we repeat the whole calculation for D_1, we have

$$
D_1(1) = \xi \int_{-\pi/2}^{\pi/2} \mathrm{d}u\, \mathrm{e}^{i\Delta u} B_1\left(-\mathrm{e}^{2iu}\right) [2\cos(u)]^{\nu}.
\tag{B.6}
$$

We can simply add the two contributions to arrive at (3.1).

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B.2 Even d

For even d our strategy is to make the calculation for generic, non-integer dimension d and take the limit $d \to$ even integer at the end of the calculation. The logic follows closely the HKLL paper [5].

We will use again the hypergeometric identity (B.2), which is valid for $d \neq$ even integer. The first term on the right hand side satisfies the identity

$$\begin{aligned} &{}_2F_1\left(1, 1-\alpha; 2-\frac{d}{2}; z\right) = \\ &\quad \frac{1-\frac{d}{2}}{\nu} {}_2F_1(1, 1-\alpha; 1-\nu; 1-z) + \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma(-\nu)}{\Gamma(1-\alpha)} z^{d/2-1}(1-z)^{\nu}. \end{aligned} \tag{B.7}$$

Here we have to require that $\nu \neq$ integer, but this restriction can be lifted soon. We will again establish a relation between D and B (and between D_1 and B_1).

The starting point is again (B.1). Using the identity (B.2) we notice again that the first term has no singularity inside the unit circle and has a cut starting at 1 and therefore for this term the integration contour can be shrunk to a small circle going around $z = 1$. Using the second hypergeometric identity (B.7) we see that the integral along this small contour of radius r is of the order $\mathrm{O}(r^{\nu+1})$ and therefore for $\nu > -1$ it vanishes in the limit $r \to 0$.

Thus we are left with the second term in (B.2) and the relation (B.1) becomes

$$D(w) = \frac{\kappa}{2i}\Gamma\left(1-\frac{d}{2}\right)\oint \frac{\mathrm{d}z}{z} B(wz)\left(-\frac{1}{z}\right)^{-d/2}(1-z)^{\nu}, \quad \kappa := \frac{\Gamma(1+\alpha)}{\pi\Omega_d\Gamma(\nu+1)}. \tag{B.8}$$

Similarly

$$D_1(w) = \frac{\kappa}{2i}\Gamma\left(1-\frac{d}{2}\right)\oint \frac{\mathrm{d}z}{z} B_1(wz)\left(-\frac{1}{z}\right)^{-d/2}(1-z)^{\nu}. \tag{B.9}$$

The above formulas are apparently divergent for even d because of the singular factor $\Gamma(1-d/2)$ but we can observe that the integrals

$$\oint \frac{\mathrm{d}z}{z} B(wz)\left(-\frac{1}{z}\right)^{-d_o/2}(1-z)^{\nu}, \quad \oint \frac{\mathrm{d}z}{z} B_1(wz)\left(-\frac{1}{z}\right)^{-d_o/2}(1-z)^{\nu} \tag{B.10}$$

vanish for even d_o. The reason is the same as above: in this case the integrand has no singularity (or cut starting) at $z = 0$ and therefore the integration contour can again be shrunk to a small circle around $z = 1$ and the integral vanishes for $\nu > -1$.

Using this observation we can proceed as follows. We write $d = d_o + \varepsilon$ and subtract the vanishing integral from (B.8). Reducing the contour to the unit circle and using the integration variable $z = -e^{-2iu}$, (B.8) can be written as

$$D(w) = \kappa\Gamma(1-d/2)\int_{-\pi/2}^{\pi/2} \mathrm{d}u B\left(-we^{-2iu}\right)e^{-iu\Delta}(2\cos u)^{\nu}\left(1-e^{iu\varepsilon}\right) \tag{B.11}$$

and a similar expression for $D_1(w)$. Now the $\mathrm{O}(1/\varepsilon)$ prefactors are compensated by the fact that the integrals are $\mathrm{O}(\varepsilon)$ and in the limit $d \to d_o$ we get

$$D(w) = \frac{2\kappa(-1)^{d_o/2}}{\Gamma(d_o/2)}\int_{-\pi/2}^{\pi/2} \mathrm{d}u B\left(-we^{-2iu}\right)e^{-iu\Delta}(2\cos u)^{\nu_o}(-iu), \quad \nu_o := \Delta - d_o \tag{B.12}$$

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and

$$D_1(w) = \frac{2\kappa(-1)^{d_o/2}}{\Gamma(d_o/2)} \int_{-\pi/2}^{\pi/2} \mathrm{d}u B\left(-we^{2iu}\right) e^{iu\Delta}(2\cos u)^{\nu_o}(iu). \tag{B.13}$$

Using integration along the unit circle, the vanishing first integral in (B.10) can be written as the identity

$$0 = \int_{-\pi/2}^{\pi/2} \mathrm{d}u B\left(-we^{-2iu}\right) e^{-iu\Delta}(2\cos u)^{\nu_o}. \tag{B.14}$$

Since this is true for all Δ for $\nu_o > -1$, following HKLL [5], we take its derivative with respect to Δ, which gives

$$0 = \int_{-\pi/2}^{\pi/2} \mathrm{d}u B\left(-we^{-2iu}\right) e^{-iu\Delta}(2\cos u)^{\nu_o}[\ln(2\cos u) - iu]. \tag{B.15}$$

Using this identity and putting $w = 1$ we obtain from (B.12)

$$D(1) = -\frac{2\kappa(-1)^{d_o/2}}{\Gamma(d_o/2)} \int_{-\pi/2}^{\pi/2} \mathrm{d}u B\left(-e^{-2iu}\right) e^{-iu\Delta}(2\cos u)^{\nu_o} \ln(2\cos u), \tag{B.16}$$

and similarly

$$D_1(1) = -\frac{2\kappa(-1)^{d_o/2}}{\Gamma(d_o/2)} \int_{-\pi/2}^{\pi/2} \mathrm{d}u B_1\left(-e^{2iu}\right) e^{iu\Delta}(2\cos u)^{\nu_o} \ln(2\cos u). \tag{B.17}$$

Now the two terms can be simply added and we get the final result (4.1), where we dropped the subscript o from the dimension d, which is from now on an even integer again.

C Bulk reconstruction for $\Delta > d - 2$ (middle point)

In this appendix we give the details of the derivation of the bulk reconstruction formulas for the middle point, both for the odd and even d cases, in the extended range $\Delta > d - 2$. To begin with, we rewrite (B.1) by adding and subtracting $B(w)$ under the integral as

$$\begin{aligned} D(w) = & \frac{B(w)}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z}\, {}_2F_1\left(1, \frac{d}{2}; 1+\alpha; \frac{1}{z}\right) \\ & + \frac{1}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z}[B(wz) - B(w)]\, {}_2F_1\left(1, \frac{d}{2}; 1+\alpha; \frac{1}{z}z\right). \end{aligned} \tag{C.1}$$

C.1 Odd d

For odd d, using (C.1), the manipulations in section B.1 remain valid for the extended range $\Delta > d - 2$ ($\nu > -2$) and we obtain

$$D(w) = \frac{B(w)}{\Omega_d} + \xi \int_{-\pi/2}^{\pi/2} \mathrm{d}u\, \mathrm{e}^{-iu\Delta}[2\cos(u)]^{\nu}\{B(-w\mathrm{e}^{-2iu}) - B(w)\}, \tag{C.2}$$

where singularities near $u = \pm\frac{\pi}{2}$ of the integrand become integrable for $\nu > -2$ thanks to the subtraction of $B(w)$.

We then separate the integrated boundary field $\mathcal{C}(t)$ into positive/negative frequency parts $\mathcal{C}_+(t)/\mathcal{C}_-(t)$, which are given by the two terms of (2.4). Using these definitions, we have the identity

$$e^{-i\Delta t}B(e^{-2it}) = e^{-i\frac{\Delta\pi}{2}}\mathcal{C}_+(t-\pi/2) = e^{i\frac{\Delta\pi}{2}}\mathcal{C}_+(t+\pi/2). \tag{C.3}$$

Thus (C.2) leads to

$$\begin{aligned} D(1) = &\frac{1}{\Omega_d}e^{-\frac{i\Delta\pi}{2}}\mathcal{C}_+(-\pi/2) + \xi\int_{-\pi/2}^{0}du[2\cos(u)]^\nu\{\mathcal{C}_+(u) - e^{-i(u+\pi/2)\Delta}\mathcal{C}_+(-\pi/2)\} \\ &+\xi\int_0^{\pi/2}du[2\cos(u)]^\nu\{\mathcal{C}_+(u) - e^{-i(u-\pi/2)\Delta}\mathcal{C}_+(\pi/2)\}. \end{aligned} \tag{C.4}$$

Next by adding and subtracting an integral proportional to k_+ for the first integral and k_- for the second integral, where

$$k_\pm = \xi\int_0^{\pi/2}du(2\cos u)^\nu[1 - e^{\pm i\Delta(u-\pi/2)}], \tag{C.5}$$

which are convergent for $\nu > -2$, we obtain

$$\begin{aligned} D(1) = &\left\{\frac{1}{\Omega_d} + e^{\frac{i\Delta\pi}{2}}k_+ + e^{\frac{-i\Delta\pi}{2}}k_-\right\}e^{-\frac{i\Delta\pi}{2}}\mathcal{C}_+(-\pi/2) \\ &+\xi\int_{-\pi/2}^{0}du[2\cos u]^\nu\{\mathcal{C}_+(u) - \mathcal{C}_+(-\pi/2)\} + \xi\int_0^{\pi/2}du[2\cos u]^\nu\{\mathcal{C}_+(u) - \mathcal{C}_+(\pi/2)\}. \end{aligned} \tag{C.6}$$

Analogously, repeating the calculation with D_1 and $\mathcal{C}_-$, we have

$$e^{i\Delta t}B_1(e^{2it}) = e^{i\frac{\Delta\pi}{2}}\mathcal{C}_-(t-\pi/2) = e^{-i\frac{\Delta\pi}{2}}\mathcal{C}_-(t+\pi/2) \tag{C.7}$$

and

$$\begin{aligned} D_1(1) = &\left\{\frac{1}{\Omega_d} + e^{\frac{i\Delta\pi}{2}}k_+ + e^{\frac{-i\Delta\pi}{2}}k_-\right\}e^{\frac{i\Delta\pi}{2}}\mathcal{C}_-(-\pi/2) \\ &+\xi\int_{-\pi/2}^{0}du[2\cos u]^\nu\{\mathcal{C}_-(u) - \mathcal{C}_-(-\pi/2)\} + \xi\int_0^{\pi/2}du[2\cos u]^\nu\{\mathcal{C}_-(u) - \mathcal{C}_-(\pi/2)\}. \end{aligned} \tag{C.8}$$

These results can be further simplified by using the following two identities.

$$\begin{aligned} \mathcal{C}(t-\pi/2) + \mathcal{C}(t+\pi/2) &= \mathcal{C}_+(t-\pi/2) + \mathcal{C}_-(t-\pi/2) + \mathcal{C}_+(t+\pi/2) + \mathcal{C}_-(t+\pi/2) \\ &= 2\cos\frac{\Delta\pi}{2}\{e^{-\frac{i\Delta\pi}{2}}\mathcal{C}_+(t-\pi/2) + e^{\frac{i\Delta\pi}{2}}\mathcal{C}_-(t-\pi/2)\}, \end{aligned} \tag{C.9}$$

$$\begin{aligned} &\int_0^{\pi/2}du(2\cos u)^A\left\{\cos\frac{B\pi}{2} - \cos Bu\right\} \\ &\qquad = \frac{\pi}{2}\Gamma(1+A)\left\{\frac{\cos\frac{B\pi}{2}}{\Gamma^2(1+A/2)} - \frac{1}{\Gamma(1+\frac{A-B}{2})\Gamma(1+\frac{A+B}{2})}\right\} \end{aligned} \tag{C.10}$$

for $A > -2$, $B = A + d$, $d = 3, 5, 7, \cdots$. Using the second identity we find

$$e^{\frac{i\Delta\pi}{2}} k_+ + e^{-\frac{i\Delta\pi}{2}} k_- = \xi \int_0^{\pi/2} du (2\cos u)^{\Delta - d} \Big\{ 2\cos\frac{\Delta\pi}{2} - 2\cos\Delta u \Big\} = \frac{1}{\Omega_d} \Big\{ \eta \cos\frac{\Delta\pi}{2} - 1 \Big\}. \tag{C.11}$$

Finally, adding $D(1)$ and $D_1(1)$ we obtain the final result (3.6).

C.2 Even d

For the even d case by adding and subtracting a constant we rewrite the starting formula (B.1) as

$$D(1) = \frac{B(1)}{\Omega_d} + \frac{1}{2\pi i \Omega_d} \oint \frac{dz}{z} B_o(z)(1-z)\, {}_2F_1(1, d/2; 1+\alpha; 1/z) \tag{C.12}$$

and similarly

$$D_1(1) = \frac{B_1(1)}{\Omega_d} + \frac{1}{2\pi i \Omega_d} \oint \frac{dz}{z} B_{1o}(z)(1-z)\, {}_2F_1(1, d/2; 1+\alpha; 1/z), \tag{C.13}$$

where we introduced the formally holomorphic fields $B_o(z)$, $B_{1o}(z)$ by

$$\hat{B}(z) = B(z) - B(1) = (1-z) B_o(z), \qquad \hat{B}_1(z) = B_1(z) - B_1(1) = (1-z) B_{1o}(z). \tag{C.14}$$

We also introduce $\Delta_o = \Delta + 1$ and note that $\Delta_o > d - 1$.

Now we can copy the results of our calculation valid for the original range ($\Delta > d-1$) with the following modifications: there is the extra constant term for both $D(1)$ and $D_1(1)$, the role of $B(z)$ and $B_1(z)$ is played by $B_o(z)$ and $B_{1o}(z)$, respectively, and we put Δ_o in place of Δ. We obtain

$$D(1) = \frac{B(1)}{\Omega_d} + \frac{\kappa}{2i} \Gamma(1 - d/2) \oint \frac{dz}{z} B_o(z) \left(-\frac{1}{z}\right)^{-d/2} (1-z)^{\Delta_o - d} \tag{C.15}$$

and

$$D_1(1) = \frac{B_1(1)}{\Omega_d} + \frac{\kappa}{2i} \Gamma(1 - d/2) \oint \frac{dz}{z} B_{1o}(z) \left(-\frac{1}{z}\right)^{-d/2} (1-z)^{\Delta_o - d}. \tag{C.16}$$

At this point we still have to regularize the dimension as $d = d_o + \epsilon$, where d_o is an even integer. After carrying out the $\epsilon \to 0$ limit and restoring $\hat{B}(z)$ and $\hat{B}_1(z)$ we find

$$D(1) = \frac{B(1)}{\Omega_d} - \frac{2\kappa(-1)^{d_o/2}}{\Gamma(d_o/2)} \int_{-\pi/2}^{\pi/2} du\, \hat{B}\left(-e^{-2iu}\right) e^{-iu\Delta} (2\cos u)^{\nu_o} \ln(2\cos u) \tag{C.17}$$

and

$$D_1(1) = \frac{B_1(1)}{\Omega_d} - \frac{2\kappa(-1)^{d_o/2}}{\Gamma(d_o/2)} \int_{-\pi/2}^{\pi/2} du\, \hat{B}_1\left(-e^{2iu}\right) e^{iu\Delta} (2\cos u)^{\nu_o} \ln(2\cos u). \tag{C.18}$$

There are also identities of the form

$$\int_{-\pi/2}^{\pi/2} du\, \hat{B}\left(-e^{-2iu}\right) e^{-iu\Delta} (2\cos u)^{\nu_o} = \int_{-\pi/2}^{\pi/2} du\, \hat{B}_1\left(-e^{2iu}\right) e^{iu\Delta} (2\cos u)^{\Delta - d_o} = 0. \tag{C.19}$$

From now on we drop the subscript from d_o and use the notation d for the dimension.

The identities (C.19) take the form

$$\int_{-\pi/2}^{\pi/2} \mathrm{d}u\, k_0(u) \left\{\mathcal{C}_+(u) - B(1)e^{-iu\Delta}\right\} = \int_{-\pi/2}^{\pi/2} \mathrm{d}u\, k_0(u) \left\{\mathcal{C}_-(u) - B_1(1)e^{iu\Delta}\right\} = 0, \quad \text{(C.20)}$$

whereas $D(1)$, $D_1(1)$ are given by

$$D(1) = \frac{B(1)}{\Omega_d} + \tilde{\xi} \int_{-\pi/2}^{\pi/2} \mathrm{d}u\, k_1(u) \left\{\mathcal{C}_+(u) - B(1)e^{-iu\Delta}\right\}, \quad \text{(C.21)}$$

$$D_1(1) = \frac{B_1(1)}{\Omega_d} + \tilde{\xi} \int_{-\pi/2}^{\pi/2} \mathrm{d}u\, k_1(u) \left\{\mathcal{C}_-(u) - B_1(1)e^{iu\Delta}\right\}. \quad \text{(C.22)}$$

We recall that

$$\tilde{\xi} = -\frac{2\kappa(-1)^{d/2}}{\Gamma(d/2)} = \left(-\frac{1}{\pi}\right)^{1+d/2} \frac{\Gamma(1+\alpha)}{\Gamma(\nu+1)}. \quad \text{(C.23)}$$

By adding and subtracting terms we can show that

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} \mathrm{d}u\, k_i(u) \left\{\mathcal{C}_+(u) - B(1)e^{-iu\Delta}\right\} \\ &\quad = \int_{(\mathrm{sub})} \mathrm{d}u\, k_i(u)\mathcal{C}_+(u) + 2B(1) \int_0^{\pi/2} \mathrm{d}u\, k_i(u) \left(\cos\frac{\pi\Delta}{2} - \cos u\Delta\right) \end{aligned} \quad \text{(C.24)}$$

and similarly

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} \mathrm{d}u\, k_i(u) \left\{\mathcal{C}_-(u) - B_1(1)e^{iu\Delta}\right\} \\ &\quad = \int_{(\mathrm{sub})} \mathrm{d}u\, k_i(u)\mathcal{C}_-(u) + 2B_1(1) \int_0^{\pi/2} \mathrm{d}u\, k_i(u) \left(\cos\frac{\pi\Delta}{2} - \cos u\Delta\right). \end{aligned} \quad \text{(C.25)}$$

Adding the two identities written in this form we get

$$\int_{(\mathrm{sub})} \mathrm{d}u\, k_0(u)\mathcal{C}(u) + 2\left[B(1) + B_1(1)\right] \int_0^{\pi/2} \mathrm{d}u\, k_0(u) \left(\cos\frac{\pi\Delta}{2} - \cos u\Delta\right) = 0. \quad \text{(C.26)}$$

The bulk field at the middle point is given by

$$\begin{aligned} \Phi(Y_o) &= \tilde{\xi} \int_{(\mathrm{sub})} \mathrm{d}u\, k_1(u)\mathcal{C}(u) \\ &\quad + \left[B(1) + B_1(1)\right] \left\{\frac{1}{\Omega_d} + 2\tilde{\xi} \int_0^{\pi/2} \mathrm{d}u\, k_1(u) \left(\cos\frac{\pi\Delta}{2} - \cos u\Delta\right)\right\}. \end{aligned} \quad \text{(C.27)}$$

To calculate the remaining integrals we will use

$$\int_0^{\pi/2} \mathrm{d}u\, (2\cos u)^A \left(\cos\frac{\Delta\pi}{2} - \cos u\Delta\right) = \cos\frac{\Delta\pi}{2}\, g(A) - \frac{\pi}{2} \frac{\Gamma(1+A)}{\Gamma\left(1+\frac{A-\Delta}{2}\right)\Gamma\left(1+\frac{A+\Delta}{2}\right)}, \quad \text{(C.28)}$$

which leads to

$$\int_0^{\pi/2} \mathrm{d}u\, k_0(u) \left(\cos\frac{\Delta\pi}{2} - \cos u\Delta\right) = \cos\frac{\Delta\pi}{2}\, g(\nu), \quad \text{(C.29)}$$

since the second term does not contribute in this case. Thus the final form of the identity becomes

$$\int_{(\mathrm{sub})} \mathrm{d}u\, k_0(u)\mathcal{C}(u) + g(\nu)\left(\mathcal{C}(\pi/2) + \mathcal{C}(-\pi/2)\right) = 0. \tag{C.30}$$

Next we calculate

$$\int_0^{\pi/2} \mathrm{d}u\, k_1(u)\left(\cos\frac{\Delta\pi}{2} - \cos u\Delta\right) = \cos\frac{\Delta\pi}{2}\, g'(\nu) - \frac{\pi}{2}\frac{\partial}{\partial A}\left(\frac{\Gamma(1+A)}{\Gamma\left(1+\frac{A-\Delta}{2}\right)\Gamma\left(1+\frac{A+\Delta}{2}\right)}\right)\Bigg|_{A=\Delta-d}. \tag{C.31}$$

For $A = \Delta - d - \epsilon$

$$\frac{1}{\Gamma\left(1+\frac{A-\Delta}{2}\right)} = \frac{\epsilon\Gamma(d/2)}{2}(-1)^{d/2} + O(\epsilon^2) \tag{C.32}$$

and so

$$\int_0^{\pi/2} \mathrm{d}u\, k_1(u)\left(\cos\frac{\Delta\pi}{2} - \cos u\Delta\right) = \cos\frac{\Delta\pi}{2}\, g'(\nu) + \frac{\pi}{4}(-1)^{d/2}\Gamma(d/2)\frac{\Gamma(1+\nu)}{\Gamma(1+\alpha)}. \tag{C.33}$$

Using this in (C.27) we see that the final result for the bulk field at the middle point simplifies to (4.11).

D Symmetries

D.1 AdS isometry $\longrightarrow$ boundary conformal transformation

Let us use coordinates (ρ, x) for a point Y in the AdS bulk and perform an AdS isometry $Y \longrightarrow gY \sim (\bar{\rho}, \bar{x})$. In the large ρ limit, we write

$$\bar{\rho} = \rho + \sigma(g, x) + o(\rho), \quad \bar{x} = gx + o(\rho), \tag{D.1}$$

where $o(\rho)$ vanishes in the $\rho \to \infty$ limit and $x \longrightarrow gx$ is the boundary conformal transformation. For the derivatives we have

$$\frac{\partial\bar{\rho}}{\partial\rho} = 1 + o(\rho), \qquad \frac{\partial\bar{\rho}}{\partial x^A} = \frac{\partial\sigma(g,x)}{\partial x^A} + o(\rho), \qquad \frac{\partial\bar{x}^A}{\partial\rho} = o(\rho), \qquad \frac{\partial\bar{x}^A}{\partial x^B} = \frac{\partial(gx)^A}{\partial x^B} + o(\rho), \tag{D.2}$$

which gives

$$M_g^{\mathrm{AdS}}(Y) = M_g^{\mathrm{bound}}(x) + o(\rho), \tag{D.3}$$

where

$$M_g^{\mathrm{AdS}}(Y) = \det\left(\frac{\partial(gY)}{\partial Y}\right), \qquad M_g^{\mathrm{bound}}(x) = \det\left(\frac{\partial\bar{x}}{\partial x}\right). \tag{D.4}$$

The AdS line element squared and the line element at the boundary are given by

$$(\mathrm{d}s^2)^{\mathrm{AdS}} = R^2\mathrm{d}\rho^2 - R^2\cosh^2\rho\mathrm{d}t^2 + R^2\sinh^2\rho\mathrm{d}n^i\mathrm{d}n^i, \qquad (\mathrm{d}s^2)^{\mathrm{bound}} = -\mathrm{d}t^2 + \mathrm{d}n^i\mathrm{d}n^i, \tag{D.5}$$

respectively. For large ρ, a relation between measure factors, square root of corresponding metric determinants, is given by

$$\gamma^{\rm AdS}(Y) = R^{d+1}\left(\frac{e^\rho}{2}\right)^d \gamma^{\rm bound}(x). \tag{D.6}$$

Since $M_g^{\rm AdS}(Y)\gamma^{\rm AdS}(gY) = \gamma^{\rm AdS}(Y)$ for an AdS isometry g, in the $\rho\to\infty$ limit we obtain

$$H^d(g,x) = J^d(g,x)\frac{\gamma^{\rm bound}(gx)}{\gamma^{\rm bound}(x)}, \quad H(g,x) := e^{-\sigma(g,x)},\ J(g,x) := [M_g^{\rm bound}(x)]^{1/d}. \tag{D.7}$$

From the group composition property $J(gh,x) = J(g,hx)J(h,x)$ and a similar relation

$$H(gh,x) = H(g,hx)H(h,x), \tag{D.8}$$

we have

$$J(g,x) = \frac{1}{J(g^{-1},gx)}, \qquad H(g,x) = \frac{1}{H(g^{-1},gx)}. \tag{D.9}$$

If g is a boundary isometry (shift of the time coordinate t, rotation of n^i), we have[1] $J^d(g,x)\gamma(gx) = \gamma(x)$, which leads to $H(g,x) = 1$.

D.2 BDHM formula

Let us reconsider the BDHM relation

$$O(x) = \lim_{\rho\to\infty}(\sinh\rho)^\Delta\Phi(\rho,x). \tag{D.10}$$

Using the Fock space transformation property of the bulk field, $\mathrm{U}(g)\Phi(Y)U^\dagger(g) = \Phi(gY)$, we find the transformation law

$$\begin{aligned}\mathrm{U}(g)O(x)U^\dagger(g) &= \lim_{\rho\to\infty}(\sinh\rho)^\Delta\Phi(\bar\rho,\bar x) = \lim_{\rho\to\infty}\left(\frac{\sinh\rho}{\sinh\bar\rho}\right)^\Delta(\sinh\bar\rho)^\Delta\Phi(\bar\rho,\bar x)\\ &= e^{-\Delta\sigma(g,x)}O(gx) = H^\Delta(g,x)O(gx).\end{aligned} \tag{D.11}$$

D.3 Definition and properties of $I(Y,x)$

The AdS invariant $S(Y_1,Y)$ for two bulk points Y_1 and Y are given by

$$S(Y_1,Y) = \cosh\rho_1\cosh\rho\cos(t_1-t) - \sinh\rho_1\sinh\rho\, n_1^i n^i, \tag{D.12}$$

which satisfies $S(gY_1,gY) = S(Y_1,Y)$. Its bulk-boundary version is defined by[2]

$$I(Y_1,x) = \lim_{\rho\to\infty}4e^{-\rho}S(Y_1,Y) = 2[\cosh\rho_1\cos(t_1-t) - \sinh\rho_1\, n_1^i n^i]. \tag{D.13}$$

Its transformation property is as follows.

$$I(gY_1,gx) = \lim_{\bar\rho\to\infty}4e^{-\bar\rho}S(gY_1,gY) = \lim_{\bar\rho\to\infty}4e^{-\bar\rho}S(Y_1,Y) = H(g,x)I(Y_1,x). \tag{D.14}$$

[1] From now on we drop the superscript 'bound'.

[2] In our usual coordinates $I(Y,x) = 2[\cosh\rho\cos(t-\tilde t) - \sinh\rho n^i\tilde n^i]$.

D.4 Definition and properties of $T(Y, x)$

Let us recall that a bulk point Y of AdS space and a boundary point x can be connected with a past directed light-like geodesic if $\tilde{t} = T_1$ with $T_1 := t - \omega$, where $\omega := \arccos[(\tanh\rho)\, n \cdot \tilde{n}]$ and $0 < \omega < \pi$. Similarly, Y and x can be connected with a future directed light-like geodesic if $\tilde{t} = T_2$ with $T_2 := t + \omega$. Finally, Y and x can be connected with a space-like geodesic if $T_1 < \tilde{t} < T_2$.

The function $T(Y, x)$, defined by $T(Y, x) = \Theta(T_2 - \tilde{t})\Theta(\tilde{t} - T_1)$ is isometry invariant as $T(gY, gx) = T(Y, x)$, since the lightlike/spacelike nature of a curve in AdS space is isometry invariant.

E Proof of $\hat{\Phi}(g) \equiv 0$

E.1 Transformation back

The transformation g appears in several places in the definition of $\hat{\Phi}(g)$ but we can simplify its expression by doing some steps backwards. Starting from the relation

$$I^{\Delta-d}(g^{-1}Y_o, y) = I^{\Delta-d}(Y_o, gy)H^{d-\Delta}(g, y) = I^{\Delta-d}(Y_o, gy)\frac{J^d(g, y)\gamma(gy)}{\gamma(y)}H^{-\Delta}(g, y), \quad \text{(E.1)}$$

we write

$$\begin{aligned}
\hat{\Phi}(g) &= \int \mathrm{d}^d y \gamma(y) I^{\Delta-d}(Y_o, gy)\frac{J^d(g, y)\gamma(gy)}{\gamma(y)}H^{-\Delta}(g, y)T(Y_o, gy)\ln[H(g, y)]O(y) && \text{(E.2)}\\
&= \int \mathrm{d}^d x \gamma(x) I^{\Delta-d}(Y_o, x)T(Y_o, x)H^{\Delta}(g^{-1}, x)\ln[H(g, g^{-1}x)]O(g^{-1}x) && \text{(E.3)}\\
&= -\int \mathcal{D}x I^{\Delta-d}(Y_o, x)T(Y_o, x)\ln[H(g^{-1}, x)]U^\dagger(g)O(x)\mathrm{U}(g) && \text{(E.4)}\\
&= -U^\dagger(g)\hat{\Psi}(g^{-1})\mathrm{U}(g), && \text{(E.5)}
\end{aligned}$$

where

$$\hat{\Psi}(g) := \int \mathcal{D}x I^{\Delta-d}(Y_o, x)T(Y_o, x)\ln[H(g, x)]O(x). \quad \text{(E.6)}$$

From the group property (D.8) satisfied by $H(g, x)$ it follows that

$$\hat{\Psi}(hg) = \hat{\Psi}(g) + \int \mathcal{D}x I^{\Delta-d}(Y_o, x)T(Y_o, x)\ln[H(h, gx)]O(x). \quad \text{(E.7)}$$

Therefore if h is a boundary isometry (i.e. $H(h, {}^\forall y) = 1$) then

$$\hat{\Psi}(hg) = \hat{\Psi}(g). \quad \text{(E.8)}$$

E.2 The representation $g = b\Xi E$

In this subsection we will use the embedding coordinates for global AdS.

$$X^i = R\sinh\rho\, n^i \ (i = 1, \dots, d), \qquad X^0 = R\cosh\rho\cos t, \quad X^D = -R\cosh\rho\sin t. \quad \text{(E.9)}$$

The embedding coordinates satisfy $-(X^0)^2-(X^D)^2+X^iX^i=-R^2$ and transform linearly under the AdS isometry $\mathrm{SO}(d,2)$. The coordinates of the middle point $(t=0,\rho=0)$ are

$$Y_o: \quad X^i=0 \ (i=1,\dots,d), \quad X^0=R, \quad X^D=0. \tag{E.10}$$

For an arbitrary bulk point Y we will find a group element $g=b\Xi E$ such that $g^{-1}Y_o=Y$. In other words, we transform Y to Y_o in three steps: $EY=Y_2$, $\Xi Y_2=Y_1$, $bY_1=Y_o$.

The first step (E) is a constant shift of the t coordinate ($\mathrm{SO}(2)$ rotation in the $(0,D)$ plane) that brings t to zero. After this step we have

$$Y_2: \quad X^i=R\sinh\rho\, n^i \ (i=1,\dots,d), \quad X^0=R\cosh\rho, \quad X^D=0. \tag{E.11}$$

The next step (Ξ) is an $\mathrm{SO}(d)$ rotation in the $(1,\dots,d)$ space that rotates the unit vector n^i so that it becomes parallel to the d axis. Then we have

$$Y_1: \quad X^i=0 \ (i=1,\dots,d-1), \quad X^d=R\sinh\rho, \quad X^0=R\cosh\rho, \quad X^D=0. \tag{E.12}$$

The last step (b) is a boost in the $(0,d)$ plane by $\bar{X}^d=X^d\cosh\beta-X^0\sinh\beta$, $\bar{X}^0=X^0\cosh\beta-X^d\sinh\beta$, which makes $\bar{X}^d=0$, $\bar{X}^0=R$ for $\beta=\rho$.

This representation is useful because both E and Ξ are boundary isometries and by (E.8) we have

$$\hat{\Psi}(g^{-1})=\hat{\Psi}(E^{-1}\Xi^{-1}b^{-1})=\hat{\Psi}(b^{-1}). \tag{E.13}$$

Hence the problem is reduced to the calculation of $\hat{\Psi}(b^{-1})$, where b^{-1} is the AdS isometry

$$\bar{X}^i=X^i,\ \bar{X}^d=X^d\cosh\beta+X^0\sinh\beta,\ \bar{X}^0=X^0\cosh\beta+X^d\sinh\beta,\ \bar{X}^D=X^D. \tag{E.14}$$

Introducing polar coordinates θ,ω as $n^d=\cos\theta$, $n^i=\sin\theta m^i(\omega)$ $(i=1,\dots,d-1)$, where ω are $d-1$ dimensional polar angles and m^i is a $d-1$ dimensional unit vector, the transformation is given by

$$\bar{\omega}=\omega, \qquad \begin{cases} \sinh\bar{\rho}\sin\bar{\theta} &= \sinh\rho\sin\theta \\ \sinh\bar{\rho}\cos\bar{\theta} &= \sinh\rho\cos\theta\cosh\beta+\cosh\rho\cos t\sinh\beta \\ \cosh\bar{\rho}\cos\bar{t} &= \cosh\rho\cos t\cosh\beta+\sinh\rho\cos\theta\sinh\beta \\ \cosh\bar{\rho}\sin\bar{t} &= \cosh\rho\sin t \end{cases} \tag{E.15}$$

Consistency of the first two transformations determines $\bar{\rho}$:

$$\sinh^2\bar{\rho}=\sinh^2\rho\sin^2\theta+(\sinh\rho\cos\theta\cosh\beta+\cosh\rho\cos t\sinh\beta)^2. \tag{E.16}$$

(Consistency of the second pair of transformations can also be used to determine $\bar{\rho}$, but this is equivalent.) We obtain $\bar{\rho}=\rho+\sigma$ in the $\rho\to\infty$ limit

$$e^{2\sigma}=[\cosh\beta+\sinh\beta\cos(t-\theta)][\cosh\beta+\sinh\beta\cos(t+\theta)]. \tag{E.17}$$

Using the convergent power series

$$\ln(1+u)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}u^k, \tag{E.18}$$

we have

$$\ln[H(b^{-1},x)] = \ln e^{-\sigma} = -\ln[\cosh\beta] + \sum_{k=1}^{\infty} \frac{(-\tanh\beta)^k}{k} E_k(t,\theta), \tag{E.19}$$

where

$$E_k(t,\theta) = \frac{1}{2}\left[\cos^k(t+\theta) + \cos^k(t-\theta)\right]. \tag{E.20}$$

The first few coefficients are

$$\begin{aligned} E_1(t,\theta) &= \cos\theta\cos t, \quad E_2(t,\theta) = \cos^2\theta\cos^2 t + \sin^2\theta\sin^2 t, \\ E_3(t,\theta) &= \cos^3\theta\cos^3 t + 3\cos\theta\cos t\sin^2\theta\sin^2 t. \end{aligned} \tag{E.21}$$

The final result is

$$\hat{\Psi}(g^{-1}) = \sum_{k=1}^{\infty} \frac{(-\tanh\rho)^k}{k}\hat{\Psi}_k, \tag{E.22}$$

where

$$\hat{\Psi}_k = \int \mathcal{D}x I^{\Delta-d}(Y_o,x)T(Y_o,x)E_k(t,\theta)O(x). \tag{E.23}$$

E.3 Calculation of the first few terms

(A.3) gives the expansion of the boundary field:

$$O(t,\Omega) = \sum_{n\ell\underline{m}} \left[e^{-i\nu_{n\ell}t}\mathcal{B}_{n\ell\underline{m}} + e^{i\nu_{n\ell}t}\mathcal{B}^\dagger_{n\ell\underline{m}}\right] Y_{\ell\underline{m}}(\Omega), \tag{E.24}$$

where $\nu_{n\ell} = \Delta + \ell + 2n$, $n = 0,1,\ldots$, $\ell = 0,1,\ldots$ and $\mathcal{B}_{n\ell\underline{m}}$ is the rescaled Fock space operator

$$\mathcal{B}_{n\ell\underline{m}} = \sqrt{\frac{\mathcal{N}R}{2\nu_{n\ell}}}\frac{P_n(1+\alpha)}{n!}\mathcal{N}_{n\ell}\mathcal{A}_{n\ell\underline{m}}. \tag{E.25}$$

Using the coordinate system $\Omega = (\theta,\omega)$ in the previous subsection and $\mathrm{d}\Omega = \sin^{2a}\theta\mathrm{d}\theta\mathrm{d}\omega$ with $a = -1 + d/2$, we can write the d dimensional spherical harmonics in terms of $d-1$ dimensional ones and Gegenbauer polynomials as

$$Y_{\ell\underline{m}}(\Omega) = Y_{\ell\lambda\underline{\tilde{m}}}(\theta,\omega) = K_{\ell\lambda\underline{\tilde{m}}}\, C^{a+\lambda}_{\ell-\lambda}(\cos\theta)\sin^\lambda\theta\, Y_{\lambda\underline{\tilde{m}}}(\omega), \tag{E.26}$$

where the multi-index $\underline{m}$ is decomposed as $\lambda\,\underline{\tilde{m}}$, $K_{\ell\lambda\underline{\tilde{m}}}$ are some normalization constants to ensure orthonormality, and the orthogonality of Gegenbauer polynomial is given by

$$\int_0^\pi \mathrm{d}\theta\sin^{2a}\theta\, C^a_\ell(\cos\theta)C^a_{\ell'}(\cos\theta) = \mu_\ell\delta_{\ell\ell'}. \tag{E.27}$$

With this choice we have

$$O(t,\Omega) = \sum_{n\ell\lambda\underline{\tilde{m}}} K_{\ell\lambda\underline{\tilde{m}}}\left[e^{-i\nu_{n\ell}t}\mathcal{B}_{n\ell\lambda\underline{\tilde{m}}} + e^{i\nu_{n\ell}t}\mathcal{B}^\dagger_{n\ell\lambda\underline{\tilde{m}}}\right] C^{a+\lambda}_{\ell-\lambda}(\cos\theta)\sin^\lambda\theta\, Y_{\lambda\underline{\tilde{m}}}(\omega). \tag{E.28}$$

Putting this expansion into (E.23) we obtain

$$\begin{aligned} \hat{\Psi}_k = &\sum_{n\ell\lambda\underline{\tilde{m}}} K_{\ell\lambda\underline{\tilde{m}}}\int_{-\pi/2}^{\pi/2}\mathrm{d}t(2\cos t)^{\Delta-d}\left[e^{-i\nu_{n\ell}t}\mathcal{B}_{n\ell\lambda\underline{\tilde{m}}} + e^{i\nu_{n\ell}t}\mathcal{B}^\dagger_{n\ell\lambda\underline{\tilde{m}}}\right] \\ &\times\int_0^\pi \mathrm{d}\theta\sin^{2a}\theta\, C^{a+\lambda}_{\ell-\lambda}(\cos\theta)\sin^\lambda\theta E_k(t,\theta)\int\mathrm{d}\omega Y_{\lambda\underline{\tilde{m}}}(\omega). \end{aligned} \tag{E.29}$$

The last integral simply gives $\sqrt{\Omega_{d-1}}\delta_{\lambda 0}\delta_{\underline{\tilde{m}}\underline{\tilde{0}}}$, where Ω_{d-1} is the $d-1$ dimensional volume element. The formula further simplifies with $\hat{b}_{n\ell} = K_{\ell 0\underline{\tilde{0}}}\sqrt{\Omega_{d-1}}\mathcal{B}_{n\ell 0\underline{\tilde{0}}}$ as

$$\hat{\Psi}_k = \sum_{n\ell}\int_{-\pi/2}^{\pi/2} \mathrm{d}t(2\cos t)^{\Delta-d}\left[e^{-i\nu_{n\ell}t}\hat{b}_{n\ell} + e^{i\nu_{n\ell}t}\hat{b}^\dagger_{n\ell}\right]\int_0^\pi \mathrm{d}\theta \sin^{2a}\theta\, C^a_\ell(\cos\theta)\, E_k(t,\theta). \tag{E.30}$$

We then write $\hat{\Psi}_k = \hat{\Psi}^{\mathrm{ann}}_k + \left(\hat{\Psi}^{\mathrm{ann}}_k\right)^\dagger$ with

$$\hat{\Psi}^{\mathrm{ann}}_k = \sum_{\ell=0}^{k}\oint \frac{\mathrm{d}z}{z}(-z)^{d/2}(1-z)^{\Delta-d}\, B_\ell(z)P_{k\ell}(z), \tag{E.31}$$

where $z = -e^{-2it}$, and

$$\mu_\ell e^{i\ell t}P_{k\ell}(z) := \int_0^\pi \mathrm{d}\theta \sin^{2a}\theta\, C^a_\ell(\cos\theta)\, E_k(t,\theta), \quad B_\ell(z) := \frac{\mu_\ell}{2i}\sum_{n=0}^{\infty}\hat{b}_{n\ell}(-z)^n. \tag{E.32}$$

Note that $P_{k\ell}(z) = 0$ if $k+\ell$ is odd.

Let us calculate the first few terms. For this we need the Gegenbauer polynomials and their inverse relations[3] such as

$$C_0(w) = 1, \qquad C_1(w) = 2aw, \qquad C_2(w) = a[2(a+1)w^2 - 1], \tag{E.33}$$

$$w = \frac{1}{2a}C_1(w), \qquad w^2 = \frac{1}{2(a+1)}\left[\frac{1}{a}C_2(w) + 1\right], \tag{E.34}$$

to obtain

$$P_{11}(z) = \frac{1-z}{4a}, \quad P_{22}(z) = \frac{1+z^2}{4a(a+1)}, \quad P_{20}(z) = \frac{1}{2} + \frac{a}{4(a+1)}\left(z + \frac{1}{z}\right). \tag{E.35}$$

Using these results we see that

$$\hat{\Psi}^{\mathrm{ann}}_1 = \frac{1}{4a}\oint \frac{\mathrm{d}z}{z}(-z)^{d/2}(1-z)^{\Delta-d}B_1(z)(1-z) = 0, \tag{E.36}$$

because the integrand is analytic inside the unit circle. Similarly

$$\hat{\Psi}^{\mathrm{ann}}_2 = \oint \frac{\mathrm{d}z}{z}(-z)^{d/2}(1-z)^{\Delta-d}\left\{B_2(z)\frac{1+z^2}{4a(a+1)} + B_0(z)\left[\frac{1}{2} + \frac{a}{4(a+1)}\left(z+\frac{1}{z}\right)\right]\right\} = 0, \tag{E.37}$$

since the $1/z$ pole coming from P_{20} is compensated by the factor $\frac{1}{z}(-z)^{d/2}$ for even integer $d > 2$. In general, $P_{k\ell}(z)$ cannot be more singular than z^{-a}, as we will see.

E.4 General proof

Using

$$E_k(t,\theta) = \frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}e^{i(2r-k)t}\cos(2r-k)\theta, \tag{E.38}$$

[3]From here on we drop the superscript a.

$P_{k\ell}(z)$ in (E.32) is evaluated as

$$
\begin{aligned}
P_{k\ell}(z) &= \frac{1}{\mu_\ell}\frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}e^{i(2r-k-\ell)t}\int_0^\pi d\theta\cos(2r-k)\theta\sin^{2a}\theta C_\ell^a(\cos\theta) \\
&= \frac{1}{\mu_\ell}\frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}(-z)^{\frac{\ell+k}{2}-r}I_\ell^{(2r-k)},
\end{aligned}
\tag{E.39}
$$

where

$$
I_\ell^{(n)} = \frac{1}{2}\int_{-\pi}^{\pi} d\theta e^{in\theta}\sin^{2a}\theta C_\ell^a(\cos\theta). \tag{E.40}
$$

Since $\sin^{2a}\theta C_\ell^a(\cos\theta)$ can be written as a Laurent polynomial in $e^{i\theta}$ of maximal degree $2a+\ell$,

$$
I_\ell^{(n)} = 0 \quad \text{for} \quad n > 2a+\ell, \tag{E.41}
$$

which implies that $P_{k\ell}(z)$ cannot be more singular than z^{-a} as

$$
P_{k\ell}(z) = \frac{1}{\mu_\ell}\frac{1}{2^k}\sum_{n=-a}^{j}\Theta(k-\ell+2n+1)\binom{k}{j-n}(-z)^n I_\ell^{(\ell-2n)}, \quad j := \frac{k+\ell}{2}. \tag{E.42}
$$

F Recursion relations

F.1 Recursion relation for $J_1(\nu,\omega)$

Using the simple identity

$$
\frac{d}{du}(\cos u + u\sin u) = u\,\cos u \tag{F.1}
$$

we can perform a partial integration in the definition of $J_1(\nu,\omega)$ as follows. Here we require that $\nu > 1$ so that all subsequent manipulations are well-defined.

$$
\begin{aligned}
J_1(\nu,\omega) &= \frac{1}{\Gamma(\nu+1)}\int_0^\omega du(u\cos u - u\cos\omega)(\cos u-\cos\omega)^{\nu-1} \\
&= -\frac{\cos\omega}{\nu}J_1(\nu-1,\omega) - \frac{(\cos u-\cos\omega)^{\nu-1}}{\Gamma(\nu+1)} \\
&\quad + \frac{\nu-1}{\Gamma(\nu+1)}\int_0^\omega du(\cos u + u\sin u)(\cos u-\cos\omega)^{\nu-2}\sin u \\
&= -\frac{\cos\omega}{\nu}J_1(\nu-1,\omega) - \frac{(1-\cos\omega)^{\nu-1}}{\Gamma(\nu+1)} + \frac{\nu-1}{\Gamma(\nu+1)}\int_0^\omega du\{u\sin^2 u(\cos u-\cos\omega)^{\nu-2}\} \\
&\quad + \frac{\nu-1}{\Gamma(\nu+1)}\int_0^\omega du\,\cos u(\cos u-\cos\omega)^{\nu-2}\sin u.
\end{aligned}
\tag{F.2}
$$

In the next to last line the integral, using $\sin^2 u = 1-\cos^2 u$, can be represented as

$$
\Gamma(\nu-1)\sin^2\omega J_1(\nu-2,\omega) - \Gamma(\nu+1)J_1(\nu,\omega) - 2\cos\omega\Gamma(\nu)J_1(\nu-1,\omega) \tag{F.3}
$$

The integral in the last line can be done explicitly and gives

$$
\frac{1}{\nu}(1-\cos\omega)^\nu + \frac{\cos\omega}{\nu-1}(1-\cos\omega)^{\nu-1}. \tag{F.4}
$$

Putting everything together, after some simplifications we get

$$\sin^2\omega J_1(\nu-2,\omega) = \nu^2 J_1(\nu,\omega) + (2\nu-1)\cos\omega J_1(\nu-1,\omega) + \frac{1}{\Gamma(\nu+1)}(1-\cos\omega)^\nu. \tag{F.5}$$

Making the shift $\nu \to \nu+2$ (so that the result is now valid for $\nu > -1$), we finally arrive at the recursion (3.22).

F.2 Recursion relation for $P_1(\nu,\omega)$

Using integration by parts we obtain

$$\begin{aligned} P_1(\nu,\omega) &= \frac{1}{\Gamma(\nu+1)}\int_0^\omega \mathrm{d}u\,(\cos u-\cos\omega)(\cos u-\cos\omega)^{\nu-1}\ln(\cos u-\cos\omega) \\ &= -\frac{\cos\omega}{\nu}P_1(\nu-1,\omega) + \int_0^\omega \mathrm{d}u\,\frac{\sin^2 u}{\Gamma(\nu+1)}\Big\{(\nu-1)(\cos u-\cos\omega)^{\nu-2}\ln(\cos u-\cos\omega) \\ &\quad + (\cos u-\cos\omega)^{\nu-2}\Big\}, \end{aligned} \tag{F.6}$$

and use the identity

$$\sin^2 u = \sin^2\omega - 2\cos\omega(\cos u-\cos\omega) - (\cos u-\cos\omega)^2 \tag{F.7}$$

to get

$$\begin{aligned} P_1(\nu,\omega) = &-\frac{\cos\omega}{\nu}P_1(\nu-1,\omega) + \sin^2\omega\left[\frac{P_1(\nu-2,\omega)}{\nu} + \frac{K_1(\nu-2,\omega)}{\nu(\nu-1)}\right] \\ &- \frac{2\cos\omega}{\nu}\left[(\nu-1)P_1(\nu-1,\omega)+K_1(\nu-1,\omega)\right] - (\nu-1)P_1(\nu,\omega) - K_1(\nu,\omega). \end{aligned} \tag{F.8}$$

We here again assume $\nu > 1$. After some rearrangements we obtain

$$\begin{aligned} \sin^2\omega\, P_1(\nu-2,\omega) &= (2\nu-1)\cos\omega P_1(\nu-1,\omega) + \nu^2 P_1(\nu,\omega) \\ &\quad + 2\cos\omega K_1(\nu-1,\omega) + \nu K_1(\nu,\omega) - \frac{\sin^2\omega}{\nu-1}K_1(\nu-2,\omega). \end{aligned} \tag{F.9}$$

Finally we make the shift $\nu \to \nu+2$ (so that the recursion is valid for $\nu > -1$) to obtain (4.20).

G AdS Green's functions

In this appendix we construct the space-like Green's function in AdS space, which will be useful (see appendix H) in an alternative method [5, 17] of the bulk reconstruction. Here, for simplicity, we will restrict our considerations to the range $\Delta > d-2$ only.

A Green's function of the massive scalar wave equation satisfies

$$(\mathcal{D}-m^2)\mathcal{G}(Y,Y') = \frac{1}{\sqrt{|g|}}\delta(Y,Y'), \quad \mathcal{D} := \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}g^{\alpha\beta}\partial_\beta), \tag{G.1}$$

where the Laplacian $\mathcal{D}$ acts on Y, and the mass is parametrized as $m^2 = \Delta(\Delta - d)$ (from now on we are using units where the AdS radius is unity). The metric $g_{\alpha\beta}$ (and its determinant g and inverse $g^{\alpha\beta}$) in Lorentzian AdS is encoded by the line element $\mathrm{d}s^2$ as

$$\mathrm{d}s^2 = (\mathrm{d}\rho)^2 - \cosh^2\rho\,(\mathrm{d}t)^2 + \sinh^2\rho\,\mathrm{d}n^i\mathrm{d}n^i \tag{G.2}$$

in global coordinates, or

$$\mathrm{d}s^2 = -(1+y^2)(\mathrm{d}t)^2 + \left(\delta_{ij} - \frac{y^i y^j}{1+y^2}\right)\mathrm{d}y^i\mathrm{d}y^j \tag{G.3}$$

in flat coordinates, where $y^i = n^i \sinh\rho$ and $y := \sqrt{y^i y^i} = \sinh\rho$.

In the Green's function method the AdS invariant (D.12)

$$\sigma(Y,Y') = \cos(t-t')\cosh\rho\,\cosh\rho' - \underline{n}\cdot\underline{n}'\sinh\rho\,\sinh\rho' \tag{G.4}$$

will play an important role, where $\underline{n}$, $\underline{n}'$ are d-dimensional vectors. If we are looking for a σ-dependent Green's function $\mathcal{G}(Y,Y') = g(\sigma(Y,Y'))$, then $g(\sigma)$ has to satisfy the differential equation

$$(\sigma^2-1)g''(\sigma) + D\sigma g'(\sigma) + \Delta(d-\Delta)g(\sigma) = \frac{1}{\sqrt{|g|}}\delta(Y,Y'). \tag{G.5}$$

If $Y' = Y_o$, we can take the more general ansatz in flat coordinates as $\mathcal{G}(Y,Y_o) = \mathcal{H}(t,y)$, which should satisfy

$$-m^2\mathcal{H}(t,y) - \frac{1}{1+y^2}\partial_t^2\mathcal{H}(t,y) + \partial_i\left[\frac{y^i}{y}(1+y^2)\partial_y\mathcal{H}(t,y)\right] = \delta(t)\delta(\underline{y}). \tag{G.6}$$

The delta function normalization of this Green's function becomes more transparent in terms of its Fourier transform

$$H(\omega,y) = \int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{i\omega t}\,\mathcal{H}(t,y). \tag{G.7}$$

We have to require

$$\begin{aligned} y>0:&\quad \left[\frac{\omega^2}{1+y^2} + \Delta(d-\Delta)\right]H(\omega,y) + \partial_i\left[\frac{y^i}{y}(1+y^2)\partial_y H(\omega,y)\right] = 0,\\ y\to 0:&\quad H(\omega,y) \approx -\frac{y^{2-d}}{(d-2)\Omega_d}. \end{aligned} \tag{G.8}$$

G.1 Hypergeometric σ-dependent solutions

A particular (properly normalized) solution of (G.5) is given by the hypergeometric solution (see [23])

$$F(\sigma) = -\frac{\Gamma(\Delta)}{2^{\Delta+1}\pi^{d/2}\Gamma(1+\alpha)}\sigma^{-\Delta}{}_2F_1\left(\frac{\Delta}{2},\frac{\Delta+1}{2};1+\alpha;\frac{1}{\sigma^2}\right), \tag{G.9}$$

which is the scalar two-point correlation function in AdS space. This solution is singular for $\sigma \to 1$ and is properly normalized as

$$F(\sigma) \sim d_* x^{-a}, \qquad a = \frac{d-1}{2}, \qquad d_* = -\frac{\Gamma(D/2)}{(D-2)(2\pi)^{D/2}} \tag{G.10}$$

for $x \to 0$ with $\sigma = 1 + x$. A solution of the homogeneous part of (G.5) can also be given [24] in terms of a hypergeometric function:

$$J(\sigma) = {}_2F_1\left(\frac{\Delta}{2}, \frac{d-\Delta}{2}; \frac{d+1}{2}; 1-\sigma^2\right). \tag{G.11}$$

This solution is regular at $\sigma = 1$: $J(1) = 1$.

Although the solution of the problem is completely given [5, 17] (see also [25]) in terms of these two special functions, it is nevertheless more transparent if we write the Green's function in an expanded form using the variable $x = \sigma - 1$. We take the ansatz

$$g(\sigma) = \psi(x) = \sum_{n=0}^{\infty} d_{n+q} x^{n+q} \tag{G.12}$$

and expand the homogeneous part of the equation written as

$$x(x+2)\psi''(x) + D(x+1)\psi'(x) + \Delta(d-\Delta)\psi(x) = 0. \tag{G.13}$$

The regular solution corresponds to the choice $q = 0$ and we write

$$J(\sigma) = h(x) = \sum_{n=0}^{\infty} c_n x^n, \tag{G.14}$$

where the expansion coefficients are given recursively as

$$c_{n+1} = -\frac{(n+\Delta)(n+d-\Delta)}{(n+1)(2n+d+1)} c_n \qquad n = 0, 1, \dots \qquad c_0 = 1. \tag{G.15}$$

The singular solution corresponds to $q = -a$ and proper normalization requires $d_{-a} = d_*$. Higher coefficients are determined from the recursion

$$d_{n-a}(n-a+\Delta)(n-a+d-\Delta) + 2d_{n-a+1}(n+1)(n+1-a) = 0, \quad n = 0, 1, \dots . \tag{G.16}$$

For even d (when a is half-integer), all higher d_{n-a} coefficients are obtained recursively from $d_{-a} = d_*$. On the other hand, for odd d (when a is integer), we first determine the coefficients $d_{-a+1}, \dots, d_{-1}$ from d_{-a} using (G.16), which are all non-zero in the range $\Delta > d - 2$. Arriving at $n = a - 1$ in the recursion (G.16), we find a contradiction unless $\Delta = d - 1$. In this case it is consistent to put $d_n = 0$, $n = 0, 1, \dots$.

For generic $\Delta \neq d-1$ and odd d, there is no singular solution within the ansatz (G.12). We therefore take a different ansatz,

$$g(\sigma) = \tilde{\psi}(x) = \psi(x) + c \ln x \, h(x), \tag{G.17}$$

which satisfies (G.13) with the coefficients $d_{-a}, \dots, d_{-1}$ as before. Then the $n = a - 1$ equation leads to

$$c = \frac{d_{-1}}{d-1}(\Delta - 1)(\Delta - d + 1), \tag{G.18}$$

and the higher coefficients can be calculated from the recursion

$$d_{n+1} = -\frac{1}{(n+1)(2n+d+1)}\Big\{(n+\Delta)(n+d-\Delta)d_n \\ + c\big[c_n(2n+d) + c_{n+1}(4n+d+3)\big]\Big\}, \quad n = 0,1,\ldots \tag{G.19}$$

(By convention) we fix $d_0 = 0$ and the general σ-dependent solution is then given by

$$g(\sigma) + pJ(\sigma), \tag{G.20}$$

where p is an arbitrary constant.

G.2 Feynman propagator

By analogy to the Minkowski case, we define the Feynman propagator in Lorentzian AdS space as

$$G(\sigma) = \mathrm{Re}\,\{ig(\sigma + i\epsilon)\}. \tag{G.21}$$

G.2.1 Odd *d*

By using

$$\mathrm{Re}\,\{i(x+i\epsilon)^n\} = 0, \qquad n \geq 0, \\ \mathrm{Re}\left\{\frac{i}{x+i\epsilon}\right\} = \pi\delta(x), \quad \mathrm{Re}\left\{\frac{i}{(x+i\epsilon)^{k+1}}\right\} = \frac{(-1)^k\pi}{k!}\delta^{(k)}(x), \tag{G.22}$$

we can write a contribution to the Feynman propagator coming from $\psi(x)$ as

$$G_{\mathrm{s}}(x) = \sum_{k=0}^{a-1} f_k \delta^{(k)}(x), \qquad f_k = \frac{(-1)^k\pi}{k!} d_{-(k+1)}, \quad k = 0,\ldots,a-1, \tag{G.23}$$

where contributions from non-singular terms vanish. For later use we rewrite the recursion relations in terms of the f_k coefficients as

$$(k+1-\Delta)(k+1+\Delta-d)f_k + (d-2k-1)f_{k-1} = 0, \qquad k = 1,\ldots,a-1, \tag{G.24}$$

$$f_{a-1} = \frac{(-1)^a\sqrt{\pi}}{\pi^{d/2}2^{\frac{d+3}{2}}}, \qquad \pi c = \frac{(\Delta-1)(\Delta-d+1)}{d-1} f_0. \tag{G.25}$$

Using the relation

$$\ln(x+i\epsilon)\, h(x+i\epsilon) = \begin{cases} (i\pi + \ln|x|)h(x) & x < 0 \\ \ln x\, h(x) & x > 0 \end{cases}, \tag{G.26}$$

we have

$$\mathrm{Re}\,[i\ln(x+i\epsilon)h(x+i\epsilon)] = -\pi h(x)\Theta(-x). \tag{G.27}$$

Thus the full Feynman propagator (for odd d in the Lorentzian AdS) becomes

$$G(\sigma) = G_{\mathrm{s}}(x) - c\pi h(x)\Theta(-x). \tag{G.28}$$

In the next section we will need the space-like Green's function, constructed as

$$\overline{g}(Y,Y') = G(\sigma) + \pi c J(\sigma) = G_{\mathrm{s}}(x) + c\pi h(x)\Theta(x). \tag{G.29}$$

This Green's function vanishes in the time-like region ($\sigma < 1$, equivalently, $x < 0$), since $G_s(x) = 0$ for $x \neq 0$.

G.2.2 Even d

In this case the expansion is in half-integer powers and the Feynman propagator is

$$G(\sigma) = \mathrm{Re}\left\{ig(\sigma + i\epsilon)\right\} = \begin{cases} 0 & x > 0 \\ -\sqrt{|x|}\sum_{n=0}^{\infty} d_{n-a} x^{n-d/2} & x < 0 \end{cases} \tag{G.30}$$

Since the half-integer powers cannot be cancelled by adding a term of the form $pJ(\sigma)$ we conclude that there is no σ-dependent space-like Green's function for even d.

H Green's function method

The starting point here is the identity involving the Green's function $\mathcal{G}(Y, Y')$ and a massive free scalar field $\Phi(Y)$:

$$\begin{aligned} \partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu \mathcal{G}\cdot\Phi - \sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\cdot\mathcal{G}\right) &= \sqrt{-g}\left(\mathcal{D}\mathcal{G}\cdot\Phi - \mathcal{D}\Phi\cdot\mathcal{G}\right) \\ &= \sqrt{-g}\left((\mathcal{D}-m^2)\mathcal{G}\cdot\Phi - (\mathcal{D}-m^2)\Phi\cdot\mathcal{G}\right) = \delta(Y, Y')\Phi(Y). \end{aligned} \tag{H.1}$$

Integrating the above relation with respect to Y and using Stokes' theorem, we obtain

$$\Phi(Y') = \int \mathrm{d}^D Y \partial_\mu X^\mu = \oint \mathrm{d}n_\mu X^\mu, \quad X^\mu := \sqrt{-g}g^{\mu\nu}(\partial_\nu\mathcal{G}\cdot\Phi - \partial_\nu\Phi\cdot\mathcal{G}), \tag{H.2}$$

where the surface integral in (H.2) must include the bulk point Y' in its interior. We now choose the space-like Green's function $\mathcal{G} = \overline{\mathcal{G}}$ for $Y' = Y_o$, and furthermore the surface is chosen to be a cylinder with symmetry axis parallel to the t coordinate axis and radius $\rho = R$. The top and bottom bases of the cylinder are at $t = \pm t_1$, where $\frac{\pi}{2} > t_1 > t_o$ and $\cos t_o = \frac{1}{\cosh R}$. The two bases of the cylinder at $t = t_1$ and $t = -t_1$ do not contribute to the integral, since $\sigma < 1$ uniformly there and the space-like Green's function $\overline{\mathcal{G}}$ vanishes. Therefore we can write

$$\Phi(Y_o) = \cosh R(\sinh R)^{d-1}\int_{-t_1}^{t_1}\mathrm{d}t\int \mathrm{d}\Omega[\partial_\rho\overline{\mathcal{G}}\cdot\Phi - \partial_\rho\Phi\cdot\overline{\mathcal{G}}]. \tag{H.3}$$

Since $\overline{\mathcal{G}}(Y, Y_o)$ depends only on t and ρ, the angular integration leads to

$$\Phi(Y_o) = \cosh R(\sinh R)^{d-1}\int_{-t_1}^{t_1}\mathrm{d}t[\partial_\rho\overline{\mathcal{G}}(t,R)\cdot D(t,R) - \partial_\rho D(t,R)\cdot\overline{\mathcal{G}}(t,R)], \tag{H.4}$$

where $D(t,\rho) := \int \mathrm{d}\Omega\,\Phi(t,\rho,\Omega)$ is the S-wave part of the scalar field. For large R, it satisfies the BDHM relation $D(t,R) \approx (\sinh R)^{-\Delta}C(t)$, where $C(t)$ is the S-wave component of the boundary conformal field. Thus the bulk reconstruction formula simplifies to

$$\Phi(Y_o) = \lim_{R\to\infty}(\cosh R)^{-\nu}\int_{-t_1}^{t_1}\mathrm{d}tC(t)[\partial_\rho\overline{\mathcal{G}}(t,R) + \Delta\overline{\mathcal{G}}(t,R)]. \tag{H.5}$$

Further simplification occurs for an odd d, because, as we have seen in appendix G, $\overline{\mathcal{G}}$ only depends on σ in this case. Therefore,

$$\partial_\rho\overline{G}(\sigma) = \cos t\sinh R\,\overline{G}'(\sigma) \approx \sigma\overline{G}'(\sigma) \tag{H.6}$$

for large R, and we can write

$$\Phi(Y_o) = \lim_{R\to\infty} (\cosh R)^{-\nu} \int_{-t_1}^{t_1} \mathrm{d}t C(t)[\sigma\overline{G}'(\sigma) + \Delta\overline{G}(\sigma)]. \tag{H.7}$$

Let us now separate the delta function and theta function parts of the representation in (H.7) as

$$\Phi(Y_o) = \lim_{R\to\infty} (\Phi_0 + \Phi_1), \tag{H.8}$$

where

$$\Phi_0 = (\cosh R)^{-\nu} \int_{-t_1}^{t_1} \mathrm{d}t C(t)[\sigma G_{\mathrm{s}}'(x) + \Delta G_{\mathrm{s}}(x) + \pi c\delta(x)] \tag{H.9}$$

with $h(0) = J(1) = 1$, and

$$\Phi_1 = \pi c(\cosh R)^{-\nu} \int_{-t_o}^{t_o} \mathrm{d}t C(t)[\sigma J'(\sigma) + \Delta J(\sigma)]. \tag{H.10}$$

The range of the t integral becomes $[-t_o, t_o]$ due to $\Theta(x)$, and we introduce $t_o = \frac{\pi}{2} - \varepsilon_o$, where $\sin\varepsilon_o = 1/\cosh R \to 0$ for large R.

For technical reasons, we will now divide Φ_1 into two parts, $\Phi_1 = \Phi_{1a} + \Phi_{1b}$, where

$$\Phi_{1a} = \pi c(\cosh R)^{-\nu} \int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon} \mathrm{d}t C(t)[\sigma J'(\sigma) + \Delta J(\sigma)], \tag{H.11}$$

and

$$\begin{aligned} \Phi_{1b} = \pi c(\cosh R)^{-\nu}\Big\{ &\int_{\pi/2-\varepsilon}^{\pi/2-\varepsilon_o} \mathrm{d}t C(t)[\sigma J'(\sigma) + \Delta J(\sigma)] \\ &+ \int_{-\pi/2+\varepsilon_o}^{-\pi/2+\varepsilon} \mathrm{d}t C(t)[\sigma J'(\sigma) + \Delta J(\sigma)]\Big\}. \end{aligned} \tag{H.12}$$

Here ε is a small but fixed parameter, while ε_o tends to zero as $R \to \infty$. We will let $\varepsilon \to 0$ at the end of the calculation.

H.1 Calculation of Φ_{1a}

As discussed in appendix I, $J(\sigma)$ has a power-law behaviour for large σ if we restrict our considerations to $\Delta > d/2$, which is relevant only for $d = 3$ with an odd d since $\Delta > d - 2$. Using the asymptotics (see appendix I)

$$\sigma J'(\sigma) + \Delta J(\sigma) \approx (\nu + \Delta)G_o\sigma^\nu \tag{H.13}$$

for $\sigma > \sin\varepsilon\cosh R \to \infty$, we have

$$\Phi_{1a} \approx \pi c G_o(2\alpha)(\cosh R)^{-\nu} \int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon} \mathrm{d}t\, C(t)(\cos t\cosh R)^\nu = 2^\nu\xi \int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon} \mathrm{d}t\, C(t)(\cos t)^\nu. \tag{H.14}$$

In the following we will continue the calculation for the cases: $\boxed{\mathcal{A}}$ $\nu > -1$, $\boxed{\mathcal{B}}$ $\nu = -1$, $\boxed{\mathcal{C}}$ $-1 > \nu > -2$, separately. For the simplest case,

$$\boxed{\mathcal{A}} \quad \Phi_{1a} \approx \xi \int_{-\pi/2}^{\pi/2} \mathrm{d}t\, C(t)(2\cos t)^\nu, \tag{H.15}$$

since the integral is convergent for $\varepsilon \to 0$ in this range, while $\boxed{\mathcal{B}}$ $\Phi_{1a} = 0$, since ξ vanishes in this spacial case. In the most complicated case we use a partial integration and obtain

$$\begin{aligned} \boxed{\mathcal{C}} \quad \Phi_{1a} &= -2^\nu \xi \int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon} \mathrm{d}t\, \dot{C}(t) g_1(t) + 2^\nu \xi \big[C(\pi/2-\varepsilon) + C(-\pi/2+\varepsilon)\big] g_1(\pi/2-\varepsilon) \\ &\approx -2^\nu \xi \int_{-\pi/2}^{\pi/2} \mathrm{d}t\, \dot{C}(t) g_1(t) + 2^\nu \xi C_+(\varepsilon) \int_\varepsilon^{\pi/2} \mathrm{d}u (\sin u)^\nu, \end{aligned} \tag{H.16}$$

where $g_1(t)$ is the primitive function

$$g_1(t) = \int_0^t \mathrm{d}u (\cos u)^\nu. \tag{H.17}$$

The last integral can be evaluated as (see appendix I)

$$\int_\varepsilon^{\pi/2} \mathrm{d}u (\sin u)^\nu = \int_\varepsilon^{\pi/2} \mathrm{d}u [(\sin u)^\nu - u^\nu] + \frac{1}{\nu+1}\left(\frac{\pi}{2}\right)^{\nu+1} - \frac{\varepsilon^{\nu+1}}{\nu+1} \approx \tilde{g}_1 - \frac{\varepsilon^{\nu+1}}{\nu+1}. \tag{H.18}$$

Putting elements of this calculation together, we finally obtain

$$\Phi_{1a} = \xi \int_{(\mathrm{sub})} \mathrm{d}t\, (2\cos t)^\nu\, C(t) + \frac{\eta}{2\Omega_d} C_+(\varepsilon) - 2^\nu \xi C_+(\varepsilon) \frac{\varepsilon^{\nu+1}}{\nu+1}, \tag{H.19}$$

where the first term is written in terms of the subtracted integral, defined in (3.8), by reversing the partial integration.

H.2 Calculation of Φ_{1b}

As a first step, we simplify Φ_{1b} as follows:

$$\Phi_{1b} \approx \pi c C_+(0) (\cosh R)^{-\nu} \int_1^M \frac{\mathrm{d}\sigma}{\sqrt{(\cosh R)^2 - \sigma^2}} \big[\sigma J'(\sigma) + \Delta J(\sigma)\big], \tag{H.20}$$

where the upper limit $M = \sin\varepsilon \cosh R$ is large. Next using

$$\frac{1}{\sqrt{1 - \frac{\sigma^2}{(\cosh R)^2}}} - 1 \le \frac{1}{\cos\varepsilon} - 1 = \mathrm{O}(\varepsilon^2) \tag{H.21}$$

we can further approximate Φ_{1b} as

$$\Phi_{1b} \approx \pi c C_+(\varepsilon) (\cosh R)^{-(\nu+1)} \mathcal{F}(\sin\varepsilon \cosh R), \quad \mathcal{F}(M) := \int_1^M \mathrm{d}\sigma \big[\sigma J'(\sigma) + \Delta J(\sigma)\big]. \tag{H.22}$$

Using the asymptotic formula $J(\sigma) \approx G_o \sigma^\nu$ again, we obtain

$$\mathcal{F}'(M) \approx (\nu + \Delta) G_o M^\nu \longrightarrow \mathcal{F}(M) \approx \frac{2\alpha G_o}{\nu+1} M^{\nu+1} + \text{const.}, \tag{H.23}$$

with which Φ_{1b} is evaluated case by case as before. In the first case the constant term is subleading and we obtain

$$\boxed{\mathcal{A}} \quad \Phi_{1b} \approx \frac{2^\nu \xi}{\nu+1} (\sin\varepsilon)^{\nu+1} C_+(\varepsilon) \to 0, \tag{H.24}$$

while $\boxed{\mathcal{B}}$ $\Phi_{1b} \approx 0$ $(c = 0)$ for the next case. For the last case, the constant term dominates and the $M^{\nu+1}$ contribution is subleading, so that we obtain

$$\boxed{\mathcal{C}} \quad \Phi_{1b} \approx \frac{2^\nu \xi}{\nu+1}(\sin\varepsilon)^{\nu+1} C_+(0) + \pi c C_+(0)(\cosh R)^{-(\nu+1)}\mathcal{F}(\infty), \tag{H.25}$$

where the results in appendix I give

$$\mathcal{F}(\infty) = -1 + (\Delta - 1)\int_1^\infty \mathrm{d}\sigma J(\sigma) = -1 - \frac{d-1}{\nu+1}. \tag{H.26}$$

We can now add up the contributions Φ_{1a} and Φ_{1b} and find

$$\boxed{\mathcal{A}} \quad \Phi_1 = \xi \int_{-\pi/2}^{\pi/2} \mathrm{d}t C(t)(2\cos t)^\nu, \tag{H.27}$$

$$\boxed{\mathcal{B}} \quad \Phi_1 = 0, \tag{H.28}$$

$$\boxed{\mathcal{C}} \quad \Phi_1 = \xi \int_{(\mathrm{sub})} \mathrm{d}t C(t)(2\cos t)^\nu + \frac{\eta}{2\Omega_d} C_+(0) + \pi c C_+(0)\mathcal{F}(\infty)(\cosh R)^{-(\nu+1)}. \tag{H.29}$$

H.3 Calculation of Φ_0

Using the delta function identity $x\delta^{(k)}(x) = -k\delta^{(k-1)}(x)$, Φ_0 can be rewritten as

$$\begin{aligned}\Phi_0 = (\cosh R)^{-\nu} \int_{-t_1}^{t_1} \mathrm{d}t\, C(t) \Big\{ & f_{a-1}\delta^{(a)}(x) + \sum_{k=1}^{a-1}[f_{k-1} + (\Delta - k - 1) f_k]\delta^{(k)}(x) \\ & + [(\Delta - 1) f_0 + \pi c]\delta(x) \Big\}.\end{aligned} \tag{H.30}$$

Using the relations

$$\int_0^{t_1} \mathrm{d}t C(t)\delta^{(k)}(x) = (-1)^k \left[\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k \frac{C(t)}{\sqrt{(\cosh R)^2 - \sigma^2}}\right]\Bigg|_{t=t_o,\sigma=1}, \quad \frac{\mathrm{d}t}{\mathrm{d}\sigma} = -\frac{1}{\sqrt{(\cosh R)^2 - \sigma^2}} \tag{H.31}$$

and

$$\int_{-t_1}^{0} \mathrm{d}t C(t)\delta^{(k)}(x) = (-1)^k \left[\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k \frac{C(t)}{\sqrt{(\cosh R)^2 - \sigma^2}}\right]\Bigg|_{t=-t_o,\sigma=1}, \quad \frac{\mathrm{d}t}{\mathrm{d}\sigma} = \frac{1}{\sqrt{(\cosh R)^2 - \sigma^2}} \tag{H.32}$$

we can evaluate (H.30) term by term. Starting with $k = 0$, the sum of the delta function integrals (H.31) and (H.32) give

$$(k=0) \quad \frac{C(t_o) + C(-t_o)}{\sinh R}, \tag{H.33}$$

$$(k=1) \quad \frac{C'(t_o) - C'(-t_o)}{\sinh^2 R} - \frac{C(t_o) + C(-t_o)}{\sinh^3 R}, \tag{H.34}$$

$$(k=2) \quad \frac{C''(t_o) + C''(-t_o)}{\sinh^3 R} - \frac{3[C'(t_o) - C'(-t_o)]}{\sinh^4 R} + \frac{C(t_o) + C(-t_o)}{\sinh^3 R} + \frac{3[C(t_o) + C(-t_o)]}{\sinh^5 R}, \tag{H.35}$$

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and so on. We see that the leading contribution is given by (H.33) and all higher contributions are subleading (of order $(\sinh R)^{-(k+1)}$). After this simplification, we find

$$\Phi_0 \approx (\cosh R)^{-(\nu+1)} C_+(0)[(\Delta-1)f_0 + \pi c], \tag{H.36}$$

which, using the results in appendix I, gives

$$\boxed{\mathcal{A}} \qquad \lim_{R\to\infty} \Phi_0 = 0, \tag{H.37}$$

$$\boxed{\mathcal{B}} \qquad c = 0 \quad (\Delta-1)f_0 = \frac{(-1)^a}{2\Omega_d} \quad \Phi_0 = \frac{(-1)^a}{2\Omega_d} C_+(0), \tag{H.38}$$

$$\boxed{\mathcal{C}} \qquad \Phi_0 \approx (\cosh R)^{-(\nu+1)} C_+(0) c\pi \left\{1 + \frac{d-1}{\nu+1}\right\}. \tag{H.39}$$

The final result of the bulk reconstruction by the Green's function method is given as

$$\begin{aligned}
&\boxed{\mathcal{A}} \quad \Phi(Y_o) = \xi \int_{-\pi/2}^{\pi/2} \mathrm{d}t\, C(t)(2\cos t)^\nu = \ (3.1), \\
&\boxed{\mathcal{B}} \quad \Phi(Y_o) = \frac{(-1)^a}{2\Omega_d} C_+(0) = \ (3.9)\ [\text{for } \ell = 0], \\
&\boxed{\mathcal{C}} \quad \Phi(Y_o) = \xi \int_{(\text{sub})} \mathrm{d}t\, C(t)(2\cos t)^\nu + \frac{\eta}{2\Omega_d} C_+(0) = \ (3.6).
\end{aligned} \tag{H.40}$$

These Green's function results are in complete agreement with those in the main text obtained by different methods, as shown in the last equalities.

I Useful relations

In this appendix we list some results which will be used in the Green's function method.

- From the asymptotics of hypergeometric functions we see that for large argument σ

  $$J(\sigma) \approx G_o \sigma^\nu, \tag{I.1}$$

  which is valid for $\Delta > d/2$ only. Since we consider the range $\Delta > d-2$ in this paper, this restriction is only relevant for $d = 3$. The coefficient in (I.1) is given by

  $$G_o = 2^{\Delta-1} \frac{\Gamma(D/2)\Gamma(\alpha)}{\sqrt{\pi}\,\Gamma(\Delta)}. \tag{I.2}$$

- The integral of the hypergeometric solution $J(\sigma)$ can be calculated with the help of the following two hypergeometric identities:

  $$(1-z)^{a+b-c} {}_2F_1(a,b;c;z) = {}_2F_1(c-a, c-b; c; z), \tag{I.3}$$

  $${}_2F_1(a,b;c;z) = \frac{c-1}{(a-1)(b-1)} \frac{\mathrm{d}}{\mathrm{d}z} {}_2F_1(a-1, b-1; c-1; z). \tag{I.4}$$

  With the substitution $\sigma = \sqrt{1+z}$ the integral is calculated to be

  $$\int_1^\infty \mathrm{d}\sigma J(\sigma) = \frac{1}{2} \int_0^\infty \frac{\mathrm{d}z}{\sqrt{1+z}} {}_2F_1\left(\frac{\Delta}{2}, \frac{d-\Delta}{2}; \frac{d+1}{2}; -z\right) = -\frac{d-1}{(\nu+1)(\Delta-1)}. \tag{I.5}$$

- For the parameter range $\nu > -2$, $\nu \neq -1$ the constant $\tilde{g}_1$ given below is well-defined and is given by

$$\tilde{g}_1 = \int_0^{\pi/2} du[(\sin u)^\nu - u^\nu] + \frac{1}{\nu+1}\left(\frac{\pi}{2}\right)^{\nu+1} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1+\nu}{2}\right)}{\Gamma(1+\nu/2)}. \tag{I.6}$$

- From the recursion (G.24) and (G.25), we can calculate the value of the coefficients

$$f_0 = 2^{-d}\frac{\Gamma(\Delta-1)(-1)^a\sqrt{\pi}}{\Gamma\left(\frac{d-1}{2}\right)\Gamma(\nu+2)\pi^{d/2}}, \qquad \pi c = 2^{-d}\frac{\Gamma(\Delta)(-1)^a\sqrt{\pi}}{2\pi^{d/2}\Gamma(D/2)\Gamma(\nu+1)}. \tag{I.7}$$

- Using the above result, we have

$$\pi c G_o(2\Delta - d) = 2^\nu \xi. \tag{I.8}$$

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