RIMS-1976

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November 2023

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Abstract

We give a restriction of the set of possible values of the 3-loop polynomials of genus 1 knots with trivial Alexander polynomial. As its special case, we present the 3-loop polynomial of any genus 1 knot with (≤ 2)-loop polynomials by using five Vassiliev invariants of the knot. Further, we give a new example of the calculation of the 3-loop polynomial.

1 Introduction

The Kontsevich invariant of knots is an universal invariant among all quantum invariants and all Vassiliev invariants. It takes its values in a space of Jacobi diagrams, which are some kinds of uni-trivalent graphs and have universal properties among simple Lie algebras and their representations. Also, each coefficient of the Kontsevich invariant is a Vassiliev invariant, so we can calculate it theoretically. The Kontsevich invariant has all information of quantum invariants and Vassiliev invariants, and is a very strong invariant, so it is desirable to determine the image of it as precisely as possible. However, the value of the Kontsevich invariant is expressed as an infinite sum of Jacobi diagrams, and in general it is so hard to determine all terms of its value at the same time concretely. In other words, even though we can obtain the information of each term (of small degree) of its value one by one, it is difficult to detect the information of the infinite sum of its value.

One approach to restrict the image of the Kontsevich invariant is the loop expansion. It is conjectured in [19] that the Kontsevich invariant of a knot is expanded in the form of the loop expansion, and it is shown in [11] that the Kontsevich invariant of a knot can be expanded in this form, and it is shown in [7] that the loop expansion is a knot invariant. The loop expansion of the Kontsevich invariant is calculated by using the rational version of the Aarhus integral [7, 11]. By the loop expansion, we can see that when we fix a loop number, each loop part is presented by some finite number of polynomials, where "loop number" means the first Betti number of graphs. In particular, up to 3-loop part, it is known that each loop part can be presented by a single polynomial (or rational form). The 1-loop part is presented by the Alexander polynomial, the 2-loop part is presented by the 2-loop polynomial, and the 3-loop part is presented by the 3-loop polynomial (the 3-loop invariant). Thus n-loop part (n-loop polynomial) in the loop expansion has all information of the infinite sum of n-loop diagrams in the image of the Kontsevich invariant.

The 2-loop polynomial is calculated in many cases. For example, the 2-loop polynomial for knots with up to 7 crossings is calculated in [19], for torus knots in [12, 13, 16], for untwisted Whitehead doubles in [10], and for genus 1 knots in [17]. Further, in [9], it is shown that the 2-loop polynomial of knots with minimal Seifert rank can be computed in terms of a few Vassiliev invariants of degree 3, 5. As its special case, we can present the 2-loop polynomial of genus 1 knots with trivial Alexander polynomial, see also [17]. This fact indicates that the information of the infinite sum of 2-loop part (hence, it contains all 2-loop diagrams of all degrees in the image of the Kontsevich invariant) of those knots are determined and presented by only the information of a few 2-loop diagrams of degree 3, 5 of those knots. On the other hand, the 3-loop polynomial is calculated in a few cases, for example, the 3-loop polynomial of some class of genus 1 knots is calculated in [21].

In this paper, we consider the 3-loop polynomial of genus 1 knots with trivial Alexander polynomial. More concretely, in Theorem 3.1, we give a restriction of the set of possible values of the 3-loop polynomials of genus 1 knots with trivial Alexander polynomial. In general, the values of the 3-loop polynomial of knots with trivial Alexander polynomial belong to (the $\mathbb{Q}$-vector space) $\mathbb{Q}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1)$, but we show that the values of the 3-loop polynomial of genus 1 knots with trivial Alexander polynomial belong to a rather narrow (finitely generated) subspace, and we give its generators concretely. Further, in Theorem 3.3, we present the 3-loop polynomial of any genus 1 knot with trivial (≤ 2)-loop polynomials (namely, trivial Alexander polynomial and trivial 2-loop polynomial) by using five Vassiliev invariants of the knot. This result indicates that the information of the infinite sum of 3-loop part (hence, it contains all 3-loop diagrams of all degrees in the image of the Kontsevich invariant) of those knots are determined and presented by only the information of several 3-loop diagrams of those knots. Moreover, in Section 6, we give a new example of the calculation of the 3-loop polynomial. By this example, we may distinguish two knots in this class which cannot be distinguished by the (≤ 2)-loop polynomials. In general, it is so complicated to calculate the 3-loop polynomial. Thus, in order to calculate as easily as possible, we take a Seifert surface with minimal Seifert rank for a genus 1 knot with trivial Alexander polynomial. Then, we perform the rational version of the Aarhus integral, and present the 3-loop polynomial by using the Vassiliev invariants of a representing tangle of the Seifert surface. After that, we obtain the results.

This paper is organized as follows. In Section 2, we review the Kontsevich invariant, its loop expansion, the 2-loop polynomial and the 3-loop polynomial. In Section 3, we state the main theorems of this paper; we give a restriction of the set of possible values of the 3-loop polynomials of genus 1 knots with trivial Alexander polynomial, and we present the 3-loop polynomials of genus 1 knots with trivial (≤ 2)-loop polynomials. In Section 4, we review the rational version of the Aarhus integral and how to compute the loop expansion of the Kontsevich invariant. In Section 5, we prove the main theorems. In Section 6, we give a new example of calculation of the 3-loop polynomial. In Appendix, we prove several lemmas for some calculations in the proof of the main Theorems.

This work was supported by JST SPRING, Grant Number JPMJSP2110. The author would like to thank Advisor Tomotada Ohtsuki for encouragement and valuable comments, and Professor Stavros Garoufalidis for helpful comments.

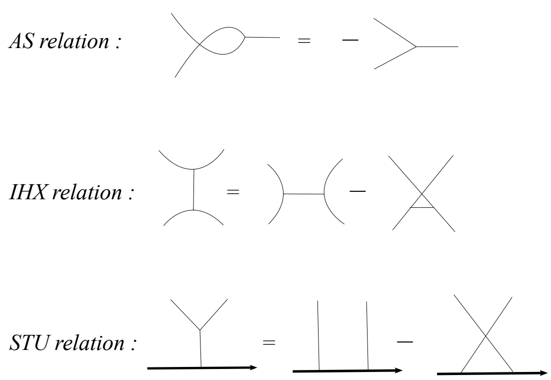
2 The Kontsevich invariant and its loop expansion

In this section, we review the Kontsevich invariant and the 3-loop invariant of knots. In this paper, all the knots are oriented and framed, unless otherwise noted. For the definition of the Kontsevich invariant, see for example [14, 15].

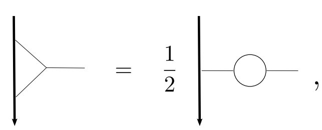
2.1 Jacobi diagrams and the Kontsevich invariant

In this section, we review Jacobi diagrams and the Kontsevich invariant.

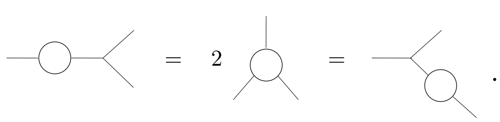
A *Jacobi diagram* on an oriented compact 1-manifold X is an uni-trivalent graph such that univalent vertices are distinct points of X, and each trivalent vertex is *vertex-oriented*, namely, a cyclic order of the three edges around each trivalent vertex is fixed. When drawing a Jacobi diagram on X, we often draw X by thick lines and uni-trivalent graphs by thin lines, and each trivalent vertex is vertex-oriented in the counterclockwise order. Further, we define the *degree* of a Jacobi diagram to be half the number of all vertices of the graph of the diagram. We define $\mathcal{A}(X)$ to be the quotient $\mathbb{Q}$-vector space spanned by Jacobi diagrams on X subject to *the AS, IHX, and STU relations*.



We note that the following equations are obtained by the above relations;



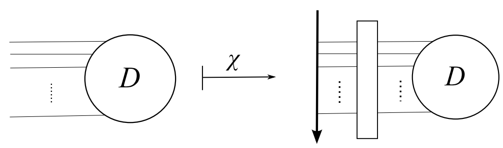
and



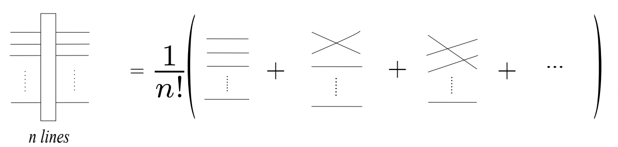
It can be shown (see for example [14, 15]) that $\mathcal{A}(S^1)$ forms a commutative algebra whose product is given by connected sum of copies of S^1. We can also show that $\mathcal{A}(\downarrow)$ forms a commutative algebra whose product is given by vertical concatenation of two copies of $\downarrow$. We can see that $\mathcal{A}(S^1)$ and $\mathcal{A}(\downarrow)$ are naturally isomorphic as commutative algebras by the isomorphism given by connecting two end points of $\downarrow$.

An *open Jacobi diagram* is a vertex-oriented uni-trivalent graph. We sometimes call an edge containing a univalent vertex a *leg*. We define $\mathcal{B}$ to be the quotient $\mathbb{Q}$-vector space spanned by open Jacobi diagrams subject to the AS, IHX relations. It can be shown that $\mathcal{B}$ forms a commutative algebra whose product is given by disjoint union. For an open Jacobi diagram, we call it a *n-loop diagram* if the first Betti number of the diagram is n. We denote $\mathcal{B}_{\text{conn}}$ by the subspace of $\mathcal{B}$ spanned by connected diagrams, and $\mathcal{B}_{\text{conn}}^{(n\text{-loop})}$ by the subspace of $\mathcal{B}_{\text{conn}}$ spanned by n-loop open Jacobi diagrams.

The *PBW isomorphism* $\chi : \mathcal{B} \to \mathcal{A}(\downarrow)$ is defined by



for any diagram $D \in \mathcal{B}$, where the box means the symmetrizer,



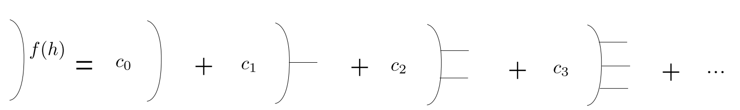
We note that the PBW map is a vector space isomorphism but not an algebra isomorphism.

For a (oriented and framed) knot K, the *Kontsevich invariant* $Z(K)$ is an invariant defined to be in $\mathcal{A}(S^1)(\cong \mathcal{A}(\downarrow))$, formally speaking is in the completion by degree of $\mathcal{A}(S^1)(\cong \mathcal{A}(\downarrow))$, for its concrete definition, see for example [14, 15]. It is known that $Z(K)$ and $\chi^{-1}Z(K)$ can be presented as exponentials of series of connected diagrams.

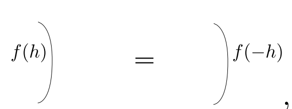
2.2 The loop expansion of the Kontsevich invariant

In this section, we review the loop expansion of the Kontsevich invariant.

For the description about the loop expansion of the Kontsevich invariant, we introduce the labeling of an edge of an open Jacobi diagram by a formal power series, as follows. Let $f(h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + \cdots$ be a power series on the variable h. Then, we define a labeling on one side of an edge of a Jacobi diagram by $f(h)$ by

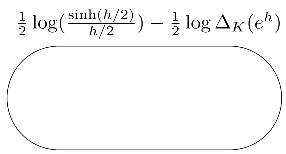


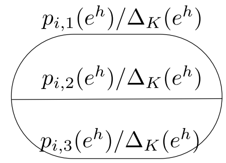
Note that

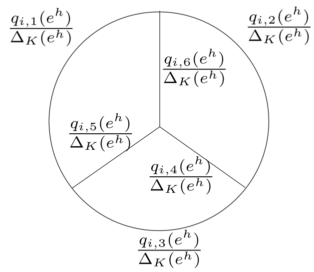


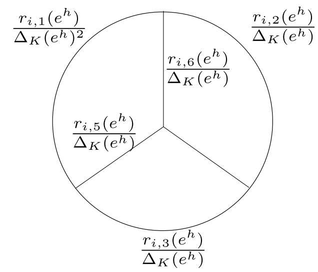
by the AS relation. The loop expansion of the Kontsevich invariant of knot K is a presentation of the following form [7, 11],

Theorem 2.1. *[7, 11] Let K be a 0-framed knot. Then, $\log(\chi^{-1}Z(K))$ is presented as follows,*

$\log(\chi^{-1}Z(K)) =$ 

$+ \sum_i^{finite}$ 

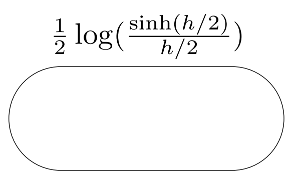
$+ \sum_i^{finite}$ 

$+ \sum_i^{finite}$ 

$+$ *(terms of (> 3)-loop parts),*

where $\Delta_K(t)$ denotes the Alexander polynomial, and $p_{i,j}(e^h), q_{i,j}(e^h), r_{i,j}(e^h)$ are polynomials in $e^{\pm h}$.

For the 3-loop part of the Kontsevich invariant, see [18, 21]. Further, it is known [4] that



$= \chi^{-1}\nu, \qquad (1)$

where we denote $\nu = Z(\text{unknot}) \in \mathcal{A}(S^1)$.

2.3 The 3-loop invariant of knots

In this section, we define the 3-loop invariant (3-loop polynomial) of knots.

Definition 2.2. Let K be a 0-framed knot. The *3-loop invariant* of K is the rational form defined by

$$\begin{aligned}
&\Lambda_K(t_1,t_2,t_3,t_4)\\
&=\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}}\Big(\frac{q_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})}\\
&\qquad\qquad\times\frac{q_{i,4}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(3)}^{-\mathrm{sgn}\tau})q_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(1)}^{-\mathrm{sgn}\tau})q_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})}\Big)\\
&+\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}}\frac{r_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(1)}^{-\mathrm{sgn}\tau})r_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_{\tau(1)}t_{\tau(4)}^{-1})^2\Delta_K(t_{\tau(2)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(1)}^{-1})\Delta_K(t_{\tau(1)}t_{\tau(2)}^{-1})}\\
&\in\frac{1}{\hat{\Delta}^2}\cdot\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4,t_1t_2t_3t_4=1),
\end{aligned}$$

where we put

$$\hat{\Delta}=\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1}).$$

In particular, if $\Delta_K(t)=1$, then $\Lambda_K(t_1,t_2,t_3,t_4)$ is a polynomial, so in this case, we call it the *3-loop polynomial*.

For details about the 3-loop part of the Kontsevich invariant, see [18, 21].

Remark 2.3. For a 0-framed knot K, the 2-loop polynomial of K is the polynomial defined by

$$\Theta_K(t_1,t_2,t_3)=\sum_{\substack{i\\ \epsilon=\pm1\\ \sigma\in\mathfrak{S}_3}}p_{i,1}(t_{\sigma(1)}^{\epsilon})p_{i,2}(t_{\sigma(2)}^{\epsilon})p_{i,3}(t_{\sigma(3)}^{\epsilon})\in\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1}]/(\mathfrak{S}_3\times\mathbb{Z}/2\mathbb{Z},t_1t_2t_3=1),$$

see for example [14, 17]

3 The 3-loop polynomial of genus 1 knots with trivial Alexander polynomial

In this section, we state the main theorems of this paper.

We consider genus 1 knots with trivial Alexander polynomial. We denote $u_{mn}=t_mt_n^{-1}+t_m^{-1}t_n-2$, where $m,n\in\{1,2,3,4\}$).

Theorem 3.1. *The value of the 3-loop polynomial of any genus 1 knot with trivial Alexander polynomial belongs to the $\mathbb{Q}$-vector space generated by the set,*

$$\{\sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}, \sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}u_{\tau(2)\tau(4)}, \sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}u_{\tau(2)\tau(3)}, \sum_{\tau\in\mathfrak{S}_4} (u_{\tau(1)\tau(4)})^2,$$
$$\sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}u_{\tau(2)\tau(4)}u_{\tau(3)\tau(4)}, \sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}u_{\tau(2)\tau(4)}u_{\tau(2)\tau(3)}, \sum_{\tau\in\mathfrak{S}_4} (u_{\tau(1)\tau(4)})^2 u_{\tau(2)\tau(4)},$$
$$\sum_{\tau\in\mathfrak{S}_4} (u_{\tau(1)\tau(4)})^2 u_{\tau(2)\tau(3)}, \sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}u_{\tau(2)\tau(4)}u_{\tau(2)\tau(3)}u_{\tau(3)\tau(1)}, \sum_{\tau\in\mathfrak{S}_4} (u_{\tau(1)\tau(4)})^2 u_{\tau(2)\tau(4)}u_{\tau(3)\tau(4)},$$
$$\sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}(u_{\tau(2)\tau(4)})^2 u_{\tau(2)\tau(3)}\}. \tag{2}$$

In particular, its 3-loop polynomial is determined by eleven Vassiliev invariants (hence, Vassiliev invariants up to degree 12) of the knot.

Remark 3.2. Each element in (2) can be written by using the basis vectors in $\mathbb{Q}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1)$. For example, we put

$$\begin{aligned}
T_{1,3} &= t_1t_2^3 + t_1t_3^3 + t_2t_1^3 + t_2t_3^3 + t_3t_1^3 + t_3t_2^3 \\
&\quad + t_1^2t_2^{-1}t_3^{-1} + t_2^2t_1^{-1}t_3^{-1} + t_3^2t_1^{-1}t_2^{-1} + t_1^{-2}t_2^{-3}t_3^{-3} + t_2^{-2}t_1^{-3}t_3^{-3} + t_3^{-2}t_1^{-3}t_2^{-3}, \\
T_{2,2} &= t_1^2t_2^2 + t_2^2t_3^2 + t_3^2t_1^2 + t_1^{-2}t_2^{-2} + t_2^{-2}t_3^{-2} + t_3^{-2}t_1^{-2}, \\
T_{1,1,2} &= t_1t_2t_3^2 + t_2t_3t_1^2 + t_3t_1t_2^2 + t_1^{-1}t_2^{-1}t_3^{-2} + t_2^{-1}t_3^{-1}t_1^{-2} + t_3^{-1}t_1^{-1}t_2^{-2} \\
&\quad + t_1t_2^{-1} + t_1t_3^{-1} + t_2t_1^{-1} + t_2t_3^{-1} + t_3t_1^{-1} + t_3t_2^{-1}, \\
T_{2,2,4} &= t_1^2t_2^2t_3^4 + t_2^2t_3^2t_1^4 + t_3^2t_1^2t_2^4 + t_1^{-2}t_2^{-2}t_3^{-4} + t_2^{-2}t_3^{-2}t_1^{-4} + t_3^{-2}t_1^{-2}t_2^{-4} \\
&\quad + t_1^2t_2^{-2} + t_1^2t_3^{-2} + t_2^2t_1^{-2} + t_2^2t_3^{-2} + t_3^2t_1^{-2} + t_3^2t_2^{-2}, \\
T_{2,3,3} &= t_1^2t_2^3t_3^3 + t_2^2t_3^3t_1^3 + t_3^2t_1^3t_2^3 + t_1t_2t_3^{-2} + t_2t_3t_1^{-2} + t_3t_1t_2^{-2} \\
&\quad + t_1^{-1}t_2^{-3} + t_1^{-1}t_3^{-3} + t_2^{-1}t_1^{-3} + t_2^{-1}t_3^{-3} + t_3^{-1}t_1^{-3} + t_3^{-1}t_2^{-3},
\end{aligned}$$

then, the first four elements in (2) are written as (these equations in [21] are written a little mistakenly)

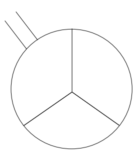
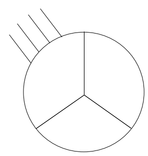
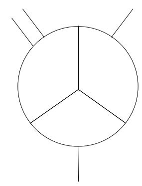
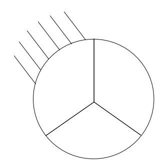
$$\begin{aligned}
&\sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)} = 4(u_{12} + u_{13} + u_{14} + u_{23} + u_{24} + u_{34}) = 4T_{1,1,2} - 48, \\
&\sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}u_{\tau(2)\tau(4)} = 2(u_{14}u_{24} + u_{14}u_{34} + u_{24}u_{34} + u_{13}u_{23} + u_{13}u_{43} + u_{23}u_{43} \\
&\qquad + u_{12}u_{32} + u_{12}u_{42} + u_{32}u_{42} + u_{21}u_{31} + u_{21}u_{41} + u_{31}u_{41}) \\
&= 2T_{2,3,3} + 2T_{1,3} - 12T_{1,1,2} + 96, \\
&\sum_{\tau\in\mathfrak{S}_4} u_{\tau(1)\tau(4)}u_{\tau(2)\tau(3)} = 8(u_{12}u_{34} + u_{13}u_{24} + u_{14}u_{23}) = 8T_{2,2} - 16T_{1,1,2} + 96,
\end{aligned}$$

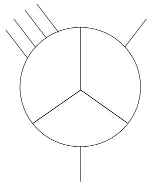
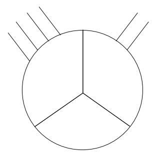
$$\sum_{\tau\in\mathfrak{S}_4}(u_{\tau(1)\tau(4)})^2 = 4(u_{12}^2+u_{13}^2+u_{14}^2+u_{23}^2+u_{24}^2+u_{34}^2) = 4T_{2,2,4}-16T_{1,1,2}+144,$$

In Section 6, we give a new example of calculation of the 3-loop polynomial of genus 1 knots with trivial Alexander polynomial.

As its special case, we consider genus 1 knots with trivial ($\leq$ 2)-loop polynomials (namely, trivial Alexander polynomial and trivial 2-loop polynomial) K. We denote

$$\chi^{-1}Z(K) = \exp\left(\sum_{i\geq 1} a_i\lambda_i + (\text{terms of } (>3)\text{-loop parts})\right),$$

where $\lambda_1 =$  , $\lambda_2 =$  , $\lambda_3 =$  , $\lambda_4 =$  ,

$\lambda_5 =$  , $\lambda_6 =$  , and λ_i's ($i > 6$) are basis vectors of $\mathcal{B}_{\text{conn}}^{(\text{3-loop})}$ of degree>8. It is known [6] that λ_i's ($1 \leq i \leq 6$) are basis vectors of $\mathcal{B}_{\text{conn}}^{(\text{3-loop})}$ of degree$\leq$8.

Theorem 3.3. *Let K be a genus 1 knot with trivial ($\leq$ 2)-loop polynomials. Then, its 3-loop polynomial $\Lambda_K(t_1,t_2,t_3,t_4)$ is determined by five Vassiliev invariants of K. More concretely, we can present $\Lambda_K(t_1,t_2,t_3,t_4)$ by*

$$\begin{aligned}
&\Lambda_K(t_1,t_2,t_3,t_4)\\
&= 4a_1(u_{12}+u_{13}+u_{14}+u_{2,3}+u_{24}+u_{34})\\
&\quad+\left(-\frac{a_1}{3}+4a_2+2a_3\right)(u_{14}u_{24}+u_{14}u_{34}+u_{24}u_{34}+u_{13}u_{23}+u_{13}u_{43}+u_{23}u_{43}\\
&\qquad\quad+u_{12}u_{32}+u_{12}u_{42}+u_{32}u_{42}+u_{21}u_{31}+u_{21}u_{41}+u_{31}u_{41})\\
&\quad-4a_3(u_{12}u_{34}+u_{13}u_{24}+u_{14}u_{23})\\
&\quad+\left(\frac{a_1}{10}+a_3-36a_4-6a_5\right)(u_{14}u_{24}u_{34}+u_{13}u_{23}u_{43}+u_{12}u_{32}u_{42}+u_{21}u_{31}u_{41})\\
&\quad+\left(\frac{a_1}{30}+\frac{2}{3}a_3-12a_4-4a_5\right)(u_{13}u_{43}u_{42}+u_{23}u_{43}u_{41}+u_{12}u_{42}u_{43}+u_{32}u_{42}u_{41}\\
&\qquad\quad+u_{21}u_{41}u_{43}+u_{31}u_{41}u_{42}+u_{12}u_{32}u_{34}+u_{42}u_{32}u_{31}+u_{21}u_{31}u_{34}\\
&\qquad\quad+u_{41}u_{31}u_{32}+u_{31}u_{21}u_{24}+u_{41}u_{21}u_{23}).
\end{aligned}$$

Corollary 3.4. *For $i > 5$, each coefficient a_i can be presented by a linear sum of $a_1, \cdots, a_5$. In particular, we have* $a_6 = -\frac{1}{180}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 - 8a_4 - 2a_5$.

In general, it is very hard to detect the information of the infinite sum of the image of the Kontsevich invariant. On the other hand, thanks to the existence of the form of the loop expansion, we can say that the 3-loop polynomial contains all the information of the infinite sum of 3-loop part of the Kontsevich invariant. Thus, Theorem 3.1 implies that the information of the infinite sum of 3-loop part of genus 1 knots with trivial Alexander polynomial is determined by only eleven (primitive) Vassiliev invariants of them, and we obtain a strong restriction of the set of possible values of the 3-loop polynomials of them. Further, Theorem 3.3 and Corollary 3.4 imply that the information of the infinite sum of 3-loop part of genus 1 knots with trivial (≤ 2)-loop polynomials is determined by only five (primitive) Vassiliev invariants of them, and we obtain the concrete presentation of the 3-loop polynomial of them.

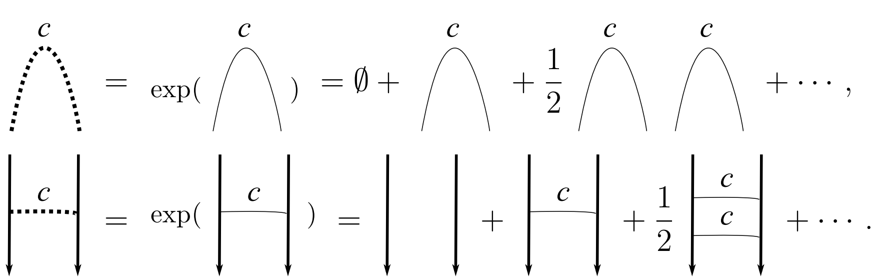
Remark 3.5. It is shown in [21] that for two knots K_1, K_2 with trivial Alexander polynomial, we have $\Lambda_{K_1 \# K_2}(t_1, t_2, t_3, t_4) = \Lambda_{K_1}(t_1, t_2, t_3, t_4) + \Lambda_{K_2}(t_1, t_2, t_3, t_4)$, where $K_1 \# K_2$ is the connected sum of K_1 and K_2. Thus Theorems 3.1, 3.3 and Corollary 3.4 hold for knots with trivial Alexander polynomial (or, with trivial (≤ 2)-loop polynomials) which are obtained from connected sums of genus 1 knots.

Remark 3.6. A similar (and more sophisticated) result for the 2-loop polynomial of knots with trivial Alexander polynomial is already known, see [9, 17] and Section 5.3.

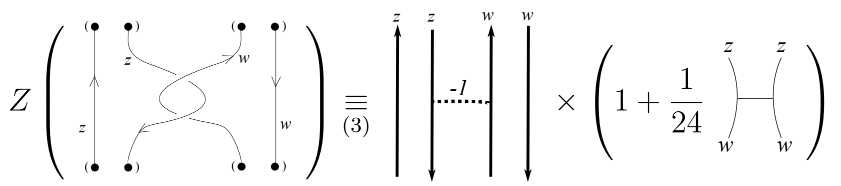
4 The rational version of the Aarhus integral and a computation of the loop expansion

In this section, we review how to compute the loop expansion of the Kontsevich invariant. Along this, we calculate the 3-loop invariant. For details, see for example, [7, 11].

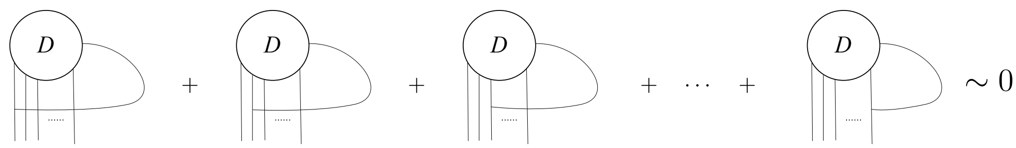
From now on, we represent "exponential" by  , for example;



For Jacobi diagrams α and β, we write $\alpha \underset{(m+1)}{\equiv} \beta$ if $\alpha - \beta$ can be expressed as a linear sum of Jacobi diagrams with more than m trivalent vertices, noting that we do not count trivalent vertices generated by attached power series. In this paper, we use "$\underset{(5)}{\equiv}$" many times, so for simplicity, we write "$\equiv$" instead of "$\underset{(5)}{\equiv}$". If the underlying uni-trivalent graph of a Jacobi diagram on a 1-manifold X has m trivalent vertices, then their univalent vertices can be placed anywhere in X modulo "$\underset{(m+1)}{\equiv}$". Thus, we can write them separately, for example,



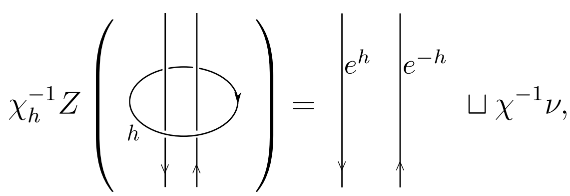
Further, we sometimes use *link relation* "$\sim$" (see [1, 2, 3]), which is defined by



and we use the equivalent relation "$\underset{(m+1)}{\sim}$", which is generated by $\underset{(m+1)}{\equiv}$ and $\sim$. It is known that under the link relation, the result of the Aarhus integral does not change (see [5]).

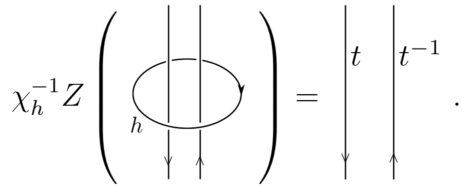
Let K be a 0-framed knot in S^3. It is known that K has a *surgery presentation* $K_0 \cup L$, such that K_0 is the unknot with 0 framing and L is a (l-components) framed link, and the linking number of K_0 and each component of L is equal to 0, further, the pair obtained from the pair (S^3, K_0) by surgery along L is homeomorphic to (S^3, K). We can obtain the loop expansion of the Kontsevich invariant of K from the Kontsevich invariant of $K_0 \cup L$, in the following way (see [11]). Let X be a finite set. We define $\mathcal{A}(*_X)$ to be the space of open Jacobi diagrams whose legs are labeled by elements of X.

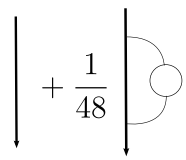
First, we compute $\chi_h^{-1} Z(K_0 \cup L)$. We label the component corresponding to K_0 by h, and label the components corresponding to L by the set $X = \{x_1, x_2, \cdots, x_l\}$. Then, we compute $\chi_h^{-1} Z(K_0 \cup L)$, where $\chi_h : \mathcal{A}(*_h \sqcup \bigsqcup_X S^1) \to \mathcal{A}(\downarrow_h \sqcup \bigsqcup_X S^1) \cong \mathcal{A}(S^1_h \sqcup \bigsqcup_X S^1)$ is defined as the PBW isomorphism. It is known that



see [4, Theorem 4], [11, Corollary 5.0.8]. We put $t = e^h$, and we write it again (omitting

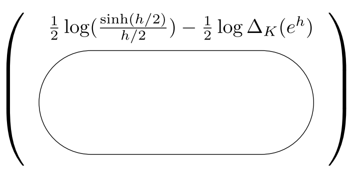
$\chi^{-1}\nu$ for simplicity, since it does not contribute to the 3-loop part),



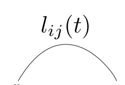
Second, we compute $\chi^{-1}\check{Z}(K_0 \cup L)$. The value $\chi_h^{-1}\check{Z}(K_0 \cup L)$ is obtained from $\chi_h^{-1}Z(K_0 \cup L)$ by connected-summing by ν to each component labeled by an element of X, where we denote $\nu = Z(\text{unknot}) \in \mathcal{A}(\downarrow)$. By (1), we can show that $\nu \underset{(3)}{\equiv}$ . Then, we compute $\chi_X^{-1}\chi_h^{-1}\check{Z}(K_0 \cup L)$, where we choose a disjoint union of the unknot and a string link $K_0 \cup \check{L}$ whose closure is isotopic to $K_0 \cup L$, and $\chi_X : \mathcal{A}(*_X) \to \mathcal{A}(\bigsqcup_X \downarrow)$ is the map defined by the composition of all PBW isomorphisms for all elements in X. We denote $\chi_X^{-1}\chi_h^{-1}\check{Z}(K_0 \cup L)$ by $\chi^{-1}\check{Z}(K_0 \cup L)$.

Third, we compute the rational version of the *Aarhus integral* (see [1, 2, 3, 11]). The Kontsevich invariant of K is computed by the rational version of the Aarhus integral as follows,

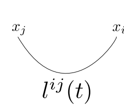
$$\chi^{-1}Z(K) = \chi^{-1}Z^{LMO}(S^3, K)$$

$$= \exp\left(\right.$$  $$\left.\right) \sqcup \frac{\langle\langle \chi^{-1}\check{Z}(K_0 \cup L)\rangle\rangle}{\langle\langle \chi^{-1}\check{Z}(U_+)\rangle\rangle^{\sigma_+}\langle\langle \chi^{-1}\check{Z}(U_-)\rangle\rangle^{\sigma_-}},$$

where $U_\pm$ denotes the unknot with ± 1 framing, and σ_+ and σ_- are the number of the positive and negative eigenvalues of the linking matrix of L. The operation "$\langle\langle\quad\rangle\rangle$" which is known as the *Aarhus integral* is defined as follows. It is known that $\chi^{-1}\check{Z}(K_0 \cup L)$ is presented by

$$\chi^{-1}\check{Z}(K_0 \cup L) = \exp\Big(\frac{1}{2}\sum_{x_i, x_j \in X}$$  $$\Big) \sqcup P\big(\chi^{-1}\check{Z}(K_0 \cup L)\big), \qquad (3)$$

where $\big(l_{ij}(t)\big)$ is an equivariant linking matrix of $L \subset S^3 \backslash K_0$ which satisfies that $l_{ji}(t) = l_{ij}(t^{-1})$, and $P\big(\chi^{-1}\check{Z}(K_0 \cup L)\big)$ is a sum of diagrams which have at least one trivalent vertex on each component. For details about an equivariant linking matrix, see for example [8]. Then, we define

$$\langle\langle \chi^{-1}\check{Z}(K_0 \cup L)\rangle\rangle = \Big\langle \exp\Big(-\frac{1}{2}\sum_{x_i, x_j \in X}$$  $$\Big), P\big(\chi^{-1}\check{Z}(K_0 \cup L)\big)\Big\rangle, \qquad (4)$$

where $\big(l^{ij}(t)\big) = \big(l_{ij}(t)\big)^{-1}$, and $\langle \quad , \quad \rangle$ is defined by

$$\langle C_1, C_2 \rangle = \left(\begin{array}{l} \text{sum of all ways gluing the } x\text{-marked legs of } C_1 \\ \text{to the } x\text{-marked legs of } C_2 \text{ for all } x \in X \end{array} \right). \tag{5}$$

For details, see [1]. Note ([5, equation (21)]) that

$$\langle\!\langle \chi^{-1} \check{Z}(U_\pm) \rangle\!\rangle = \langle \chi^{-1}\nu, \chi^{-1}\nu \rangle^{-1} \exp\left(\mp \frac{1}{16} \ominus \right) \underset{(3)}{\equiv} \exp\left(\mp \frac{1}{16} \ominus \right). \tag{6}$$

By this procedure, we can compute the loop expansion of the Kontsevich invariant of K, and in particular, we can get its 3-loop part as follows,

$$\chi^{-1} Z(K)^{(3-loop)} = \left(\frac{\langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\!\rangle}{\langle\!\langle \chi^{-1} \check{Z}(U_+) \rangle\!\rangle^{\sigma_+} \langle\!\langle \chi^{-1} \check{Z}(U_-) \rangle\!\rangle^{\sigma_-}} \right)^{(3-loop)}. \tag{7}$$

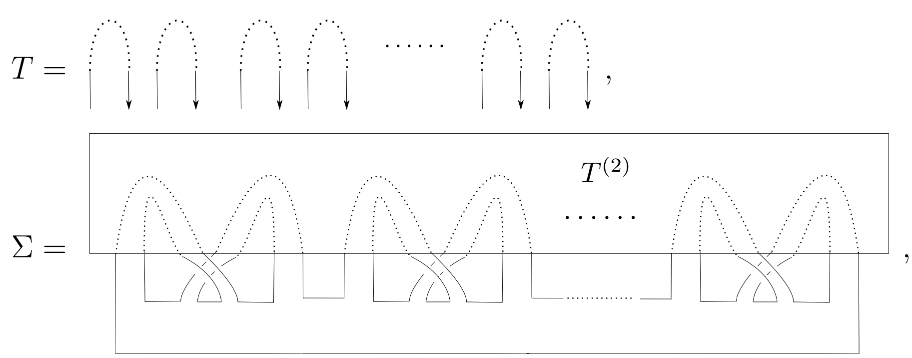
Finally, we obtain the 3-loop invariant $\Lambda_K(t_1, t_2, t_3, t_4)$.

5 Proofs of main theorems

In this section, we prove Theorems 3.1 , 3.3, and Corollary 3.4.

5.1 Representing tangles of knots

In general, for a knot K and its Seifert surface Σ with genus g, there exists a $2g$-component framed tangle T such that



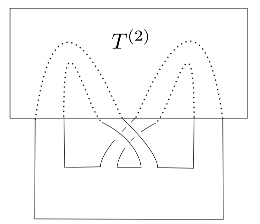
where dotted lines in the picture of T imply strands knotted or linked in some fashion, and $T^{(2)}$ denotes the double of T. From now on, we call T a representing tangle of K. Note that a representing tangle is not unique for a knot K.

5.2 Proof of Theorem 3.1

In this section, we prove Theorem 3.1. Let K be a genus 1 knot with trivial Alexander polynomial.

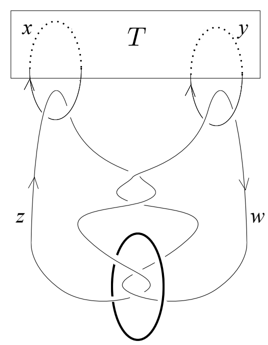
Lemma 5.1. *[9] There exists a 2-component representing tangle T of K such that each strand in T has 0-framing and the linking number of the two strands in T is equal to 0.*

This Lemma immediately follows from [9, Lemma 4.2, Corollary 4.3]. For K, we choose such a representing tangle T,

$K =$ 

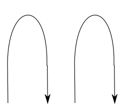
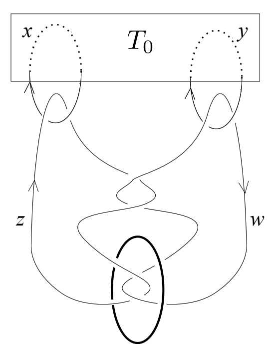
.

Then, we get the following surgery presentation of K,

$K_0 \cup L =$ 

,

where K_0 is depicted by a thick line and L is depicted by thin lines. Namely, K is obtained from $K_0 \cup L$ by surgery along L. Further, we put

$T_0 =$  , and $K_0 \cup L_0 =$ 

.

Lemma 5.2. *We have*

$$\chi^{-1} Z(K)^{\textit{(3-loop)}} = \left(\frac{\langle\langle \chi^{-1} \check{Z}\big(K_0 \cup L\big) - \chi^{-1} \check{Z}\big(K_0 \cup L_0\big) \rangle\rangle}{\langle\langle \chi^{-1} \check{Z}(U_+) \rangle\rangle^{\sigma_+} \langle\langle \chi^{-1} \check{Z}(U_-) \rangle\rangle^{\sigma_-}} \right)^{\textit{(3-loop)}},$$

Proof. By (7), we have

$$\chi^{-1} Z(K)^{(\text{3-loop})} = \left(\frac{\langle\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\rangle}{\langle\langle \chi^{-1} \check{Z}(U_+) \rangle\rangle^{\sigma_+} \langle\langle \chi^{-1} \check{Z}(U_-) \rangle\rangle^{\sigma_-}} \right)^{(\text{3-loop})}.$$

On the other hand, we can see that $K_0 \cup L_0$ is a surgery presentation of the unknot, so its 3-loop part equals to 0. Hence, we get

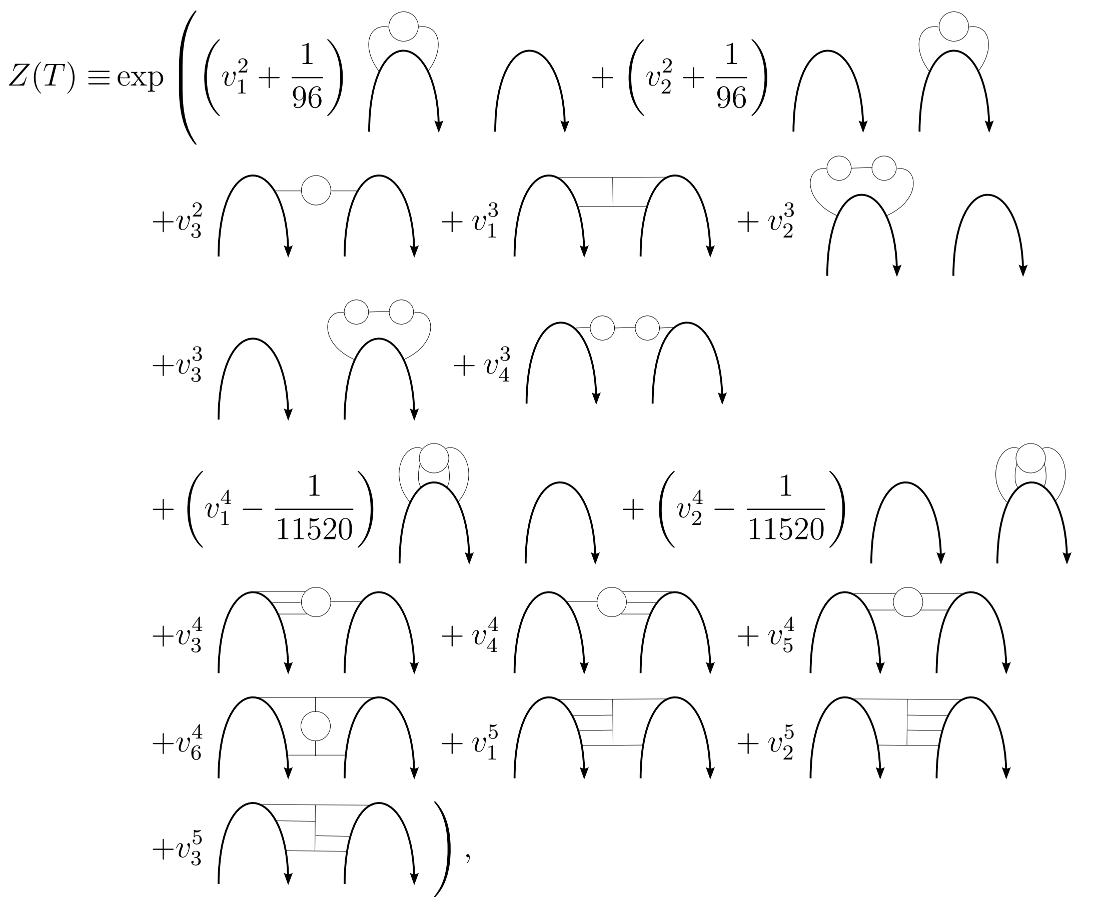
$$\begin{aligned}
&\chi^{-1} Z(K)^{(\text{3-loop})} \\
&= \left(\frac{\langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\!\rangle}{\langle\!\langle \chi^{-1} \check{Z}(U_+) \rangle\!\rangle^{\sigma_+} \langle\!\langle \chi^{-1} \check{Z}(U_-) \rangle\!\rangle^{\sigma_-}} - \frac{\langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L_0) \rangle\!\rangle}{\langle\!\langle \chi^{-1} \check{Z}(U_+) \rangle\!\rangle^{\sigma_+} \langle\!\langle \chi^{-1} \check{Z}(U_-) \rangle\!\rangle^{\sigma_-}} \right)^{(\text{3-loop})} \\
&= \left(\frac{\langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\!\rangle - \langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L_0) \rangle\!\rangle}{\langle\!\langle \chi^{-1} \check{Z}(U_+) \rangle\!\rangle^{\sigma_+} \langle\!\langle \chi^{-1} \check{Z}(U_-) \rangle\!\rangle^{\sigma_-}} \right)^{(\text{3-loop})}.
\end{aligned}$$

Since T satisfies the condition of Lemma 5.1, $L \subset S^3 \backslash K_0$ and $L_0 \subset S^3 \backslash K_0$ have the same equivariant linking matrix. Thus, we obtain

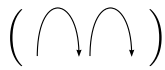
$$\langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\!\rangle - \langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L_0) \rangle\!\rangle = \langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L) - \chi^{-1} \check{Z}(K_0 \cup L_0) \rangle\!\rangle.$$

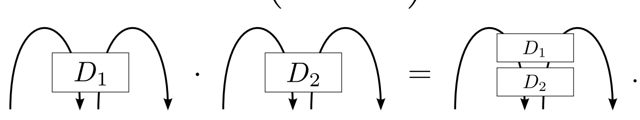
Therefore, we obtain the required equation. □

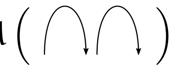
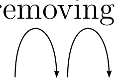
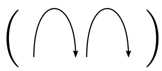
We can denote



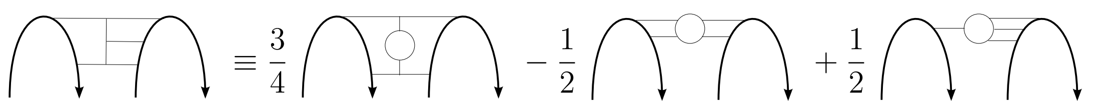
(8)

where the product of two elements in $\mathcal{A}$  is defined by



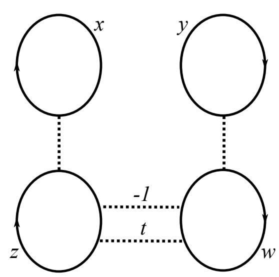
Moreover, the coproduct is defined as follows. For any $D \in \mathcal{A}\Big($  $\Big)$, we define $\hat{\Delta}(D)$ to be the sum of $D' \otimes D''$ where D' runs over all diagrams obtained as a subset of D by removing some of the thin connected components and D'' is the diagram consisting of  and the other components of thin components. It can be shown that $\mathcal{A}\Big($  $\Big)$ forms a bialgebra. Since it is known that $Z(T)$ is a group-like element, it is presented as an exponential of a primitive elements. Thus, we can denote $Z(T)$ like (8).

Remark 5.3. There are other connected diagrams up to degree 5 which are not depicted in (8), but each of those diagrams is equivalent to a sum of the diagrams depicted in (8) or is equivalent to 0 modulo "≡". For example, we can show that

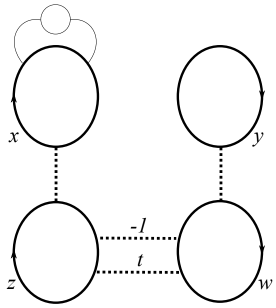
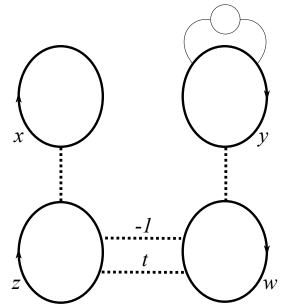
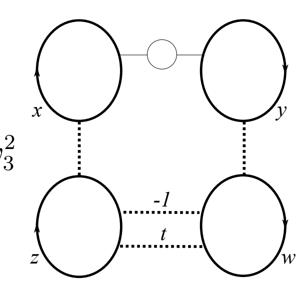


Let $\mathcal{U} \subset \mathcal{A}(*_{\{h,x,y,z,w\}})$ be the subspace spanned by diagrams which have 4 trivalent vertices (we do not count trivalent vertices generated by $t = e^h$) and satisfies $|x| < |z|$ or $|y| < |w|$ or $|x| - |z| \neq |y| - |w|$, where $|x|, |y|, |z|, |w|$ implies the number of univalent vertices labeled by x, y, z, w, respectively. (As shown later in Lemma 5.6, we note that any element in $\mathcal{U}$ does not contribute the 3-loop part of K.) We recall that $t = e^h$ and h is the label associated to K_0.

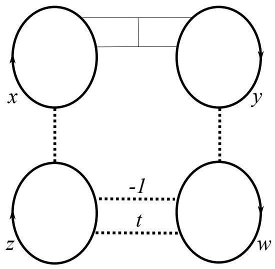
Lemma 5.4. *We have*

$$\chi_h^{-1}\check{Z}(K_0 \cup L) - \chi_h^{-1}\check{Z}(K_0 \cup L_0) \equiv \zeta +$$  $$\times (\gamma_1 + \delta_1),$$

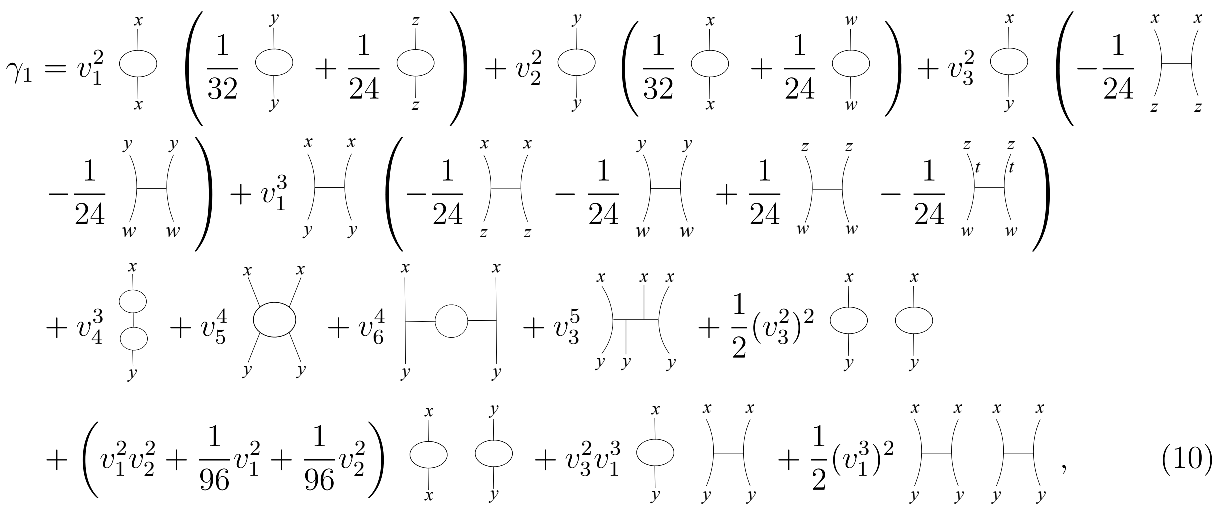
where

$\zeta = v_1^2$  $+ v_2^2$  $+ v_3^2$ 

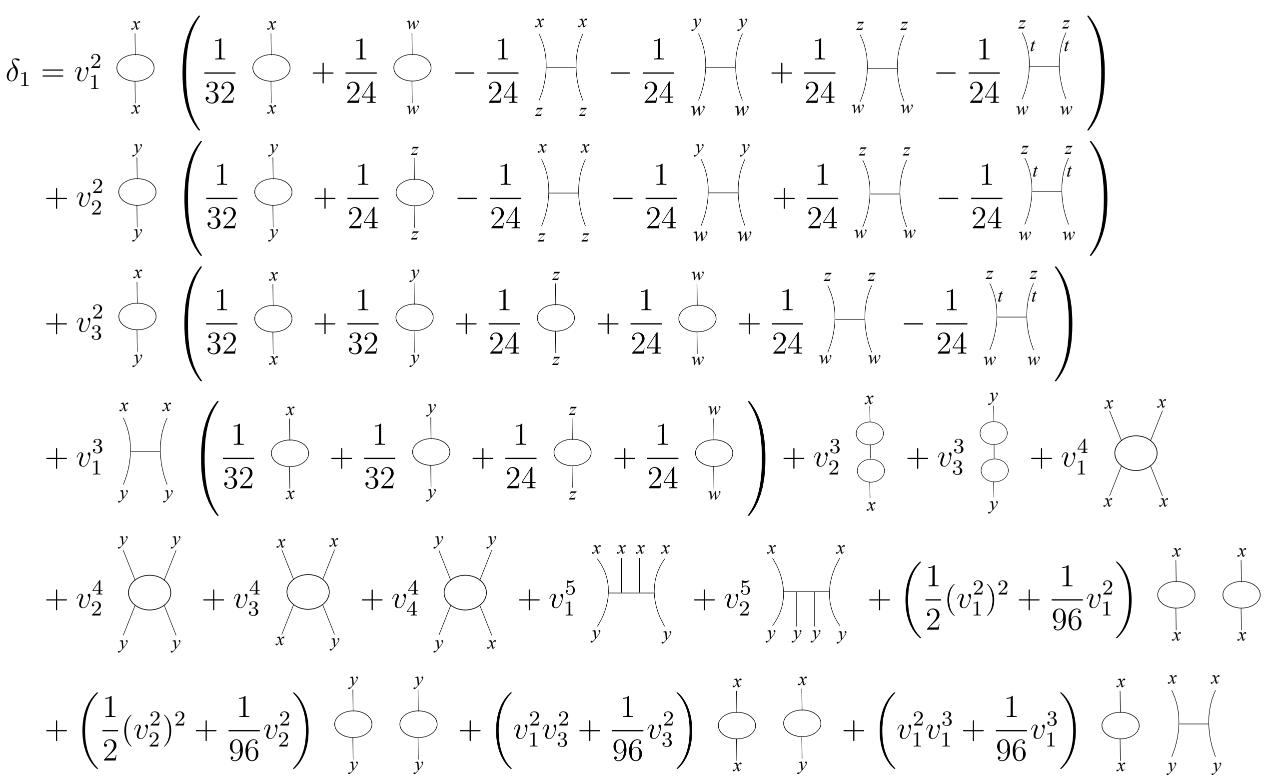
$+ v_1^3$

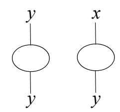
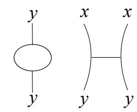


(9)



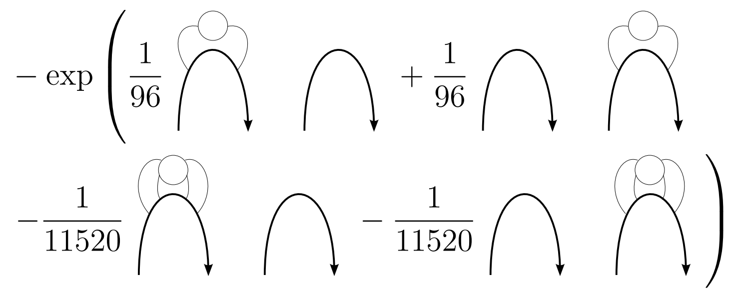
(10)



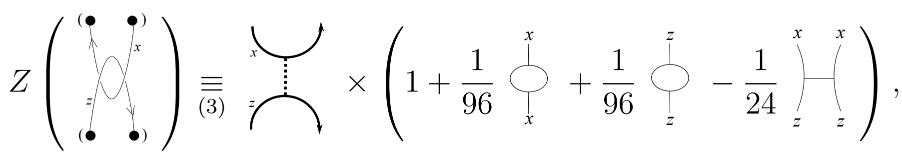
$$+\left(v_2^2v_3^2+\frac{1}{96}v_3^2\right)$$  $$+\left(v_2^2v_1^3+\frac{1}{96}v_1^3\right)$$  $$\in \mathcal{U}. \qquad (11)$$

Proof. First, we calculate $\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0)$ by decomposing into the following parts. We note that each term of the right-hand side of the formula of (12) below has at least 2 trivalent vertices, so it is sufficient to calculate other parts modulo "$\underset{(3)}{\equiv}$".

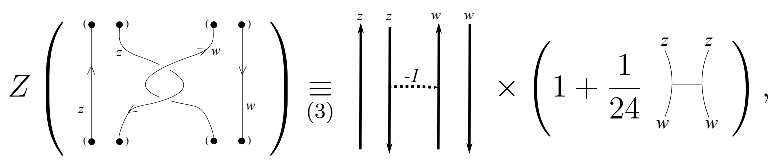
$$Z(T) - Z(T_0) \equiv \big(\text{the right hand side of (8)}\big)$$



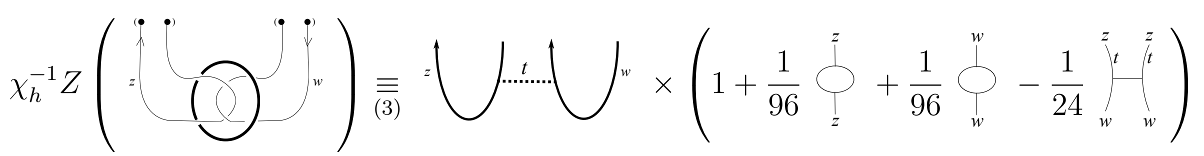
(12)



(13)

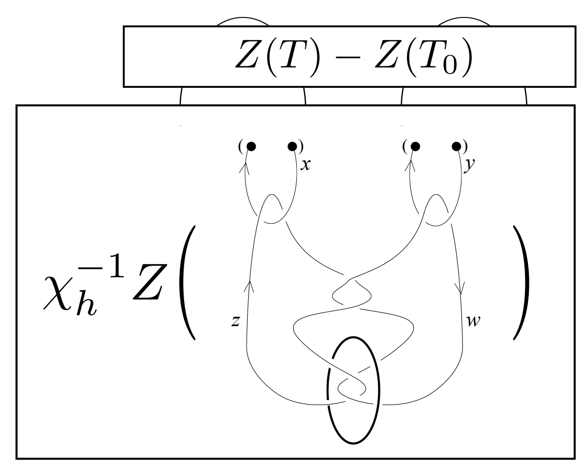


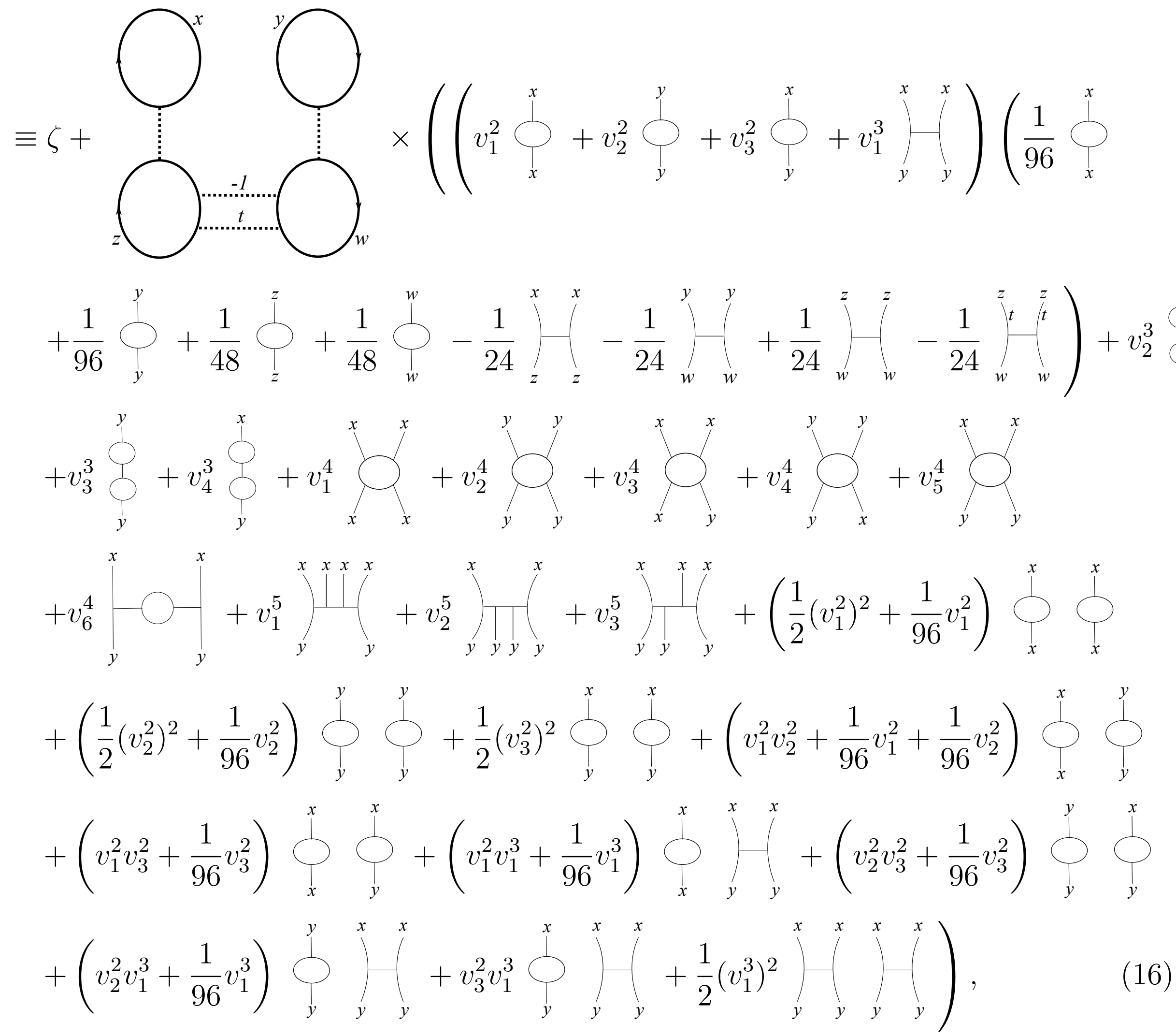
(14)

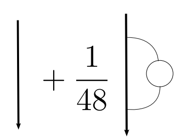


(15)

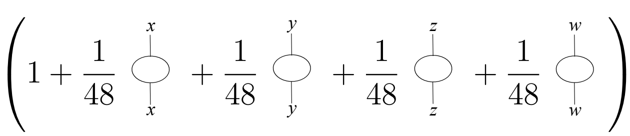
Then, by (12), (13), (14), and (15), we get

$$\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0) =$$ 



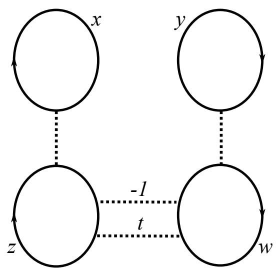
where ζ is given by (9). Now, recall that $\nu \underset{(3)}{\equiv}$  , so

$$\chi_h^{-1}\check{Z}(K_0 \cup L) - \chi_h^{-1}\check{Z}(K_0 \cup L_0)$$
$$= \big(\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0)\big)\#\nu^{\otimes 4}$$
$$\equiv \big(\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0)\big) \times$$

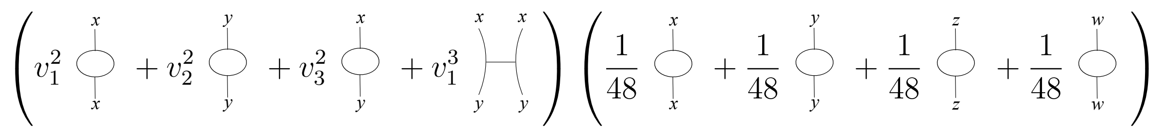


$$\equiv (\text{the right-hand side of (16)})$$

$+$



$\times$



(17)

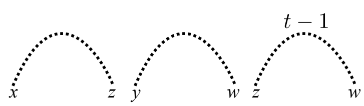
We put γ_1 as in (10) and δ_1 as in (11), noting that $\delta_1 \in \mathcal{U}$. Thus, we obtain the required formula. $\square$

From now on, we use the notation "$\equiv$" as meaning "$\underset{(5)}{\equiv}$" or "$\underset{(5)}{\overset{z,w}{\sim}}$".

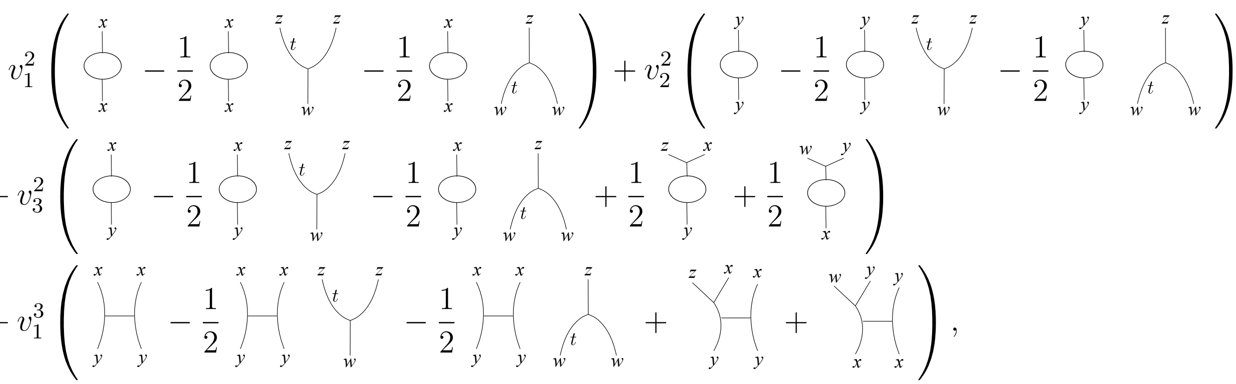
Lemma 5.5. *We have*

$$\chi^{-1}\check{Z}(K_0 \cup L) - \chi^{-1}\check{Z}(K_0 \cup L_0) \equiv \alpha \sqcup (\beta + \gamma_1 + \gamma_2 + \delta_1 + \delta_2),$$

where

$\alpha =$  ,

$\beta =$

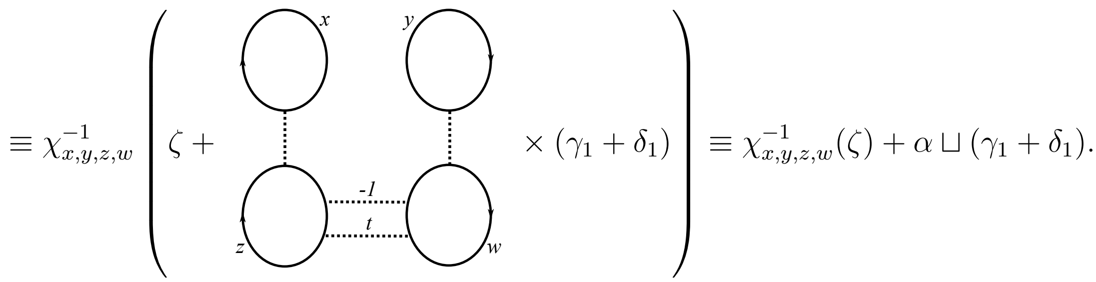


$$\gamma_2 = v_1^2\gamma_2^1 + v_2^2\gamma_2^2 + v_3^2\gamma_2^3 + v_1^3\gamma_2^4,$$
$$\delta_2 = v_1^2\delta_2^1 + v_2^2\delta_2^2 + v_3^2\delta_2^3 + v_1^3\delta_2^4 \in \mathcal{U},$$

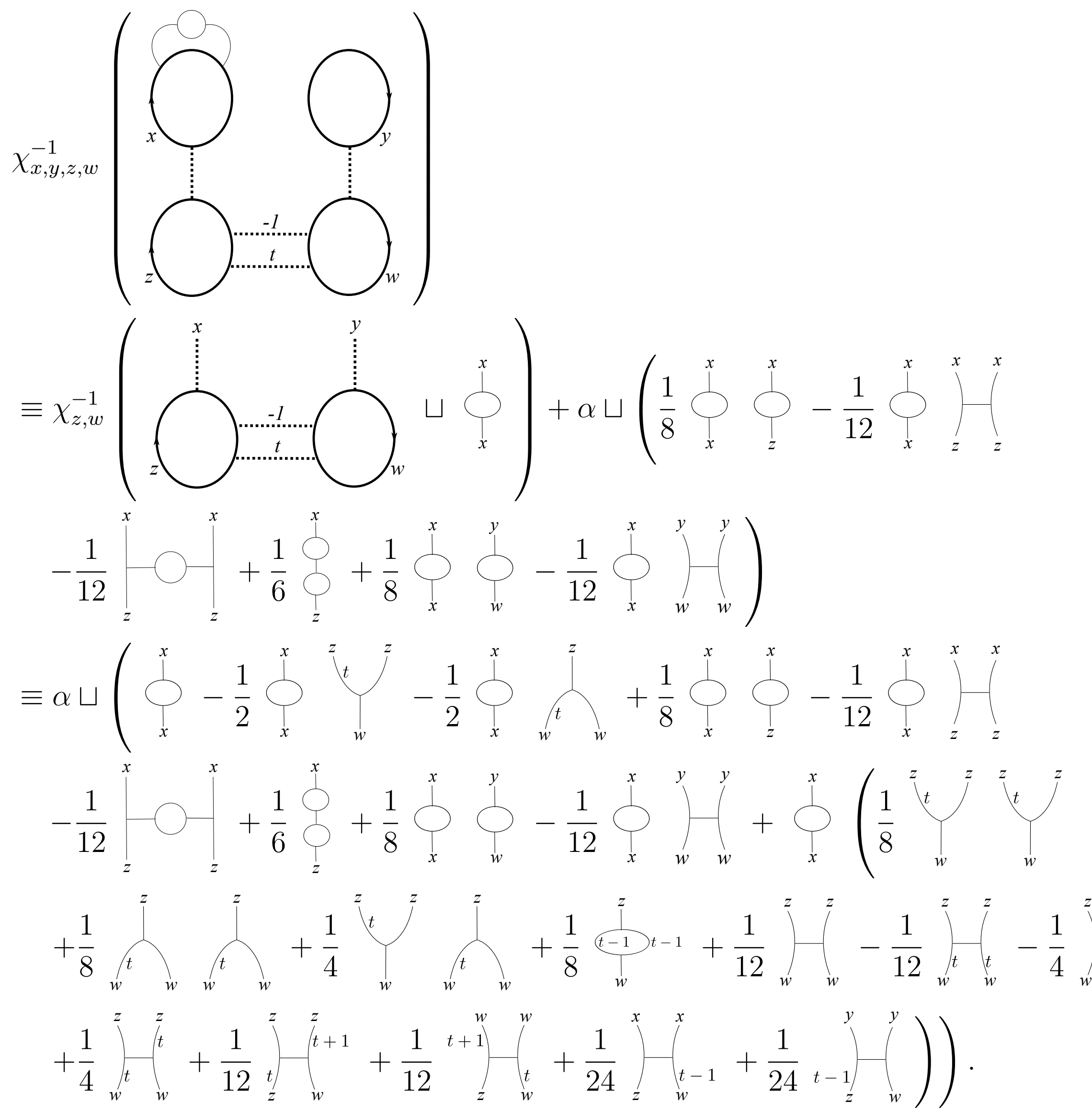
where $\gamma_2^j \quad (j = 1, 2, 3, 4)$ *and* $\delta_2^j \quad (j = 1, 2, 3, 4)$ *are given in Proof below.*

Proof. By Lemma 5.4, we have

$$\chi^{-1}\check{Z}(K_0 \cup L) - \chi^{-1}\check{Z}(K_0 \cup L_0) = \chi^{-1}_{x,y,z,w}\big(\chi_h^{-1}\check{Z}(K_0 \cup L) - \chi_h^{-1}\check{Z}(K_0 \cup L_0)\big)$$

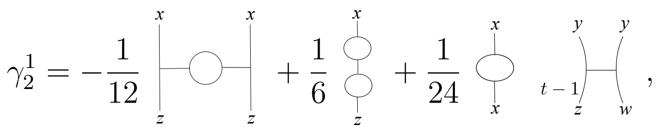


We calculate $\chi^{-1}_{x,y,z,w}(\zeta)$. By Lemmas A.1, A.2, and A.9, we have

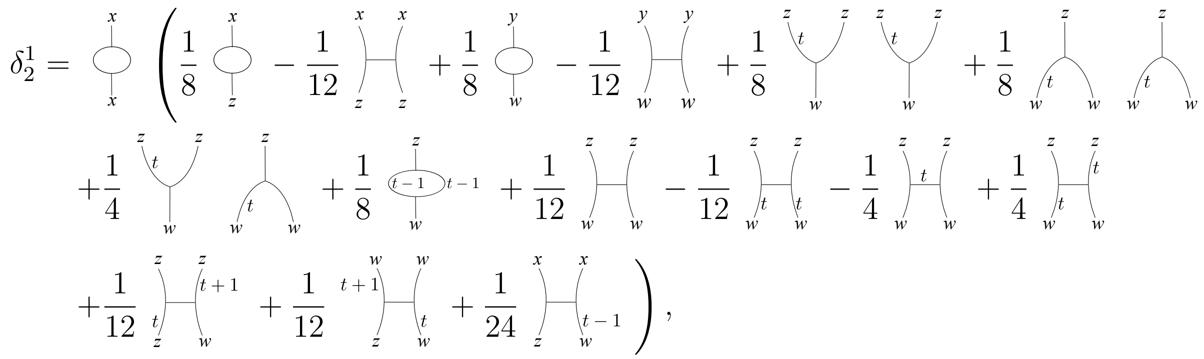


(18)

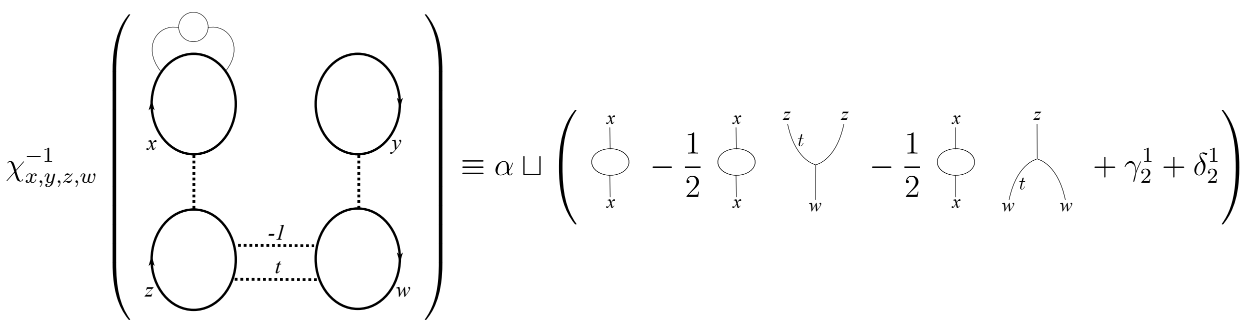
We put



(19)

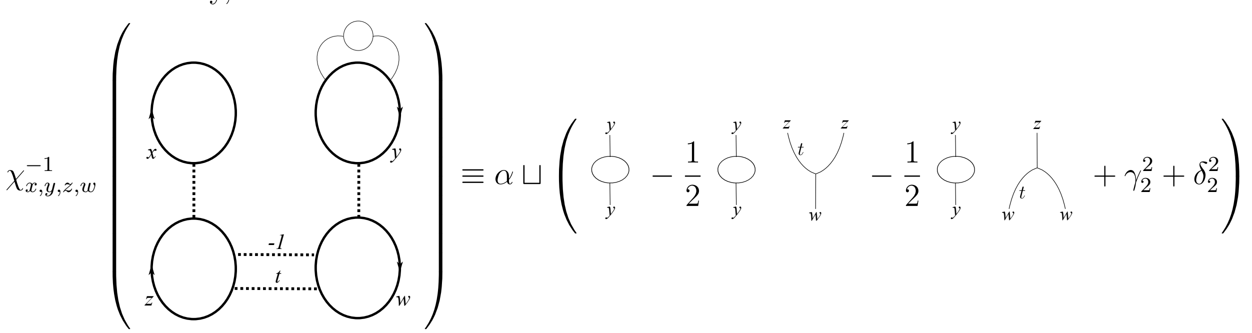


noting that $\delta_2^1 \in \mathcal{U}$. Thus, the formula (18) is rewritten,



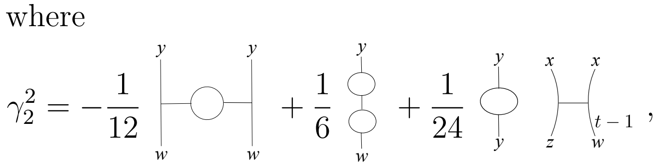
(20)

In a similar way, we have

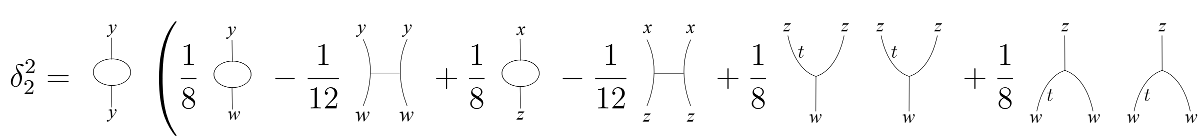


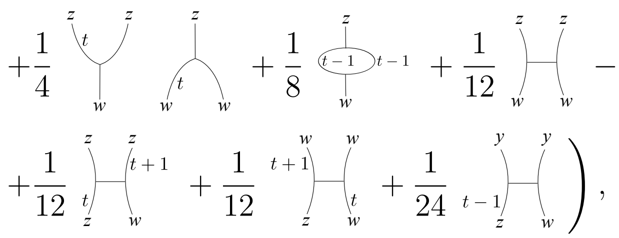
(21)

where



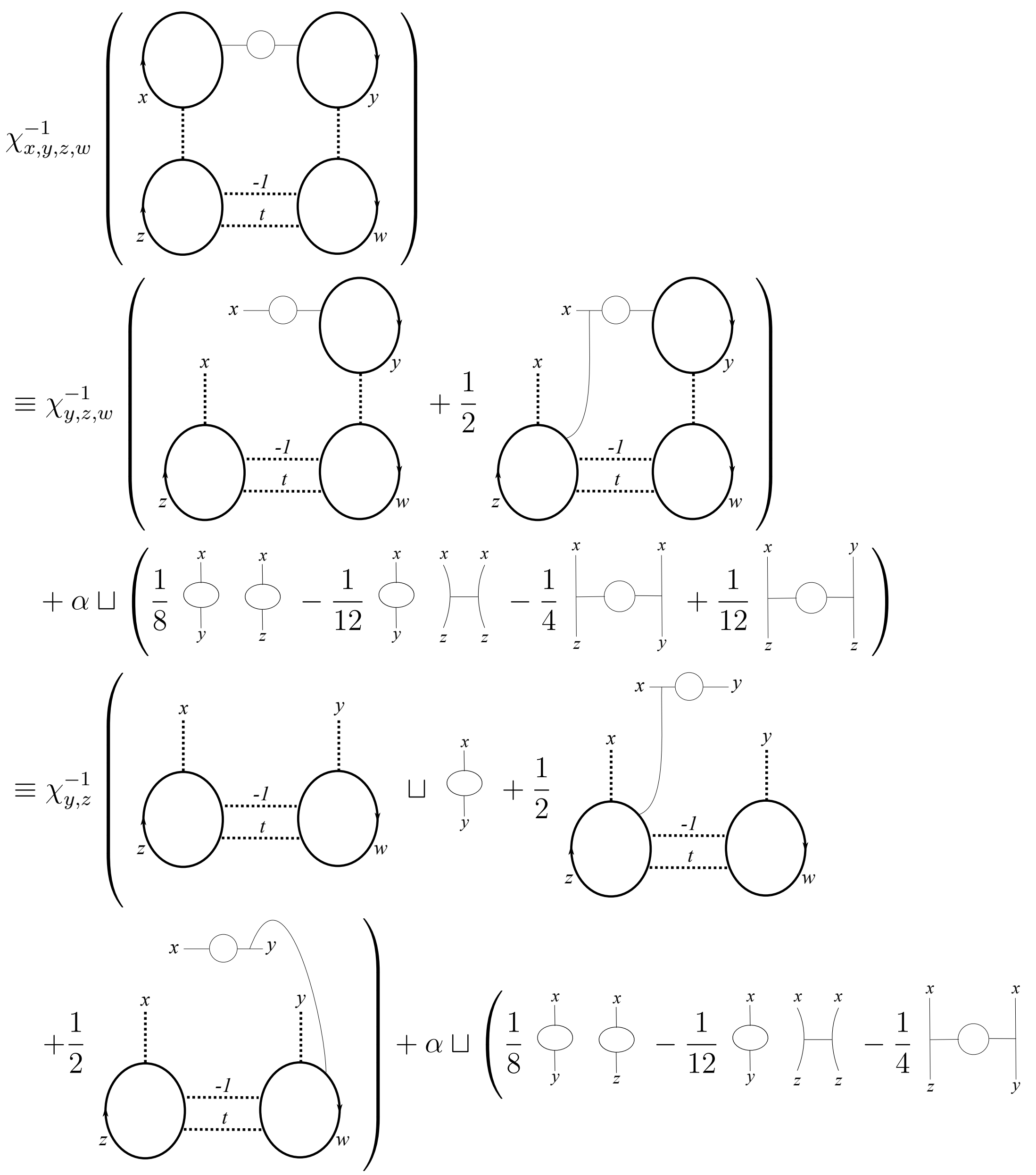
(22)

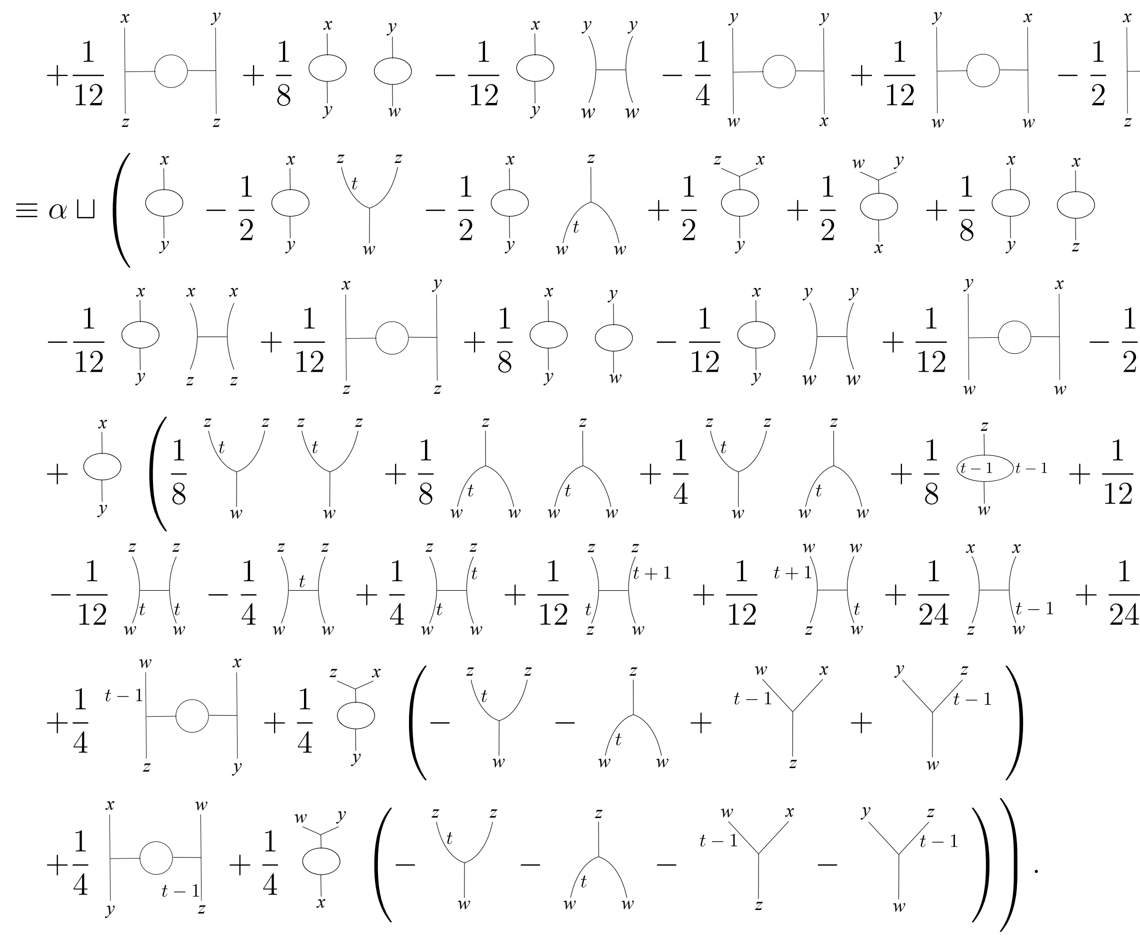




noting that $\delta_2^2 \in \mathcal{U}$.

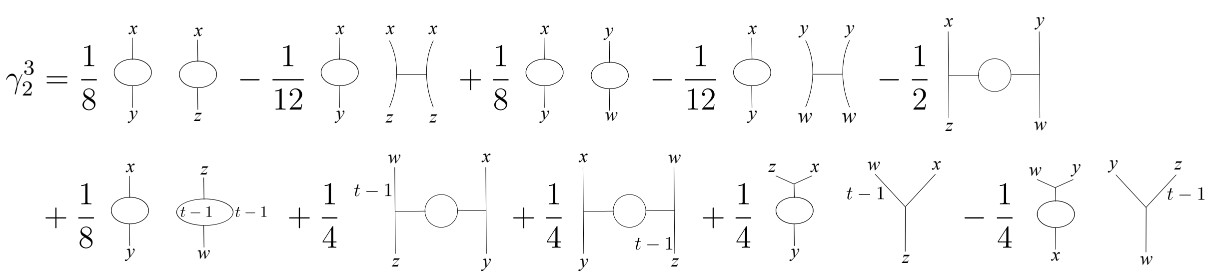
By Lemmas A.3, A.4, A.9, A.11, and A.12, we have



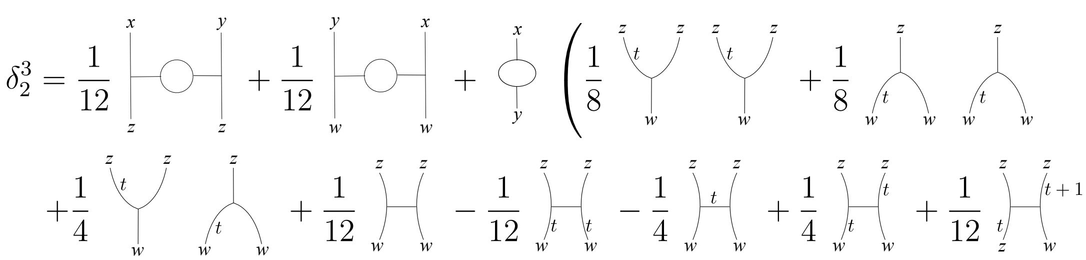


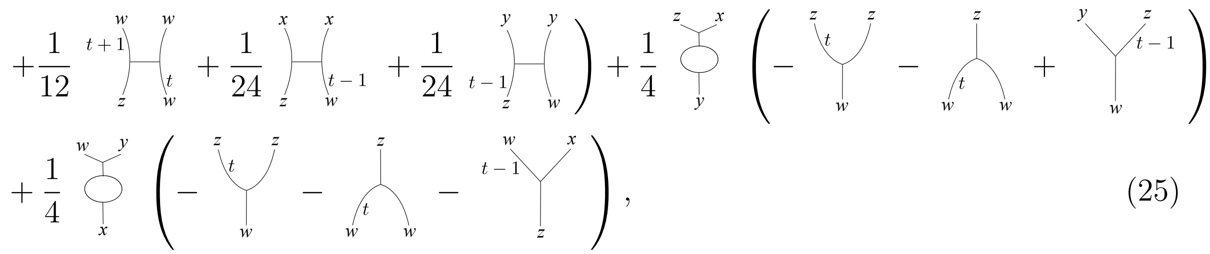
(23)

We put



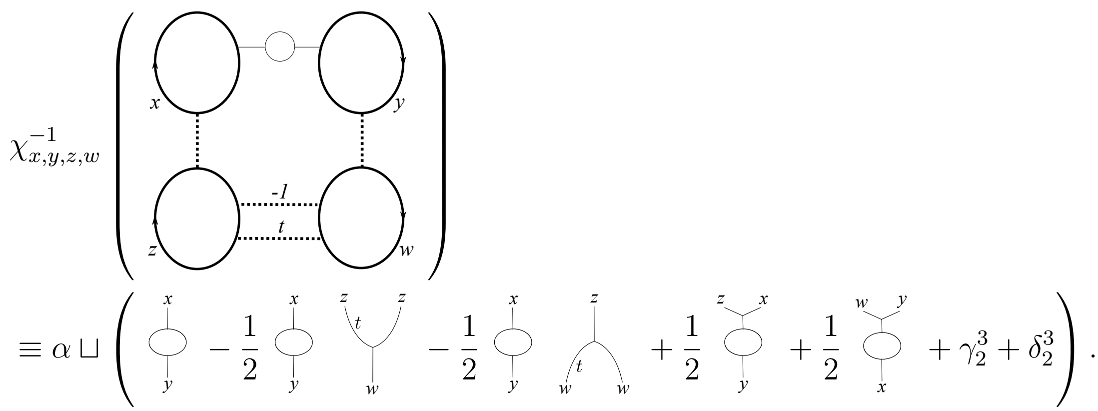
(24)





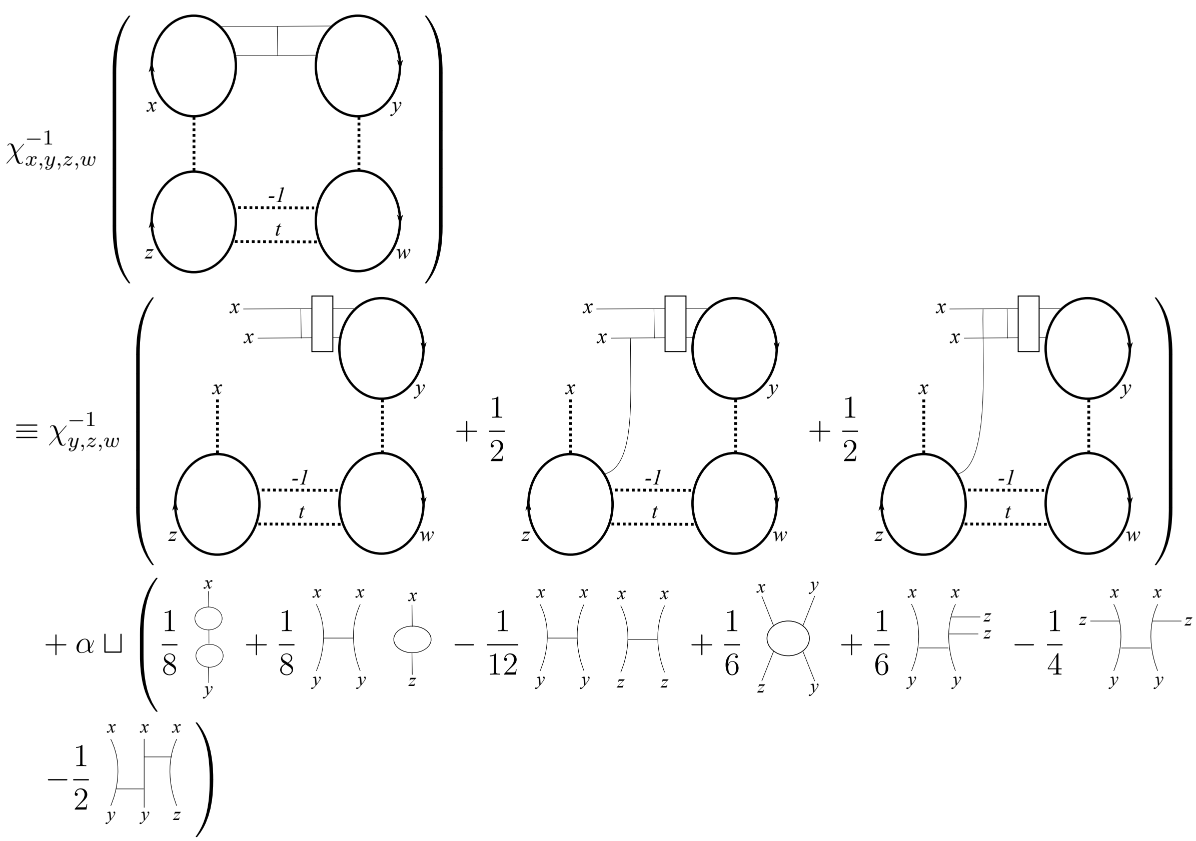
(25)

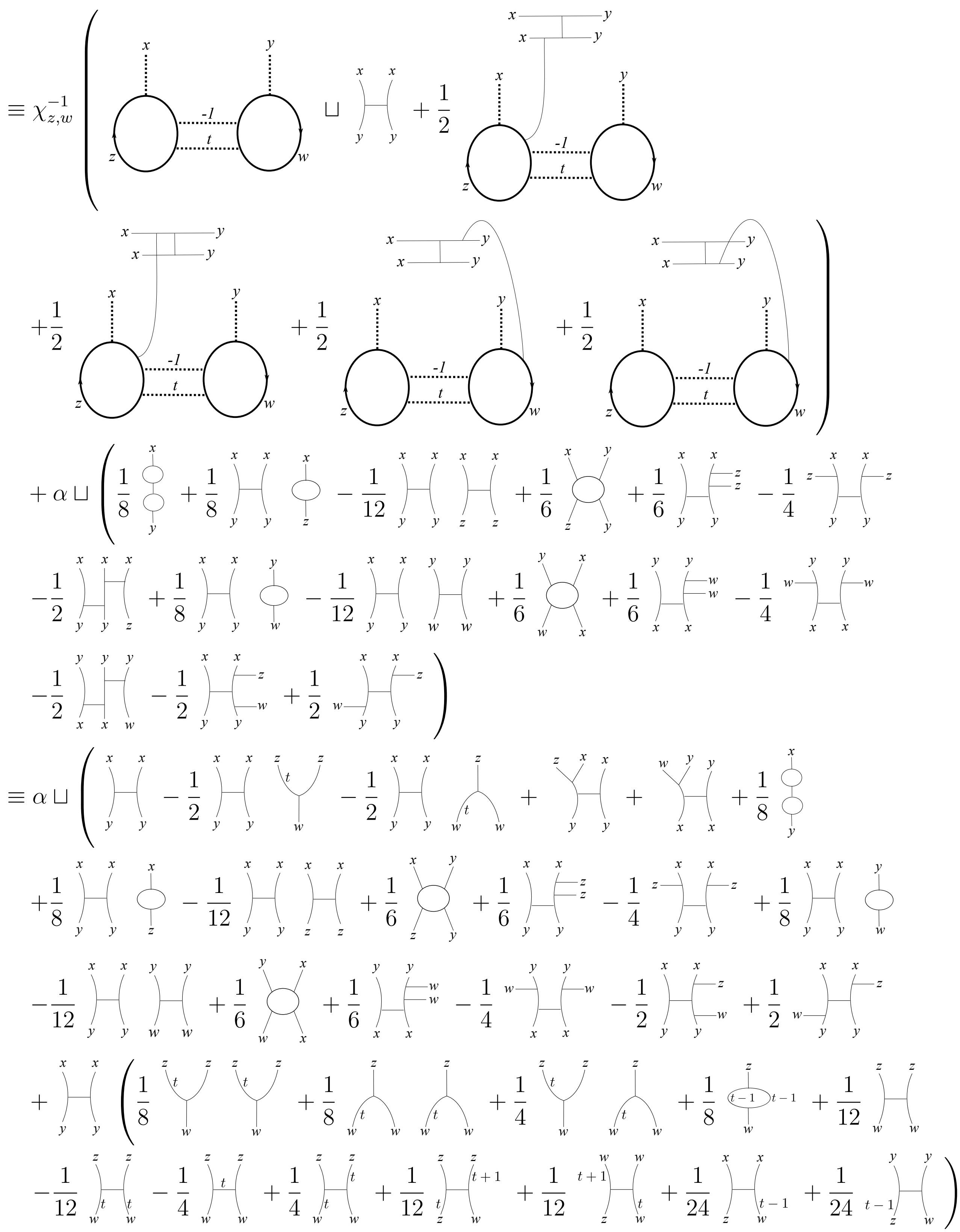
noting that $\delta_2^3 \in \mathcal{U}$. Thus, the formula (23) is rewritten,

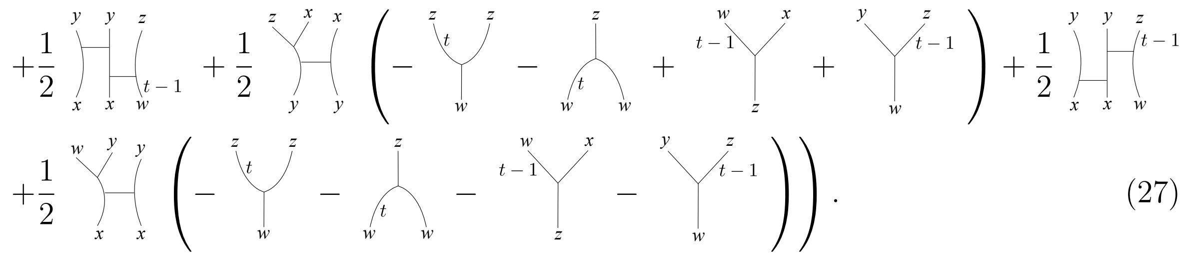


(26)

By Lemmas A.5, A.6, A.7, A.8, A.9, A.11, and A.12, we have

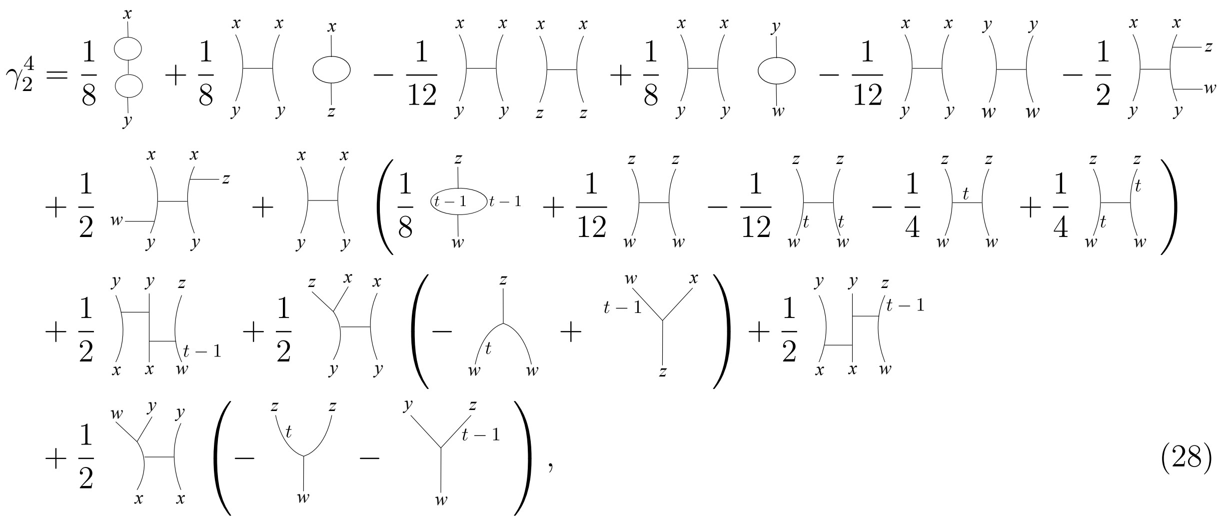




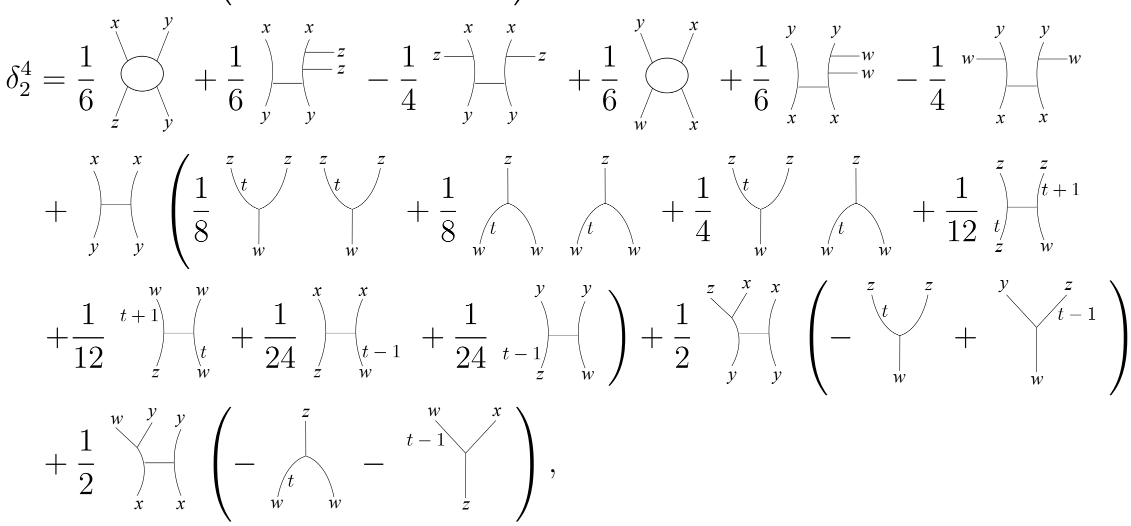


(27)

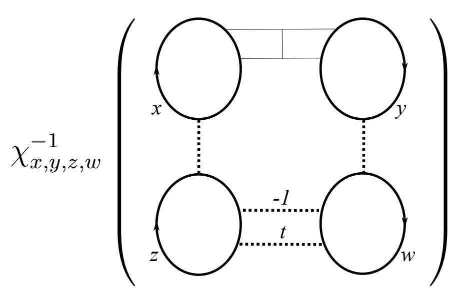
We put

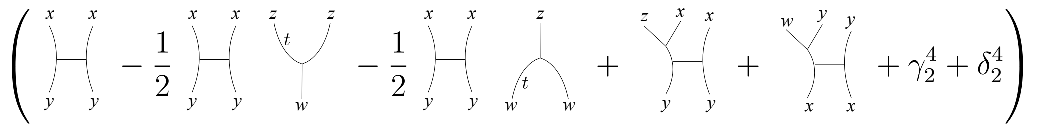


(28)



noting that $\delta_2^4 \in \mathcal{U}$. Thus, the formula (27) is rewritten,



$\equiv \alpha \sqcup$  (29)

Then, we put

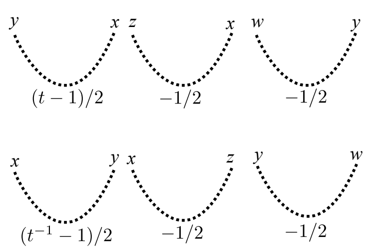
$$\gamma_2 = v_1^2\gamma_2^1 + v_2^2\gamma_2^2 + v_3^2\gamma_2^3 + v_1^3\gamma_2^4,$$
$$\delta_2 = v_1^2\delta_2^1 + v_2^2\delta_2^2 + v_3^2\delta_2^3 + v_1^3\delta_2^4.$$

Thus, by (20), (21), (26), and (29), we obtain the required formula. □

Lemma 5.6. *We have*

$$\chi^{-1} Z(K)^{(\text{3-loop})} = \langle \hat{\alpha}, (\gamma_1 + \gamma_2)\rangle_{(conn)},$$

where we denote the connected part of $\langle\ \ ,\ \ \rangle$ *by* $\langle\ \ ,\ \ \rangle_{(conn)}$, *and* $\hat{\alpha}$ *is given by*

$\hat{\alpha} =$  . (30)

Proof. The equivariant linking matrix $\big(l_{ij}(t)\big)$ of $L \subset S^3\backslash K_0$ (and $L_0 \subset S^3\backslash K_0$) is given by

$$\big(l_{ij}(t)\big) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & t-1 \\ 0 & 1 & t^{-1}-1 & 0 \end{pmatrix}.$$

Hence,

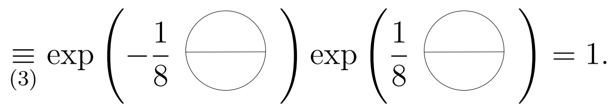
$$-\frac{1}{2}\big(l^{ij}(t)\big) = \frac{1}{2}\begin{pmatrix} 0 & t-1 & -1 & 0 \\ t^{-1}-1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Thus, by (4) and Lemma 5.5, we obtain

$$\langle\langle \chi^{-1}\check{Z}(K_0 \cup L) - \chi^{-1}\check{Z}(K_0 \cup L_0)\rangle\rangle \equiv \langle \hat{\alpha}, \beta + \gamma_1 + \gamma_2 + \delta_1 + \delta_2\rangle,$$

noting that the right-hand side contains only diagrams with at least 2 trivalent vertices. Let σ_+ and σ_- be the number of the positive and negative eigenvalues of the linking matrix of L (and L_0), and we have $\sigma_+ = \sigma_- = 2$. Then, by (6), we get

$$\langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{\sigma_+}\langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{\sigma_-} = \langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{2}\langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{2}$$

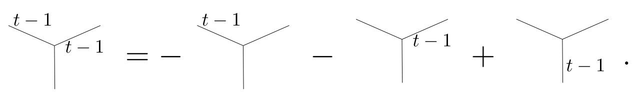


Thus, by Lemma 5.2, we have

$$\chi^{-1}Z(K) \equiv \frac{\langle\!\langle \chi^{-1}\check{Z}\big(K_0 \cup L\big) - \chi^{-1}\check{Z}\big(K_0 \cup L_0\big)\rangle\!\rangle}{\langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{\sigma_+}\langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{\sigma_-}} \equiv \langle \hat{\alpha}, \beta + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 \rangle.$$

However, since β contains only diagrams with at most 3 trivalent vertices, this does not contribute the 3-loop part. Moreover, by considering the value of $\hat{\alpha}$, we have $\langle \hat{\alpha}, \delta \rangle = 0$ for any element $\delta \in \mathcal{U}$, so we have $\langle \hat{\alpha}, \delta_1 + \delta_2 \rangle = 0$ since $\delta_1, \delta_2 \in \mathcal{U}$. Therefore, we obtain the required equation. □

Lemma 5.7. *We have*

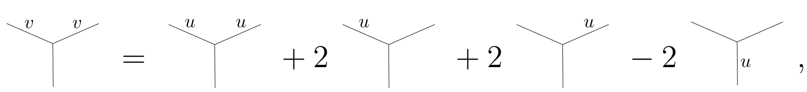


(31)

Proof. This formula immediately follows from the IHX relation. □

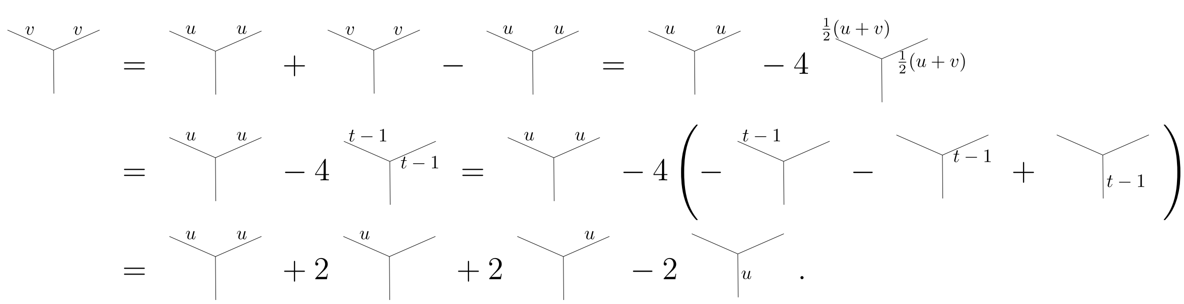
We denote $u = t + t^{-1} - 2$ and $v = t - t^{-1}$.

Lemma 5.8. *For 3-loop graphs, we have*



where these five diagrams are identical except at those local sites in the pictures and have even number of legs (when we substitute $t = e^h = 1 + h + h^2/2 + \cdots$).

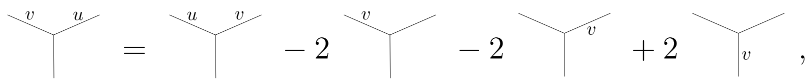
Proof. By Lemma 5.7 and the IHX relation and the assumption that they have even number of legs, we have



□

Remark 5.9. We note that the formulas in Theorem 3.2 and Corollary 3.5 in [21] can be rewritten more simply by the above formula.

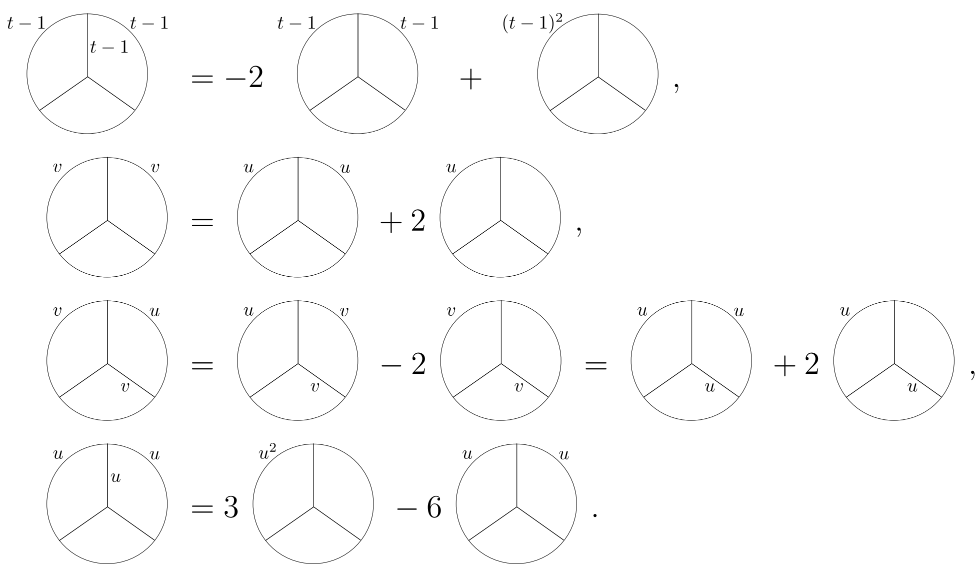
Lemma 5.10. *For 3-loop graphs, we have*



where these five diagrams are identical except at those local sites in the pictures and have even number of legs (when we substitute $t = e^h = 1 + h + h^2/2 + \cdots$*).*

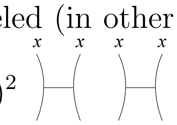
Proof. We can show this in a same way of the proof of Lemma 5.8. □

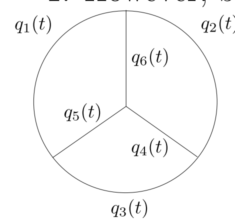
For example, by using the formula $t^{\pm 1} - 1 = \frac{1}{2}(u \pm v)$, $v^2 = u^2 + 4u$ and by Lemmas 5.7, 5.8, and 5.10, we have



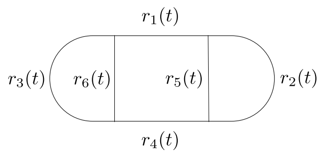
Proof of Theorem 3.1. By Lemma 5.6, we have

$$\chi^{-1} Z(K)^{(3\text{-loop})} = \langle \hat{\alpha}, (\gamma_1 + \gamma_2) \rangle_{(\mathrm{conn})} = \langle \hat{\alpha}, (\gamma_1 + v_1^2 \gamma_2^1 + v_2^2 \gamma_2^2 + v_3^2 \gamma_2^3 + v_1^3 \gamma_2^4) \rangle_{(\mathrm{conn})}, \quad (32)$$

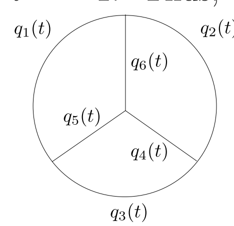
recalling that γ_1 is given by (10) and γ_2^j $(j = 1, 2, 3, 4)$ are given by (19), (22), (24), and (28). By a straightforward calculation, we can see that each term of (32) has at least three edges such that no power series on h are labeled (in other words, the power series "1" is labeled) on them, except the term $\Big\langle \hat{\alpha}, \frac{1}{2}(v_1^3)^2$  $\Big\rangle \; (\cdots (\#))$. Moreover, power series which appear in each diagram in $\hat{\alpha}$, γ_1 and γ_2^j are only t and $t^{\pm 1} - 1$, and other power series will not appear immediately after performing the Aarhus integral. Therefore, the value of $\big((32)-(\#)\big)$ belongs to the $\mathbb{Q}$-vector space generated by 3-loop graphs, at least three edges of which are labeled by 1 and others by $t^{\pm1}-1$. However, by the IHX relation, it is sufficient to consider the following two types,



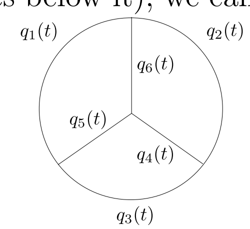
and



, where at least three of $q_j(t)$'s are 1 and others are $t^{\pm1}-1$ $\big(\cdots(*)\big)$, and the condition $(*)$ also holds for $r_j(t)$'s. For details, see [21]. If $r_1(t)=r_4(t)=1$, or $r_1(t)=t-1, r_4(t)=t^{-1}-1$, or $r_1(t)=t^{-1}-1, r_4(t)=t-1$, the second one can be deformed into a sum of the first ones with the condition $(*)$ by the IHX relation. Otherwise, it can be deformed into a sum of the first ones with the condition that at least four of $q_j(t)$'s are 1, one is $(t^{\pm1}-1)^2$ and others are $t^{\pm1}-1$. Thus, $\big((32)-(\#)\big)$ belongs to the $\mathbb{Q}$-vector space generated by 3-loop graphs



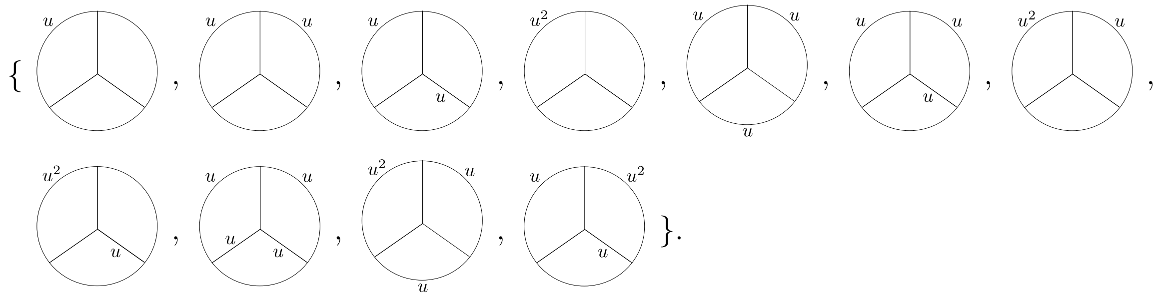
such that $q_i(t)=(t^{\pm1}-1)^{\epsilon_i}$ where $\epsilon_i=0,1,2$ and $1\le\sum_{i=1}^{6}\epsilon_i\le3$. On the other hand, by the straightforward calculation and by using Lemma 5.7 (and the examples below it), we can see that $(\#)$ belongs to the $\mathbb{Q}$-vector space generated by 3-loop graphs



such that either one of the following conditions holds,
- $q_i(t)=(t^{\pm1}-1)^{\epsilon_i}$ where $\epsilon_i=0,1,2$ and $1\le\sum_{i=1}^{6}\epsilon_i\le3$,
- $q_1(t)=t-1$, $q_2(t)=t-1$, $q_4(t)=t-1$ $q_5(t)=t^{-1}-1$, and others are equal to 1,
- $q_1(t)=(t^{\pm1}-1)^2$, $q_2(t)=t^{\pm1}-1$, $q_3(t)=t^{\pm1}-1$, and others are equal to 1,
- $q_1(t)=t^{\pm1}-1$, $q_2(t)=(t^{\pm1}-1)^2$, $q_4(t)=t^{\pm1}-1$ and others are equal to 1.

Thus, by the formula $t^{\pm1}-1=\dfrac{1}{2}(u\pm v)$, $v^2=u^2+4u$ and by Lemmas 5.7, 5.8, and 5.10 (and also see the examples below them), we can see that $\chi^{-1}Z(K)^{\text{3-loop}}$ belongs to the

$\mathbb{Q}$-vector space generated by the set



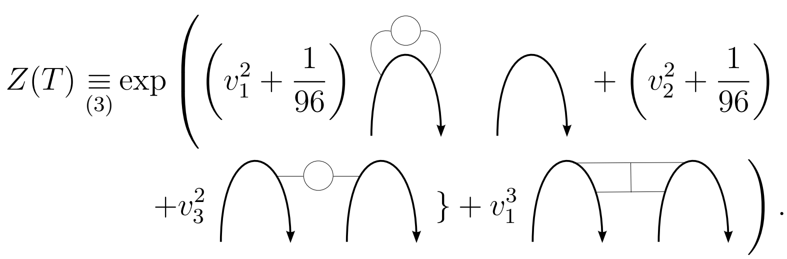
Therefore, by the definition of the 3-loop polynomial, we obtain the required statement. $\square$

5.3 The 2-loop polynomial of genus 1 knots with trivial Alexander polynomial

In this section, we consider the 2-loop polynomial of genus 1 knots with trivial Alexander polynomial, see [9, 17]. We denote $u = t + t^{-1} - 2$.

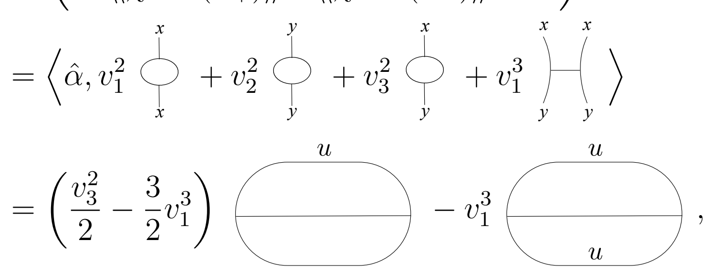
Lemma 5.11. *Let K be a genus 1 knot with trivial Alexander polynomial and T its representing tangle as in Lemma 5.1. Then, the 2-loop polynomial of K is 0 if and only if $v_3^2 = v_1^3 = 0$, where v_3^2 and v_1^3 are defined in (8).*

Proof. By (8), we have



By the same argument in Subsection 5.2, we obtain

$$\chi^{-1} Z(K)^{(\text{2-loop})} = \left(\frac{\langle\!\langle \chi^{-1}\check{Z}\big(K_0 \cup L\big) - \chi^{-1}\check{Z}\big(K_0 \cup L_0\big) \rangle\!\rangle}{\langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{\sigma_+} \langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{\sigma_-}} \right)^{(\text{2-loop})}$$

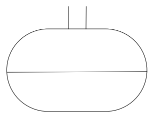
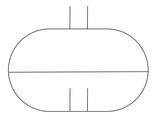


(33)

where $\hat{\alpha}$ is defined by (30). Since we can see that two 2-loop graphs in the right-hand side of (33) are linearly independent, we have $\Theta_K(t_1, t_2, t_3) = 0$ if and only if $v_3^2 = v_1^3 = 0$. $\square$

Let K be a genus 1 knot with trivial Alexander polynomial. We denote

$$\chi^{-1}Z(K) = \exp\left(\sum_{i\geq 1} b_i\theta_i + (\text{terms of } (>2)\text{-loop parts})\right),$$

where $\theta_1 =$  , $\theta_2 =$  , and θ_i's $(i > 2)$ are basis vectors of $\mathcal{B}^{(\text{2-loop})}_{\text{conn}}$ of degree>5. We can see that θ_1 and θ_2 are basis vectors of $\mathcal{B}^{(\text{2-loop})}_{\text{conn}}$ of degree≤5. The 2-loop polynomial $\Theta_K(t_1, t_2, t_3)$ is determined by two Vassiliev invariant of K, more concretely, we can present $\Theta_K(t_1, t_2, t_3)$ by

$$\begin{aligned}
&\Theta_K(t_1, t_2, t_3) \\
&= 2b_1(t_1 + t_1^{-1} + t_2 + t_2^{-1} + t_3 + t_3^{-1} - 6) + \left(2b_2 - \frac{1}{3}b_1\right)\big((t_1 + t_1^{-1} - 2)(t_2 + t_2^{-1} - 2) \\
&\quad + (t_2 + t_2^{-1} - 2)(t_3 + t_3^{-1} - 2) + (t_3 + t_3^{-1} - 2)(t_1 + t_1^{-1} - 2)\big),
\end{aligned}$$

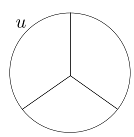
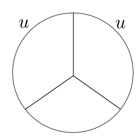
see [9, 17]. As this corollary, we can see that for $i > 2$, each coefficient b_i can be presented by a linear sum of b_1 and b_2.

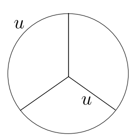
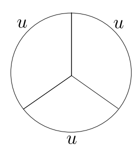
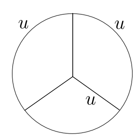
5.4 Proof of Theorem 3.3 and Corollary 3.4

In this section, we prove Theorem 3.3 and Corollary 3.4.

Let K be a genus 1 knot with trivial (≤ 2)-loop polynomials.

Proposition 5.12. *We have*

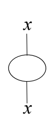
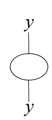
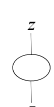
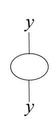
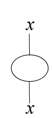
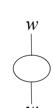
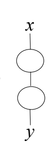
$$\chi^{-1}Z(K)^{(\text{3-loop})} = \left(v_4^3 - \frac{v_5^4}{2} - 3v_6^4 - 4v_1^2v_2^2\right)$$  $$+ \left(-2v_6^4 + \frac{15}{2}v_3^5\right)$$ 

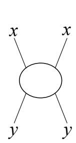
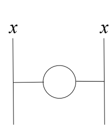
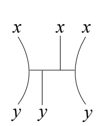
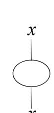
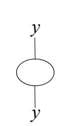
$$+ \left(\frac{v_5^4}{2} + \frac{5}{4}v_3^5\right)$$  $$+ 2v_3^5$$  $$+ 3v_3^5$$  .

Proof. By Lemmas 5.6 and 5.11, we obtain

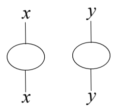
$$\chi^{-1}Z(K)^{(\text{3-loop})} = \langle \hat{\alpha}, (\gamma_1' + v_1^2\gamma_2^1 + v_2^2\gamma_2^2)\cdot\rangle_{(\text{conn})}, \tag{34}$$

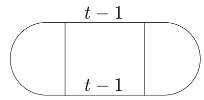
where

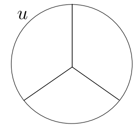
$$\gamma_1 = v_1^2$$  $$\left(\frac{1}{32}\right.$$  $$+ \frac{1}{24}$$  $$\left.\right) + v_2^2$$  $$\left(\frac{1}{32}\right.$$  $$+ \frac{1}{24}$$  $$\left.\right) + v_4^3$$ 

$$+ v_5^4$$  $$+ v_6^4$$  $$+ v_3^5$$  $$+ \left(v_1^2v_2^2 + \frac{1}{96}v_1^2 + \frac{1}{96}v_2^2\right)$$   .

In the following caluculation, we use Lemmas 5.8 and 5.10.
We calculate $\langle \hat{\alpha}, \gamma_1' \rangle_{(\mathrm{conn})}$, as follows,

$\Big\langle \hat{\alpha}, \frac{v_1^2}{32}$ 

$\Big\rangle_{(\mathrm{conn})} = \frac{v_1^2}{16}$ 

$= -\frac{v_1^2}{8}$ 

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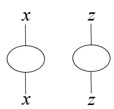
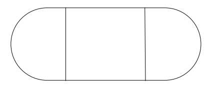
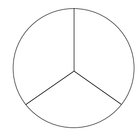
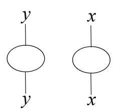
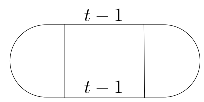
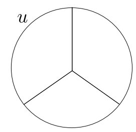
$\Big\langle \hat{\alpha}, \frac{v_1^2}{24}$ 

$\Big\rangle_{(\mathrm{conn})} = \frac{v_1^2}{12}$  $= \frac{v_1^2}{6}$  ,

$\Big\langle \hat{\alpha}, \frac{v_2^2}{32}$ 

$\Big\rangle_{(\mathrm{conn})} = \frac{v_2^2}{16}$ 

$= -\frac{v_2^2}{8}$ 

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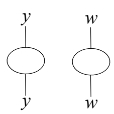
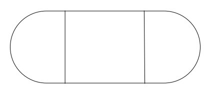
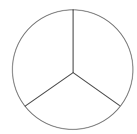
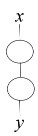
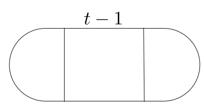
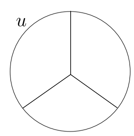
$\Big\langle \hat{\alpha}, \frac{v_2^2}{24}$ 

$\Big\rangle_{(\mathrm{conn})} = \frac{v_2^2}{12}$  $= \frac{v_2^2}{6}$  ,

$\Big\langle \hat{\alpha}, v_4^3$ 

$\Big\rangle_{(\mathrm{conn})} = v_4^3$ 

$= v_4^3$ 

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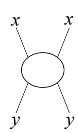
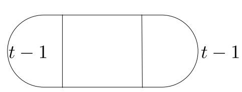
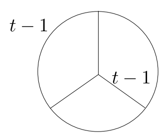
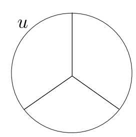
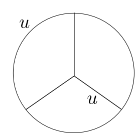
$\Big\langle \hat{\alpha}, v_5^4$ 

$\Big\rangle_{(\mathrm{conn})}$

$= v_5^4$ 

$+ v_5^4$ 

$= -\frac{v_5^4}{2}$ 

$+ \frac{v_5^4}{2}$ 

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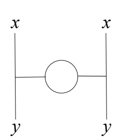
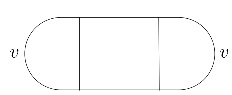
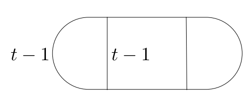
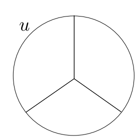
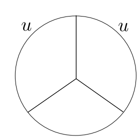
$\Big\langle \hat{\alpha}, v_6^4$ 

$\Big\rangle_{(\mathrm{conn})}$

$= -v_6^4$ 

$- v_6^4$ 

$= -3v_6^4$ 

$- 2v_6^4$ 

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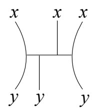
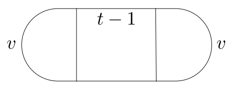
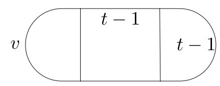
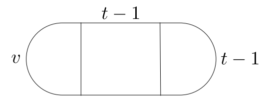
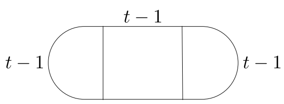
$$\Big\langle \hat{\alpha}, v_3^5$$ 

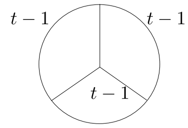
$$\Big\rangle_{(\mathrm{conn})}$$

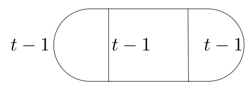
$$= v_3^5$$ 

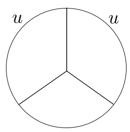
$$- v_3^5$$ 

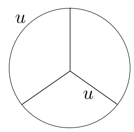
$$+ v_3^5$$ 

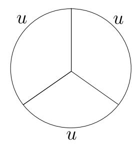
$$+ v_3^5$$ 

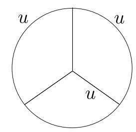
$$+ v_3^5$$ 

$$+ v_3^5$$ 

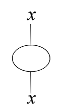
$$= \frac{15}{2} v_3^5$$ 

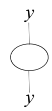
$$+ \frac{5}{4} v_3^5$$ 

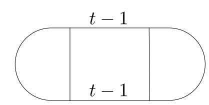
$$+ 2v_3^5$$ 

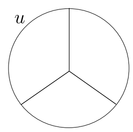
$$+ 3v_3^5$$ 

,

$$\Big\langle \hat{\alpha}, \left(v_1^2 v_2^2 + \frac{1}{96} v_1^2 + \frac{1}{96} v_2^2 \right)$$ 



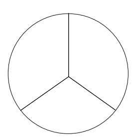
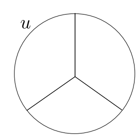
$$\Big\rangle_{(\mathrm{conn})} = \left(2v_1^2 v_2^2 + \frac{1}{48} v_1^2 + \frac{1}{48} v_2^2 \right)$$ 

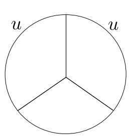
$$= \left(-4v_1^2 v_2^2 - \frac{1}{24} v_1^2 - \frac{1}{24} v_2^2 \right)$$ 

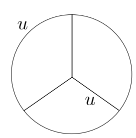
.

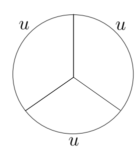
Hence, we have

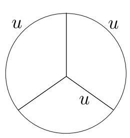
$$\langle \hat{\alpha}, \gamma_1' \rangle_{(\mathrm{conn})}$$

$$= \left(\frac{v_1^2}{6} + \frac{v_2^2}{6} \right)$$ 
$$+ \left(-\frac{v_1^2}{6} - \frac{v_2^2}{6} + v_4^3 - \frac{v_5^4}{2} - 3v_6^4 - 4v_1^2 v_2^2 \right)$$ 

$$+ \left(-2v_6^4 + \frac{15}{2} v_3^5 \right)$$ 

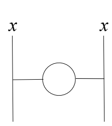
$$+ \left(\frac{v_5^4}{2} + \frac{5}{4} v_3^5 \right)$$ 

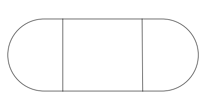
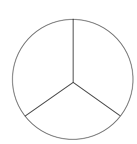
$$+ 2v_3^5$$ 

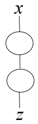
$$+ 3v_3^5$$ 

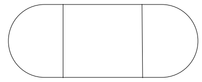
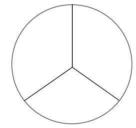
. (35)

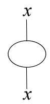
We calculate $\langle \hat{\alpha}, v_1^2 \gamma_2^1 \rangle_{(\mathrm{conn})}$, as follows,

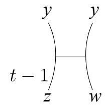
$$\Big\langle \hat{\alpha}, -\frac{v_1^2}{12}$$ 

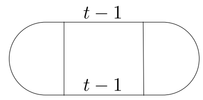
$$\Big\rangle_{(\mathrm{conn})} = \frac{v_1^2}{12}$$ 
$$= \frac{v_1^2}{6}$$ 
,

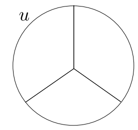
$\Big\langle \hat{\alpha}, \frac{v_1^2}{6}$ 

$\Big\rangle_{(\mathrm{conn})} = -\frac{v_1^2}{6}$  $= -\frac{v_1^2}{3}$  ,

$\Big\langle \hat{\alpha}, \frac{v_1^2}{24}$ 

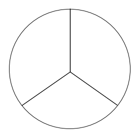
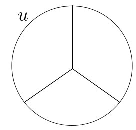


$\Big\rangle_{(\mathrm{conn})} = -\frac{v_1^2}{12}$ 

$= \frac{v_1^2}{6}$ 

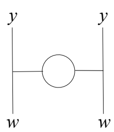
.

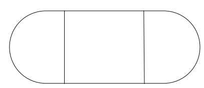
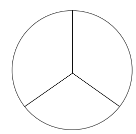
Hence, we have

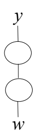
$\langle \hat{\alpha}, v_1^2 \gamma_2^1 \rangle_{(\mathrm{conn})} = -\frac{v_1^2}{6}$  $+ \frac{v_1^2}{6}$ 

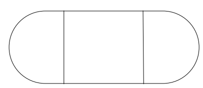
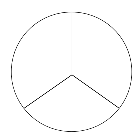
. (36)

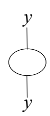
We calculate $\langle \hat{\alpha}, v_2^2 \gamma_2^2 \rangle_{(\mathrm{conn})}$, as follows,

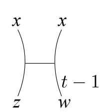
$\Big\langle \hat{\alpha}, -\frac{v_2^2}{12}$ 

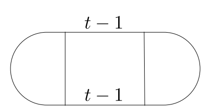
$\Big\rangle_{(\mathrm{conn})} = \frac{v_2^2}{12}$  $= \frac{v_2^2}{6}$  ,

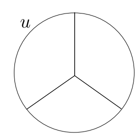
$\Big\langle \hat{\alpha}, \frac{v_2^2}{6}$ 

$\Big\rangle_{(\mathrm{conn})} = -\frac{v_2^2}{6}$  $= -\frac{v_2^2}{3}$  ,

$\Big\langle \hat{\alpha}, \frac{v_2^2}{24}$ 

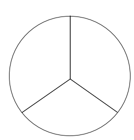
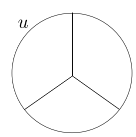


$\Big\rangle_{(\mathrm{conn})} = -\frac{v_2^2}{12}$ 

$= \frac{v_2^2}{6}$ 

.

Hence, we have

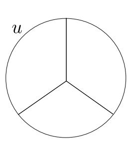
$\langle \hat{\alpha}, v_2^2 \gamma_2^2 \rangle_{(\mathrm{conn})} = -\frac{v_2^2}{6}$  $+ \frac{v_2^2}{6}$ 

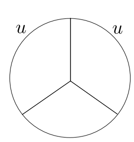
. (37)

Therefore, by applying (35), (36), (37) to (34), we obtain the required formula. □

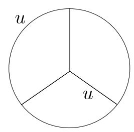
For Jacobi diagrams α and β, we write $\alpha \equiv_{d \leq 8} \beta$ if $\alpha - \beta$ is a linear sum of Jacobi diagrams with degree>8.

Lemma 5.13. *We have*

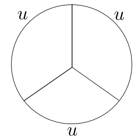


$\equiv_{d \leq 8} \lambda_1 + \frac{1}{12}\lambda_2 + \frac{1}{360}\lambda_4,$ 

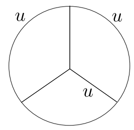
$\equiv_{d \leq 8} \frac{1}{2}\lambda_2 + \frac{1}{6}\lambda_6,$



$\equiv_{d\le 8} \lambda_2 - 2\lambda_3 - \frac{1}{3}\lambda_5 + \frac{1}{3}\lambda_6,$



$\equiv_{d\le 8} -\frac{1}{3}\lambda_4 + \lambda_5 + \frac{2}{3}\lambda_6,$

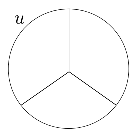


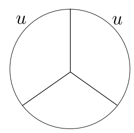
$\equiv_{d\le 8} \frac{1}{6}\lambda_4 - \lambda_5 + \frac{2}{3}\lambda_6,$

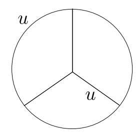
where λ_i's are defined in Section 3.

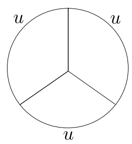
Proof. We can show these formula by the straightforward calculations, see Appendix B. □

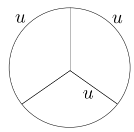
Proof of Theorem 3.3. By Proposition 5.12 and Lemma 5.13, we can denote

$\chi^{-1}Z(K)^{(\text{3-loop})} = x_1$ 

$+\, x_2$ 

$+\, x_3$ 

$+\, x_4$ 

$+\, x_5$ 

$$\begin{aligned}
&\equiv_{d\le 8} x_1\lambda_1 + \left(\frac{x_1}{12} + \frac{x_2}{2} + x_3\right)\lambda_2 - 2x_3\lambda_3 + \left(\frac{x_1}{360} - \frac{x_4}{3} + \frac{x_5}{6}\right)\lambda_4 \\
&+ \left(-\frac{x_3}{3} + x_4 - x_5\right)\lambda_5 + \left(\frac{x_2}{6} + \frac{x_3}{3} + \frac{2}{3}x_4 + \frac{2}{3}x_5\right)\lambda_6. \qquad (38)
\end{aligned}$$

Since we denote $\chi^{-1}Z(K)^{(\text{3-loop})} = \exp\left(\sum_{i\ge 1} a_i\lambda_i\right)$, we have

$$a_1 = x_1, \quad a_2 = \frac{x_1}{12} + \frac{x_2}{2} + x_3, \quad a_3 = -2x_3, \quad a_4 = \frac{x_1}{360} - \frac{x_4}{3} + \frac{x_5}{6}, \quad a_5 = -\frac{x_3}{3} + x_4 - x_5.$$

By solving this simultaneous equation, we get

$$\begin{aligned}
&x_1 = a_1, \quad x_2 = -\frac{a_1}{6} + 2a_2 + a_3, \quad x_3 = -\frac{a_3}{2}, \\
&x_4 = \frac{a_1}{60} + \frac{a_3}{6} - 6a_4 - a_5, \quad x_5 = \frac{a_1}{60} + \frac{a_3}{3} - 6a_4 - 2a_5. \qquad (39)
\end{aligned}$$

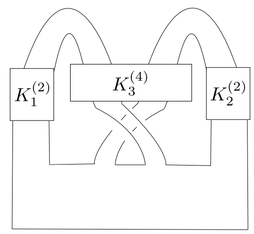
By applying (39) to (38) and by the definition of the 3-loop polynomial, we obtain the required formula. □

Proof of Corollary 3.4. Theorem 3.3 immediately implies the first statement. Further, by (38), we have $a_6 = \frac{x_2}{6} + \frac{x_3}{3} + \frac{2}{3}x_4 + \frac{2}{3}x_5$, and by substituting (39) to this, we obtain the second statement. □

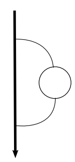
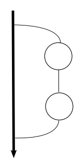
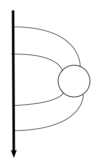
6 An example of calculation

In this section, we give a new example of calculation of the 3-loop polynomial.

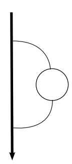
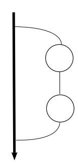
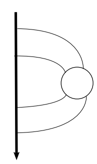
Let $D(K_1, K_2, K_3)$ be a knot defined as follows,

$$D(K_1, K_2, K_3) = $$  ,

where K_1, K_2, and K_3 are 0-framed long knots (1-tangles), and $K_1^{(2)}$, $K_2^{(2)}$ are the doubles of K_1, K_2, respectively and $K_3^{(4)}$ is the double of double of K_3. Note that $D(K_1, K_2, K_3)$ is a genus 1 knot with trivial Alexander polynomial. We denote the Kontsevich invariant of K_i $(i = 1, 2, 3)$ as follows,

$$Z(K_i) \equiv \exp\left(a_i \right.$$  $$+ b_i$$  $$+ c_i$$  $$\left. \right).$$

Remark 6.1. It is shown (see [20]) that $Z(K_i)$ is presented by

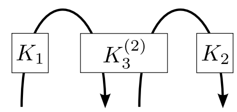
$$Z(K_i) \equiv \exp\left(-\frac{1}{2}c_2^i \right.$$  $$-\frac{1}{24}j_3^i$$  $$+\frac{1}{24}(-12c_4^i + 6(c_2^i)^2 - c_2^i)$$  $$\left. \right),$$

where c_n^i are coefficients of the Conway polynomial $\nabla_{\hat{K}_i}(z) = \sum c_n^i z^n$ and j_n^i are coefficients of the Jones polynomial $J_{\hat{K}_i}(e^t) = \sum j_n^i t^n$, where $\hat{K}_i$ is the closure of K_i. Note that the Conway polynomial is defined by $\nabla_{\hat{K}_i}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_{\hat{K}_i}(t)$. Therefore we can get

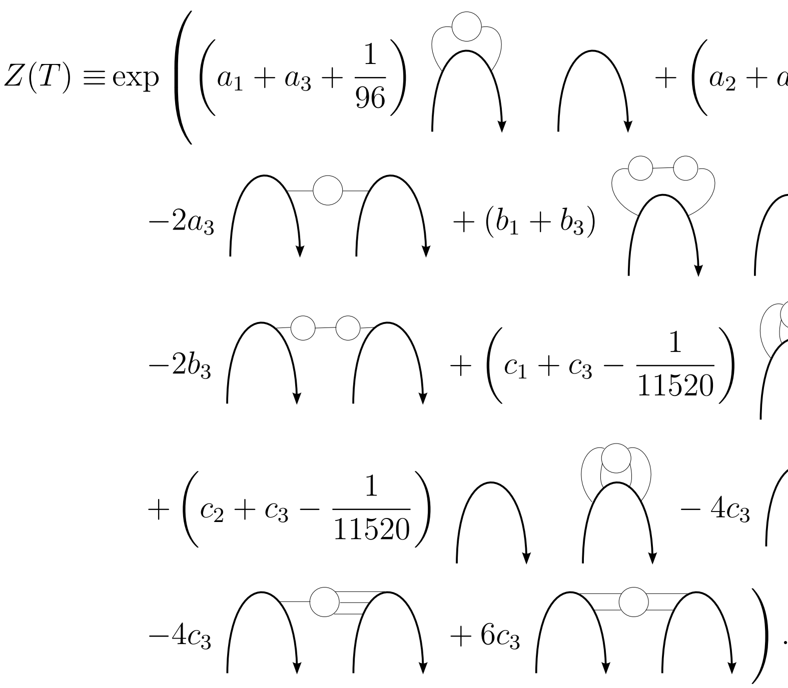
$$a_i = -\frac{1}{2}c_2^i, \quad b_i = -\frac{1}{24}j_3^i, \quad c_i = \frac{1}{24}(-12c_4^i + 6(c_2^i)^2 - c_2^i).$$

Proposition 6.2. *The 3-loop polynomial of $D(K_1, K_2, K_3)$ is presented by*

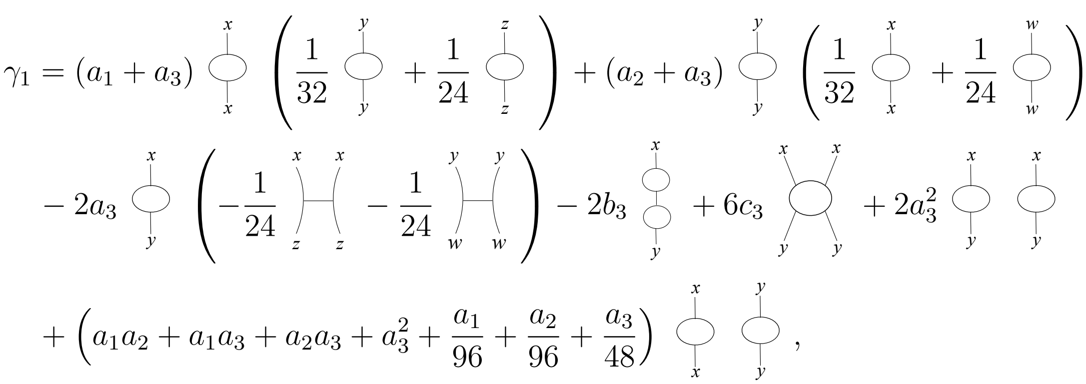
$$\begin{aligned}
&\Lambda_{D(K_1,K_2,K_3)}(t_1, t_2, t_3, t_4) \\
&= (-a_3 - 16a_1a_2 - 16a_1a_3 - 16a_2a_3 - 8b_3 - 12c_3)\,(u_{12} + u_{13} + u_{14} + u_{2,3} + u_{24} + u_{34}) \\
&\quad + 24c_3(u_{12}u_{34} + u_{13}u_{24} + u_{14}u_{23}) \\
&\quad + 8a_3^2(u_{12}^2 + u_{13}^2 + u_{14}^2 + u_{2,3}^2 + u_{24}^2 + u_{34}^2).
\end{aligned}$$

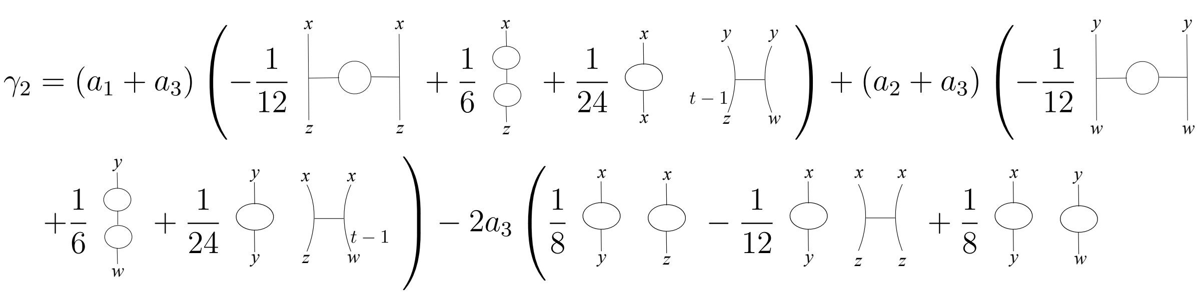
Proof. Since a representing tangle of $D(K_1, K_2, K_3)$ is given by $T =$ ,

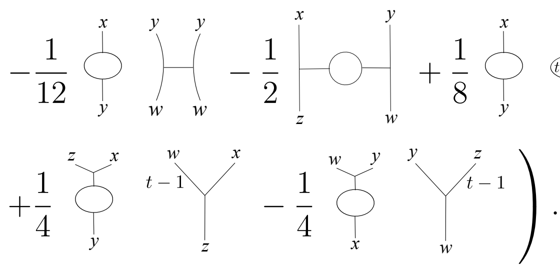
we have



For a calculation of the Kontsevich invariant of the double of a tangle, see for example [14, 15]. Thus, we have

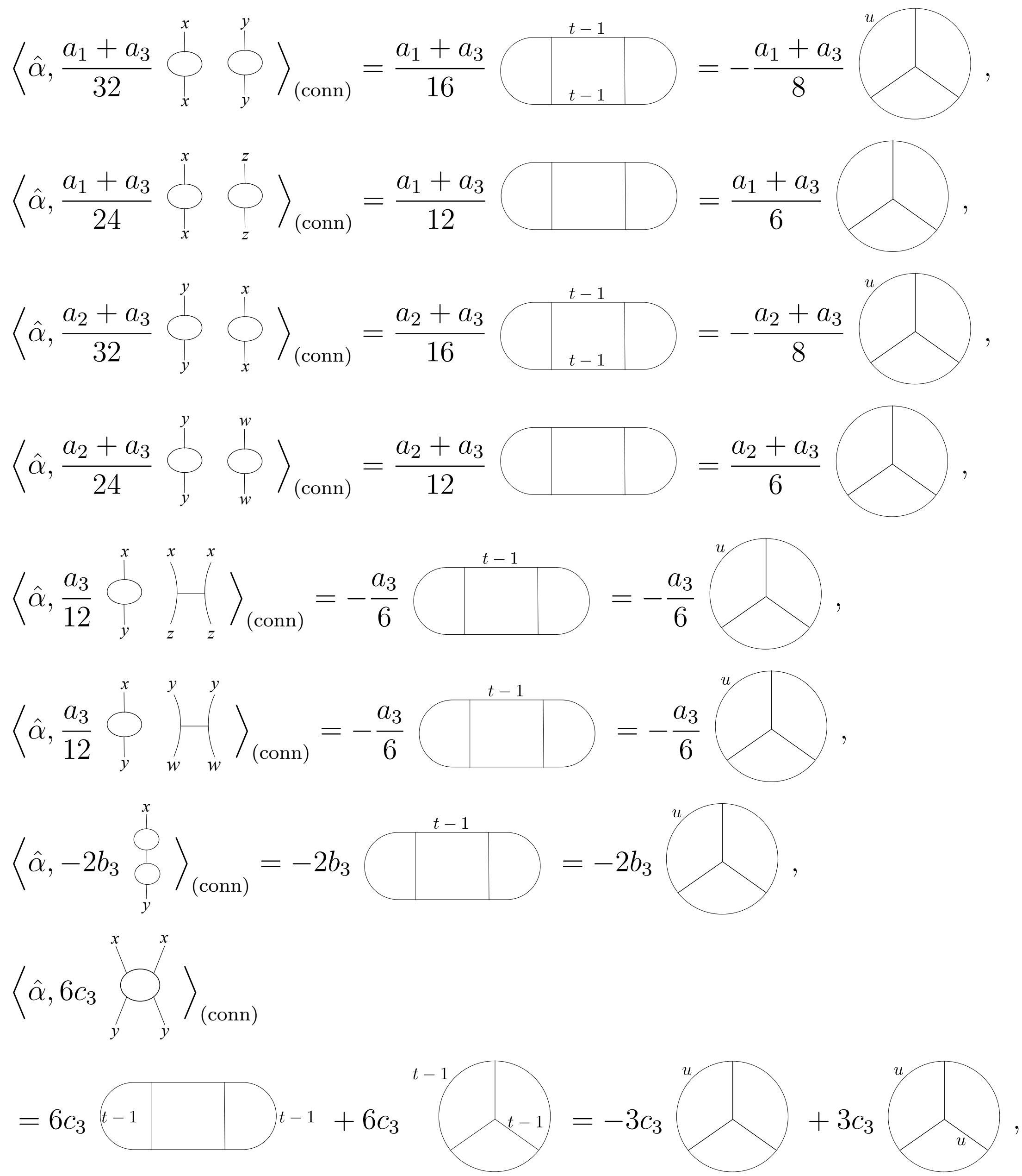


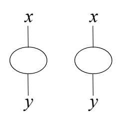
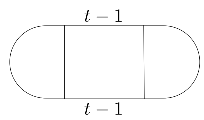
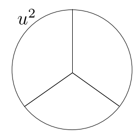
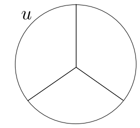


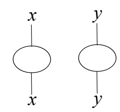


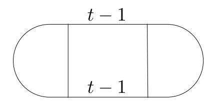
Therefore, by Lemma 5.6, we have $\chi^{-1}Z\big(D(K_1,K_2,K_3)\big)^{(\text{3-loop})} = \langle \hat{\alpha}, (\gamma_1+\gamma_2)\rangle_{(\text{conn})}$.

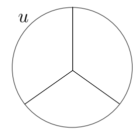
In a similar way of the calculation of Proposition 5.12, we calculate $\langle \hat{\alpha}, \gamma_1\rangle_{(\text{conn})}$ as follows,



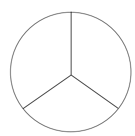
$$\left\langle \hat{\alpha}, 2a_3^2 \right.$$  $$\left.\right\rangle_{(\mathrm{conn})} = 2a_3^2$$  $$= 2a_3^2$$  $$+ 4a_3^2$$  ,

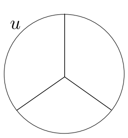
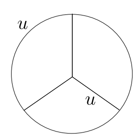
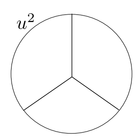
$$\left\langle \hat{\alpha}, \left(a_1a_2 + a_1a_3 + a_2a_3 + a_3^2 + \frac{a_1}{96} + \frac{a_2}{96} + \frac{a_3}{48}\right)\right.$$  $$\left.\right\rangle_{(\mathrm{conn})}$$

$$= \left(2a_1a_2 + 2a_1a_3 + 2a_2a_3 + 2a_3^2 + \frac{a_1}{48} + \frac{a_2}{48} + \frac{a_3}{24}\right)$$ 

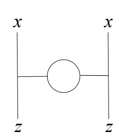
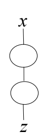
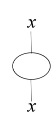
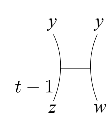
$$= \left(-4a_1a_2 - 4a_1a_3 - 4a_2a_3 - 4a_3^2 - \frac{a_1}{24} - \frac{a_2}{24} - \frac{a_3}{12}\right)$$  .

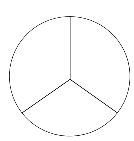
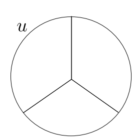
Hence, we have

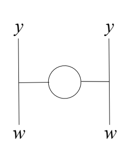
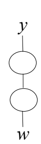
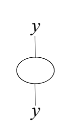
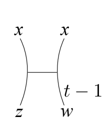
$$\langle \hat{\alpha}, \gamma_1 \rangle_{(\mathrm{conn})} = \left(\frac{a_1}{6} + \frac{a_2}{6} + \frac{a_3}{3}\right)$$  $$+ \left(-\frac{a_1}{6} - \frac{a_2}{6} - \frac{2}{3}a_3 - 4a_1a_2 - 4a_1a_3 - 4a_2a_3\right.$$

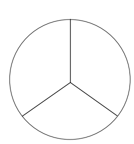
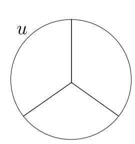
$$\left. -2b_3 - 3c_3\right)$$  $$+ 3c_3$$  $$+ 2a_3^2$$  . (40)

By (36) and (37) in Proposition 5.12, we have

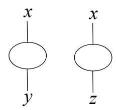
$$\left\langle \hat{\alpha}, (a_1 + a_3) \left(-\frac{1}{12} \right.\right.$$  $$+ \frac{1}{6}$$  $$+ \frac{1}{24}$$   $$\left.\left.\right)\right\rangle_{(\mathrm{conn})}$$

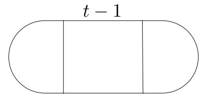
$$= \left(-\frac{a_1}{6} - \frac{a_3}{6}\right)$$  $$+ \left(\frac{a_1}{6} + \frac{a_3}{6}\right)$$  ,

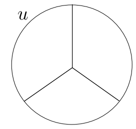
$$\left\langle \hat{\alpha}, (a_2 + a_3) \left(-\frac{1}{12} \right.\right.$$  $$+ \frac{1}{6}$$  $$+ \frac{1}{24}$$   $$\left.\left.\right)\right\rangle_{(\mathrm{conn})}$$

$$= \left(-\frac{a_2}{6} - \frac{a_3}{6}\right)$$  $$+ \left(\frac{a_2}{6} + \frac{a_3}{6}\right)$$  .

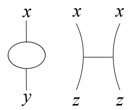
Further,

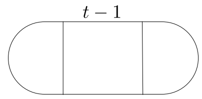
$$\left\langle \hat{\alpha}, -\frac{a_3}{4}\right.$$ 

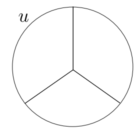
$$\left.\right\rangle_{(\mathrm{conn})} = \frac{a_3}{4}$$ 

$$= \frac{a_3}{4}$$ 

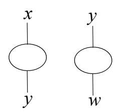
,

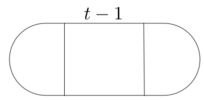
$$\left\langle \hat{\alpha}, \frac{a_3}{6}\right.$$ 

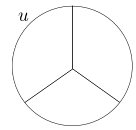
$$\left.\right\rangle_{(\mathrm{conn})} = -\frac{a_3}{3}$$ 

$$= -\frac{a_3}{3}$$ 

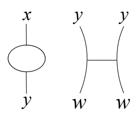
,

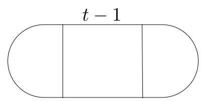
$$\left\langle \hat{\alpha}, -\frac{a_3}{4}\right.$$ 

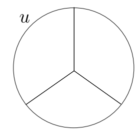
$$\left.\right\rangle_{(\mathrm{conn})} = \frac{a_3}{4}$$ 

$$= \frac{a_3}{4}$$ 

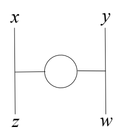
,

$$\left\langle \hat{\alpha}, \frac{a_3}{6}\right.$$ 

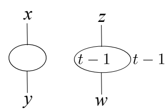
$$\left.\right\rangle_{(\mathrm{conn})} = -\frac{a_3}{3}$$ 

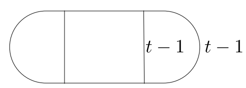
$$= -\frac{a_3}{3}$$ 

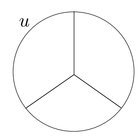
,

$$\left\langle \hat{\alpha}, a_3\right.$$ 

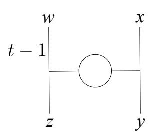
$$\left.\right\rangle_{(\mathrm{conn})} = 0,$$

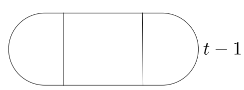
$$\left\langle \hat{\alpha}, -\frac{a_3}{4}\right.$$ 

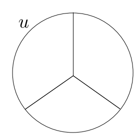
$$\left.\right\rangle_{(\mathrm{conn})} = -\frac{a_3}{4}$$ 

$$= \frac{a_3}{4}$$ 

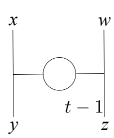
,

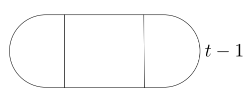
$$\left\langle \hat{\alpha}, -\frac{a_3}{2}\right.$$ 

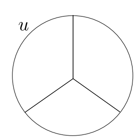
$$\left.\right\rangle_{(\mathrm{conn})} = \frac{a_3}{2}$$ 

$$= \frac{a_3}{2}$$ 

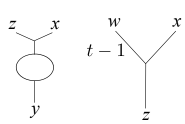
,

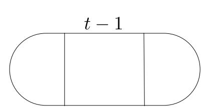
$$\left\langle \hat{\alpha}, -\frac{a_3}{2}\right.$$ 

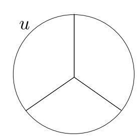
$$\left.\right\rangle_{(\mathrm{conn})} = \frac{a_3}{2}$$ 

$$= \frac{a_3}{2}$$ 

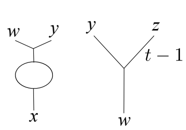
,

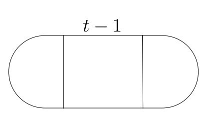
$$\left\langle \hat{\alpha}, -\frac{a_3}{2}\right.$$ 

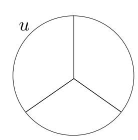
$$\left.\right\rangle_{(\mathrm{conn})} = -\frac{a_3}{2}$$ 

$$= -\frac{a_3}{2}$$ 

,

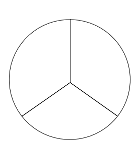
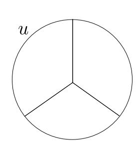
$$\left\langle \hat{\alpha}, \frac{a_3}{2}\right.$$ 

$$\left.\right\rangle_{(\mathrm{conn})} = -\frac{a_3}{2}$$ 

$$= -\frac{a_3}{2}$$ 

.

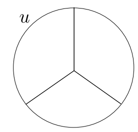
Hence, we have

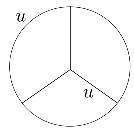
$$\langle \hat{\alpha}, \gamma_2 \rangle_{(\mathrm{conn})} = \left(-\frac{a_1}{6} - \frac{a_2}{6} - \frac{a_3}{3}\right)$$ 
$$+ \left(\frac{a_1}{6} + \frac{a_2}{6} + \frac{5}{12}a_3\right)$$ 

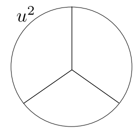
. (41)

Thus, by (40) and (41), we get

$$\chi^{-1} Z\big(D(K_1, K_2, K_3)\big)^{(\text{3-loop})}$$

$$= \Big(-\frac{1}{4}a_3 - 4a_1a_2 - 4a_1a_3 - 4a_2a_3 - 2b_3 - 3c_3\Big)$$ 

$$+ 3c_3$$ 

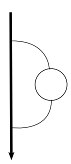
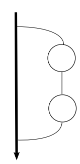
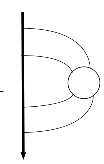
$$+ 2a_3^2$$ 

.

Therefore, by the definition of the 3-loop polynomial, we obtain the required formula. □

Remark 6.3. By Lemma 5.11, we have $\Theta_{D(K_1,K_2,K_3)}(t_1, t_2, t_3) = -2a_3(t_1 + t_1^{-1} + t_2 + t_2^{-1} + t_3 + t_3^{-1} - 6)$, which indicates that the 2-loop polynomial of $D(K_1, K_2, K_3)$ depends only on the Vassiliev invariants of K_3. Thus, if $K_3 = K_3'$, we cannot distinguish $D(K_1, K_2, K_3)$ and $D(K_1', K_2', K_3')$ for any K_1, K_2, K_1', K_2' by ($\leq$ 2)-loop polynomials. On the other hand, even if $K_3 = K_3'$, we may distinguish $D(K_1, K_2, K_3)$ and $D(K_1', K_2', K_3')$ by the 3-loop polynomial.

Example 6.4. Let T_n be the long knot such that its closure is $(2, 2n+1)$ torus knot. It can be shown that (see [21, Example 3.8])

$$Z(T_n) = \exp\left(-\frac{n(n+1)}{4} \right.$$  $$+ \frac{n(n+1)(2n+1)}{24}$$  $$\left. + \frac{n(n+1)(2n^2+2n+1)}{48} \right.$$  $$\Big).$$

Thus, by Proposition 6.2, the 3-loop polynomial of $D(T_{n_1}, T_{n_2}, T_{n_3})$ is given by

$$\begin{aligned}
&\Lambda_{D(T_{n_1},T_{n_2},T_{n_3})}(t_1, t_2, t_3, t_4) \\
&= \Bigg(\frac{n_3(n_3+1)}{4} - n_1(n_1+1)n_2(n_2+1) - n_1(n_1+1)n_3(n_3+1) - n_2(n_2+1)n_3(n_3+1) \\
&\quad -\frac{n_3(n_3+1)(6n_3^2+14n_3+7)}{12}\Bigg)(u_{12} + u_{13} + u_{14} + u_{2,3} + u_{24} + u_{34}) \\
&\quad + \frac{n_3(n_3+1)(2n_3^2+2n_3+1)}{2}(u_{12}u_{34} + u_{13}u_{24} + u_{14}u_{23}) \\
&\quad + \frac{n_3^2(n_3+1)^2}{2}(u_{12}^2 + u_{13}^2 + u_{14}^2 + u_{2,3}^2 + u_{24}^2 + u_{34}^2).
\end{aligned}$$

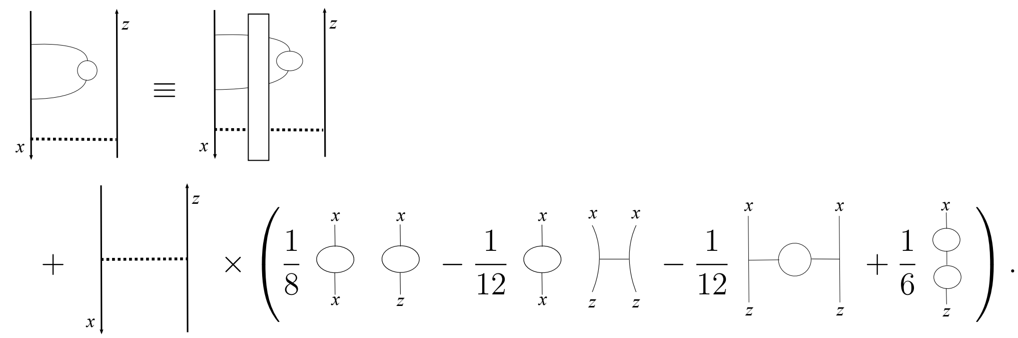
Thus, for example, for all n, m, k, l such that $n \neq m$, we have $\Delta_{D(T_n,T_k,T_l)}(t) = \Delta_{D(T_m,T_k,T_l}(t)$ and $\Theta_{D(T_n,T_k,T_l)}(t_1, t_2, t_3) = \Theta_{D(T_m,T_k,T_l)}(t_1, t_2, t_3)$, but $\Lambda_{D(T_n,T_k,T_l)}(t_1, t_2, t_3, t_4) \neq \Lambda_{D(T_m,T_k,T_l)}(t_1, t_2, t_3, t_4)$.

Appendix

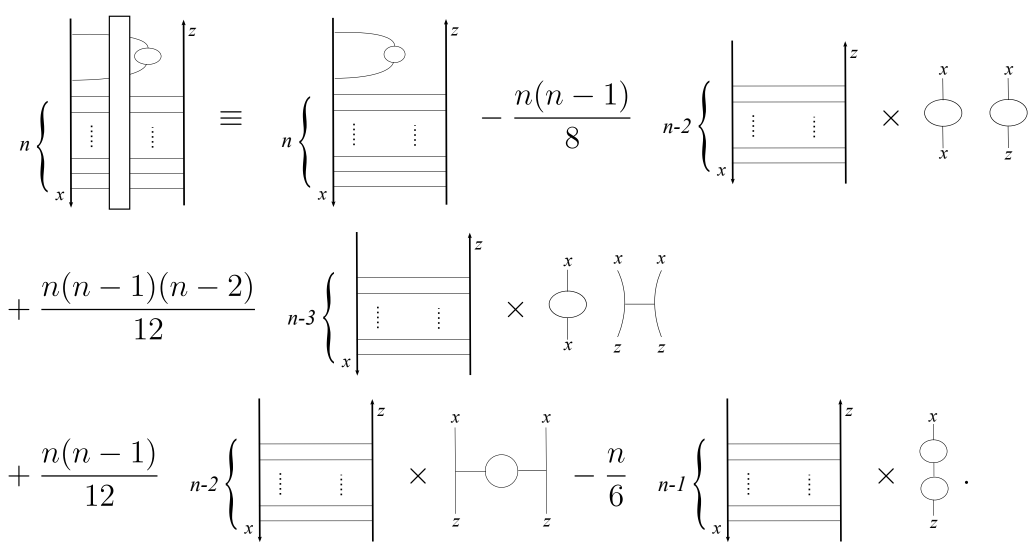
A Calculations of Jacobi diagrams with symmetrizer

In this Section, we prove several lemmas for the calculations of the inverse image by the PBW isomorphism.

Lemma A.1. *(see also [21, Lemma 6.3])*

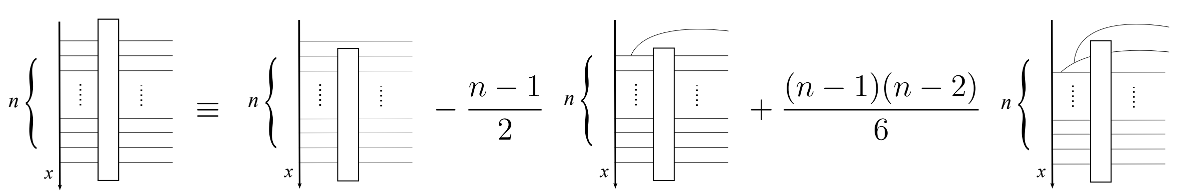


Proof. It is sufficient to show the formula,



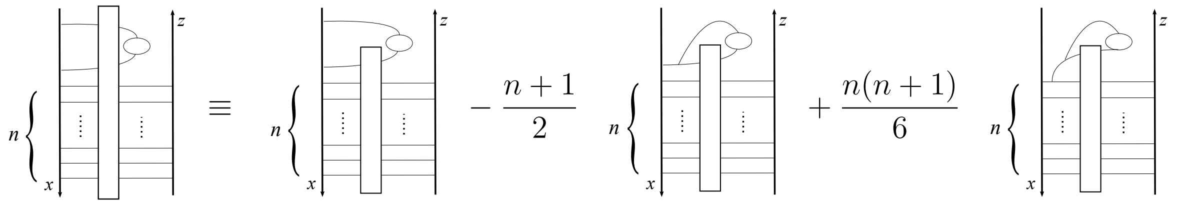
(42)

We show this. It is shown in [17, Lemma 5.1] that



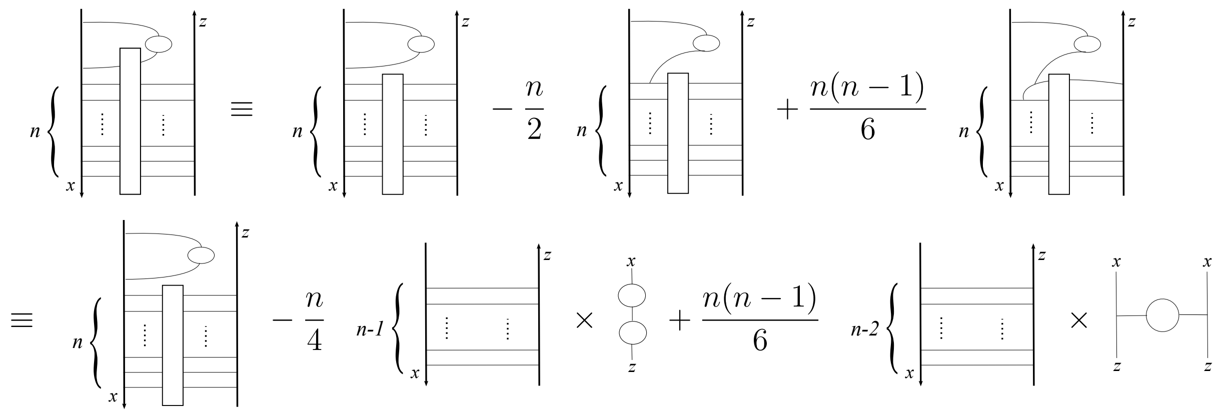
(43)

By applying (43) (replacing n with $n+2$) to the left-hand side of (42), we get



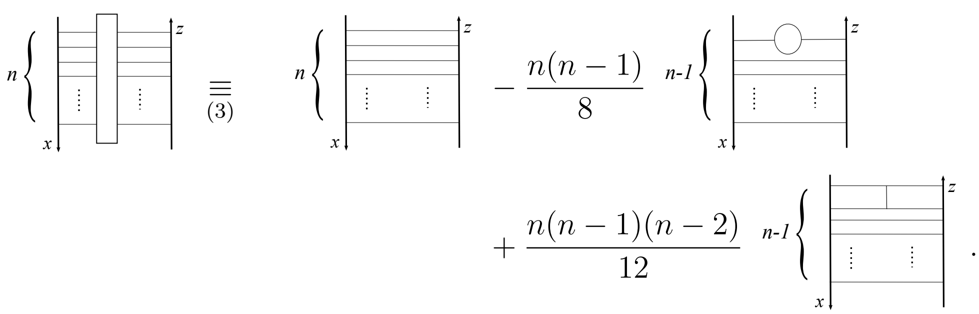
(44)

The first term of the right-hand side of (44) is calculated by applying (43) (replacing n with $n+1$) as follows,



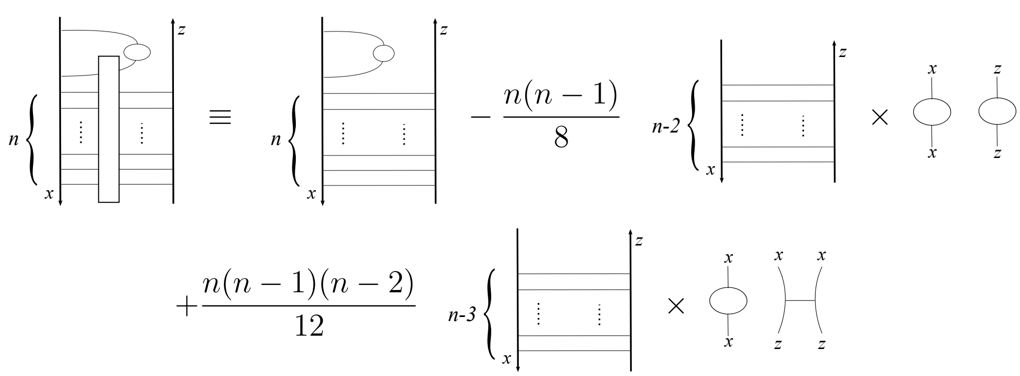
(45)

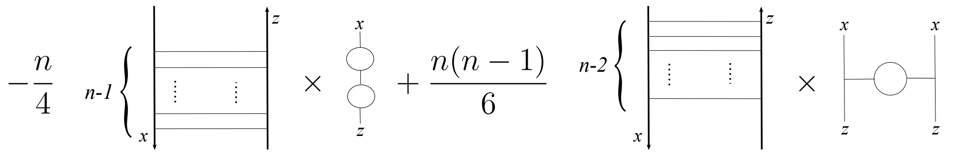
Using [17, Lemma 5.2] we can show that



(46)

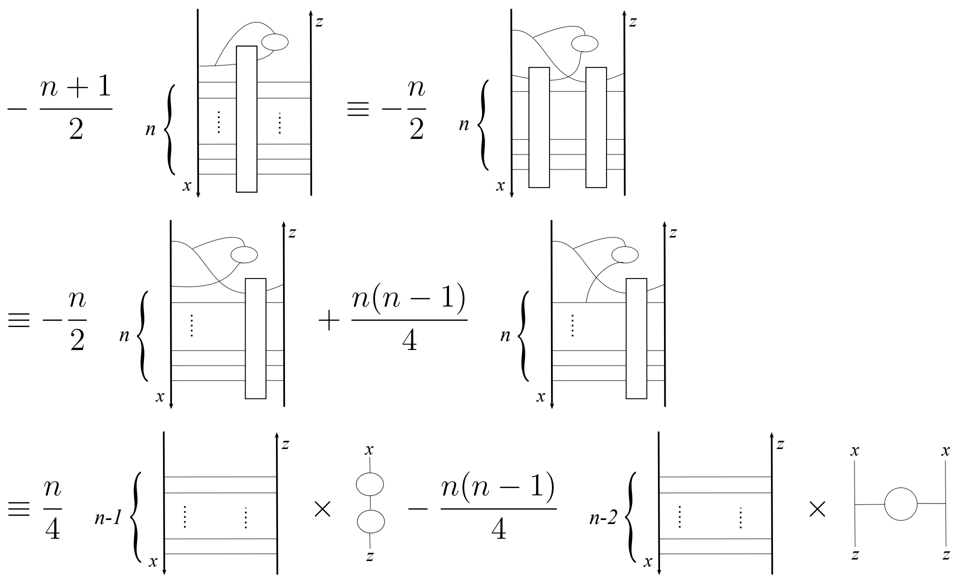
By applying (46) to (45), we obtain





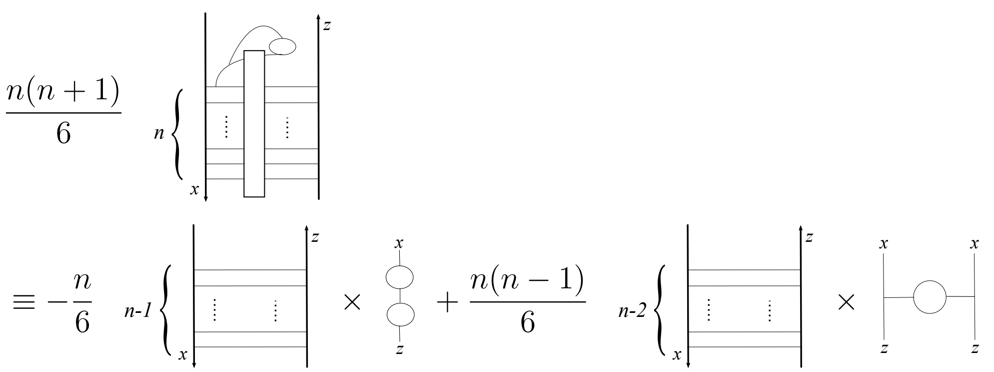
(47)

The second term of the right-hand side of (44) is calculated as follows,



(48)

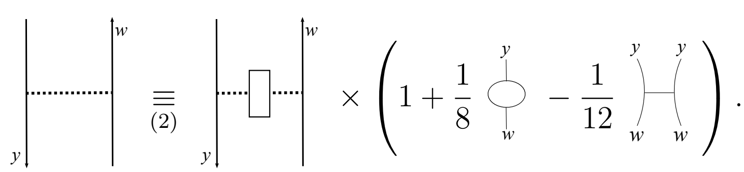
The third term of the right-hand side of (44) is calculated as follows,



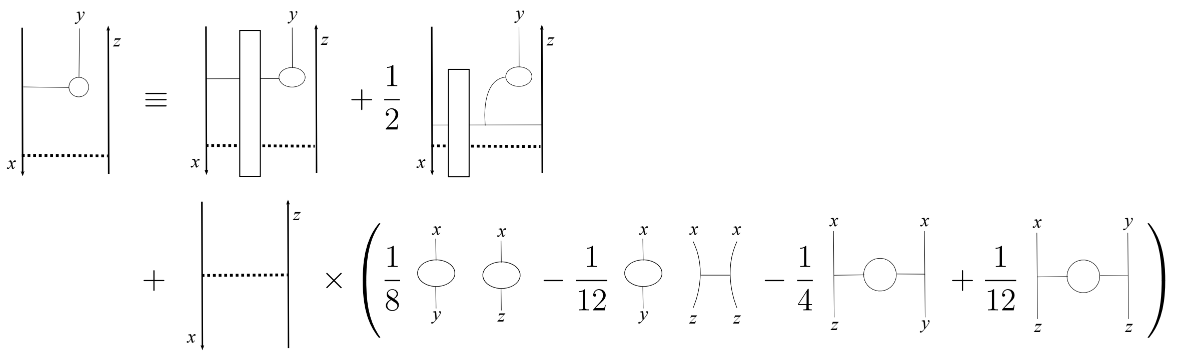
(49)

Thus, by applying (47), (48), (49) to (44), we obtain (42). Therefore, we obtain the required formula. □

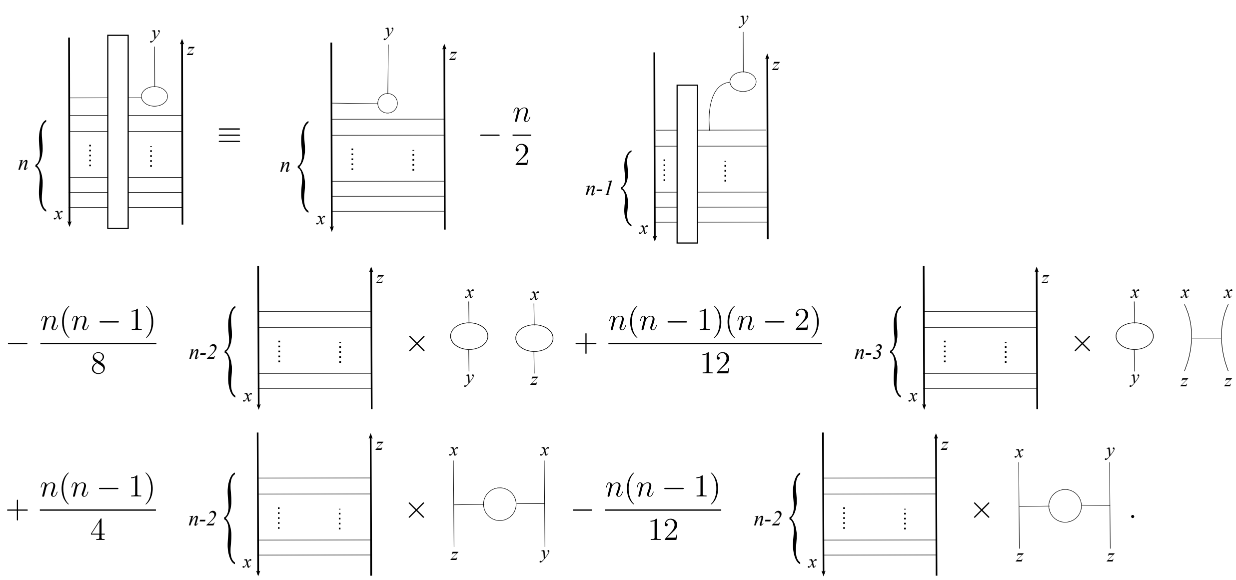
Lemma A.2. *[17, Lemma 5.2]*



Lemma A.3.

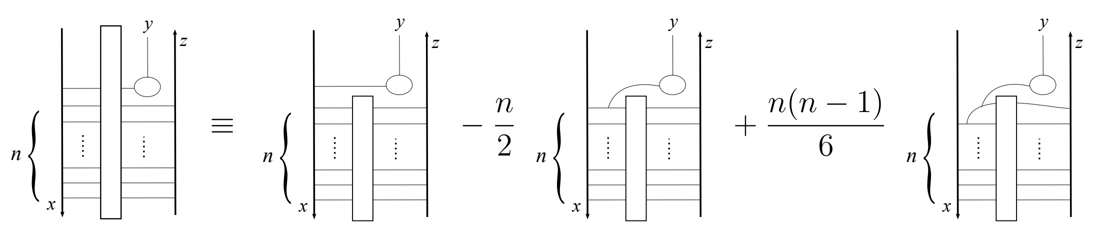


Proof. It is sufficient to show the formula,



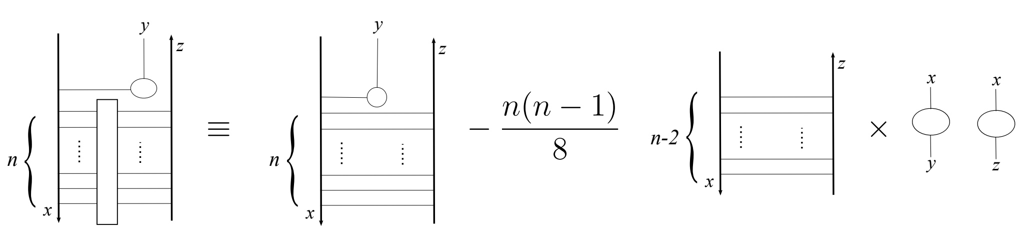
(50)

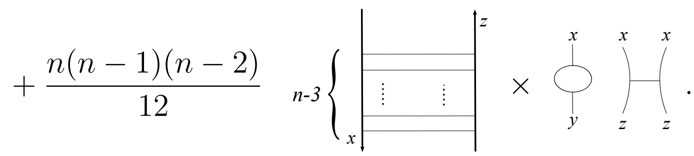
We show this. By (43), we get



(51)

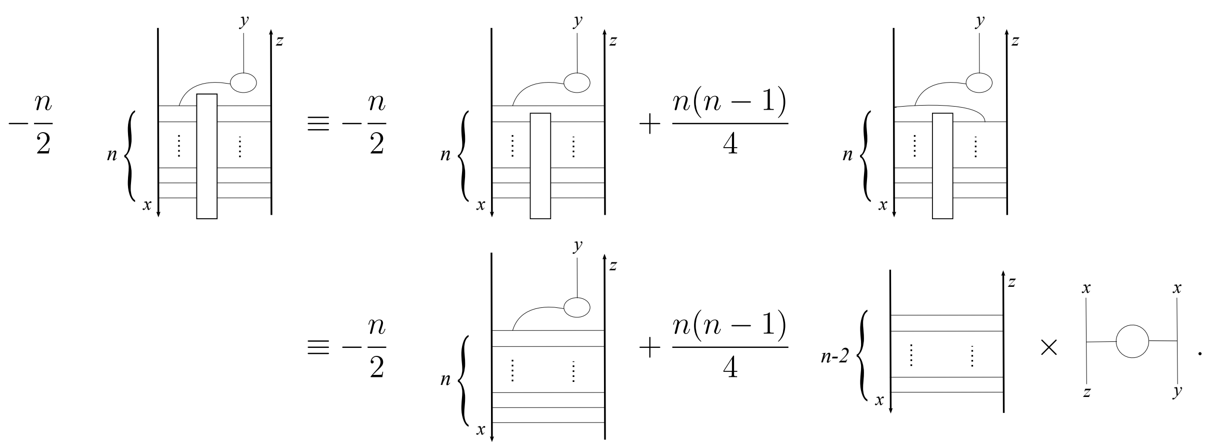
By (46), the first term of the right-hand side of (51) is calculated as follows,





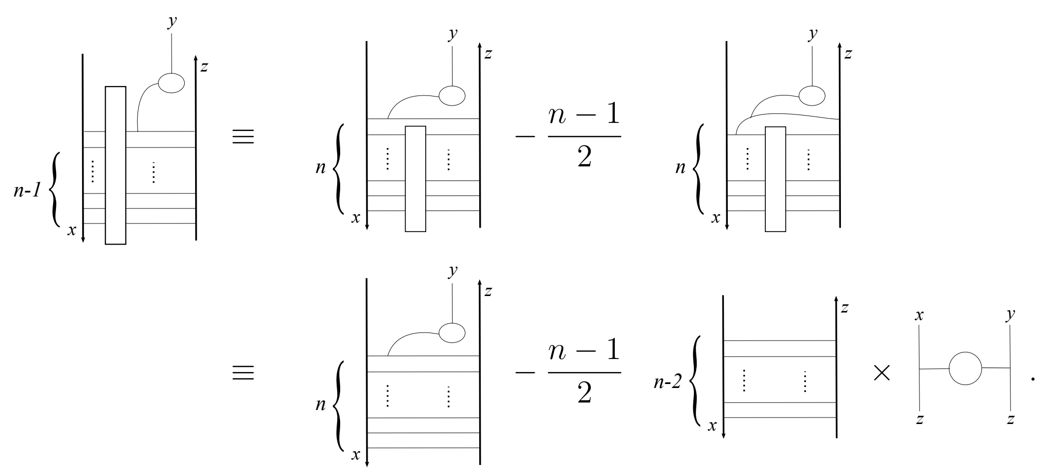
(52)

By (43) and (46), the second term of the right-hand side of (51) is calculated as follows,

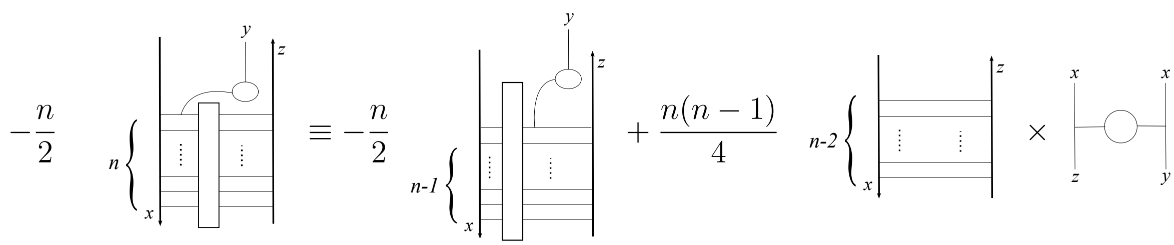


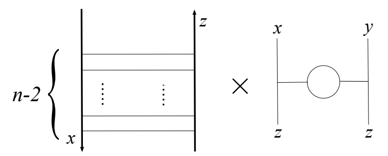
(53)

Moreover, by (43) and (46), we get

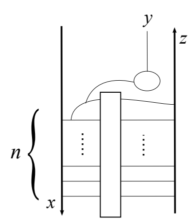
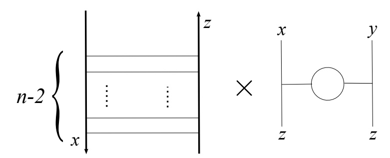


By applying this to (53), we obtain



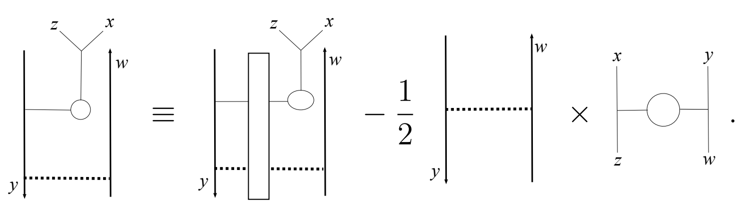
$$-\frac{n(n-1)}{4}$$  (54)

The third term of the right-hand side of (51) is calculated as follows,

$$\frac{n(n-1)}{6}$$  $$\equiv \frac{n(n-1)}{6}$$  (55)

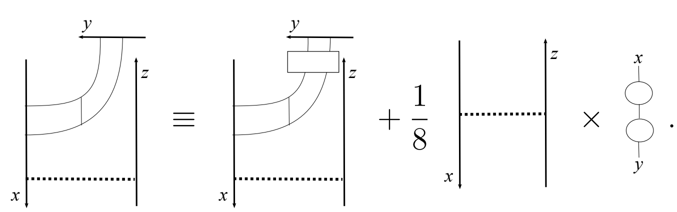
Thus, by applying (52), (54), (55) to (51), we get (50). Therefore, we obtain the required formula. □

Lemma A.4.

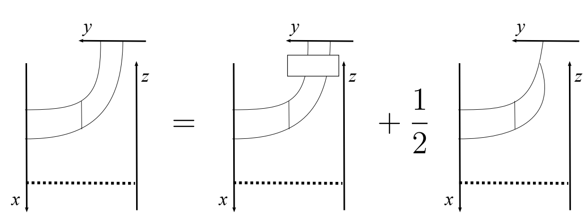


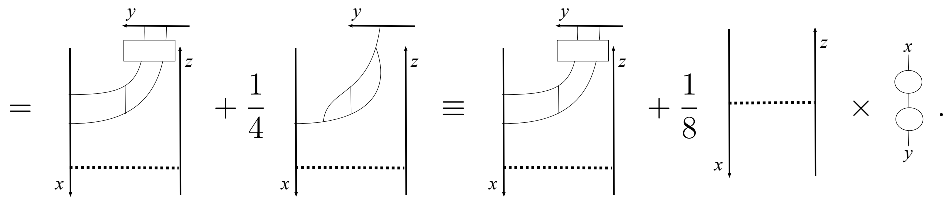
Proof. We can show this in a same way of the proof of Lemma A.3. □

Lemma A.5.



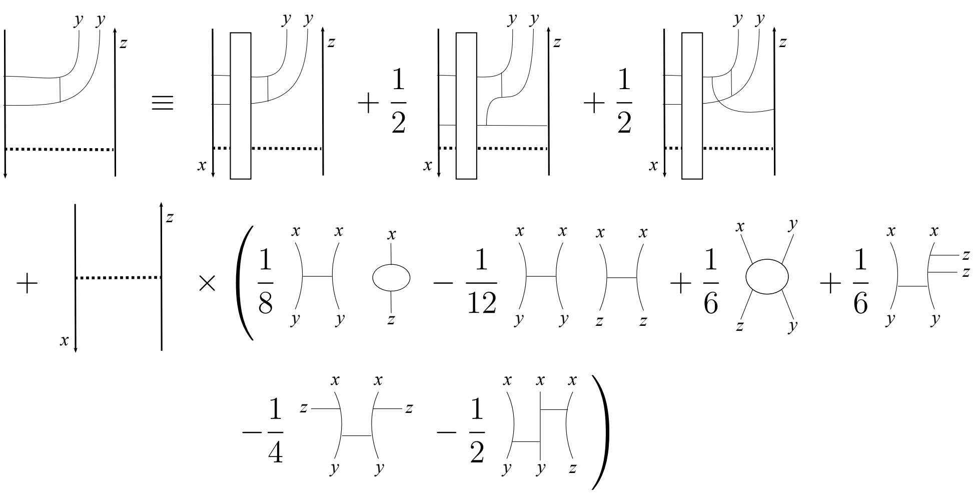
Proof. .By the IHX and STU relations, we have



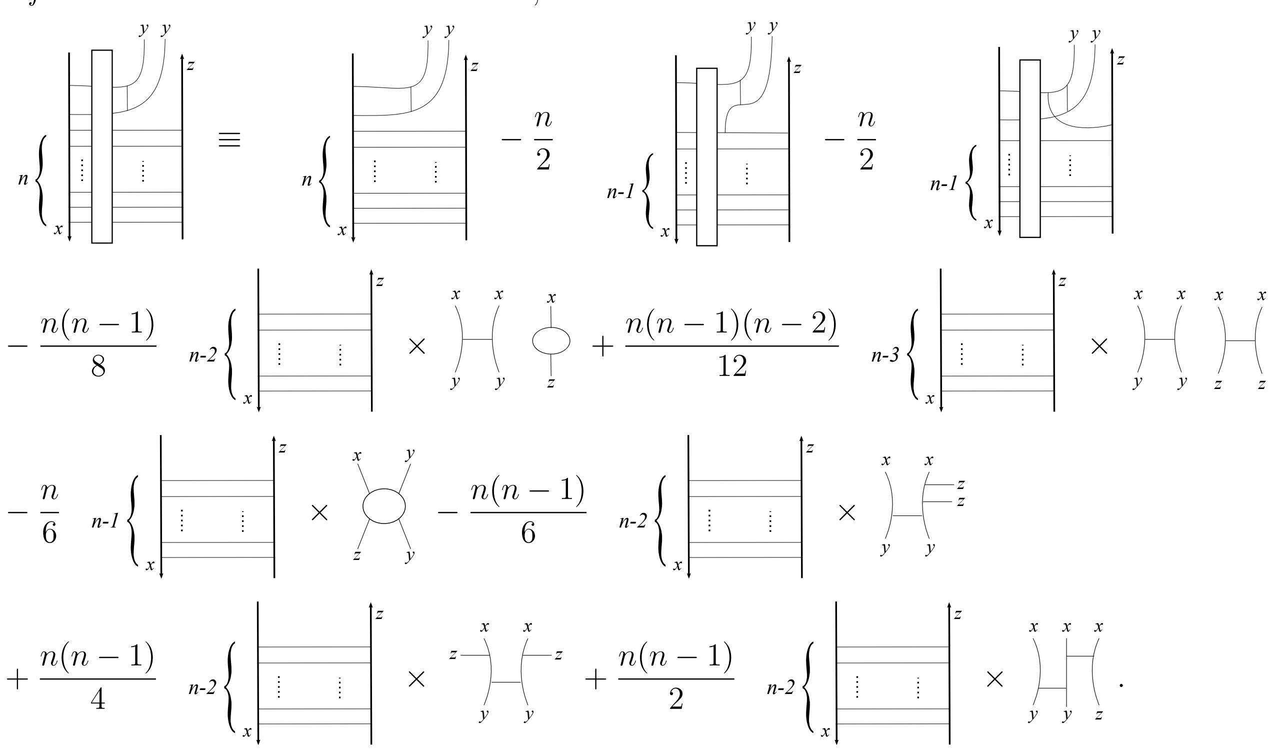


□

Lemma A.6.

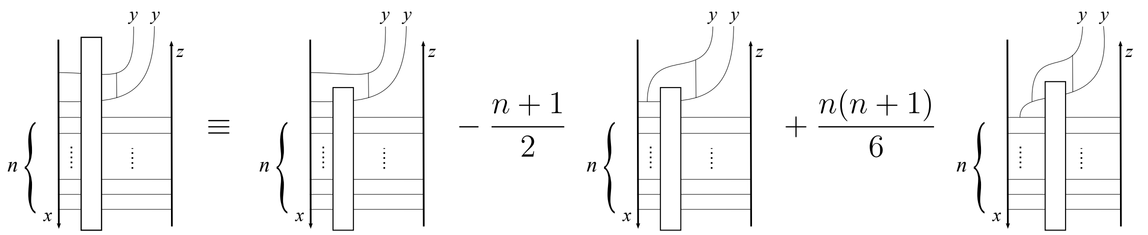


Proof. It is sufficient to show the formula,



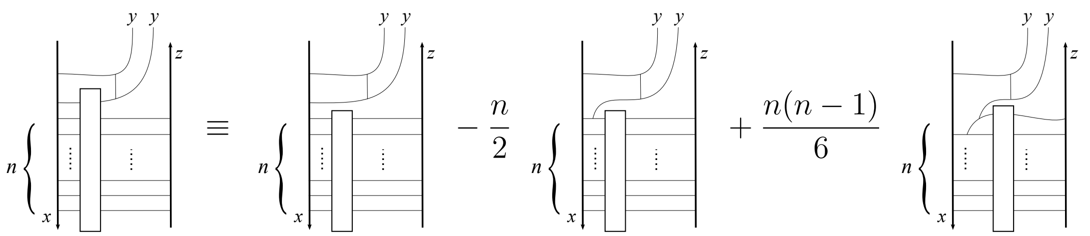
(56)

We show this. By (43), we get



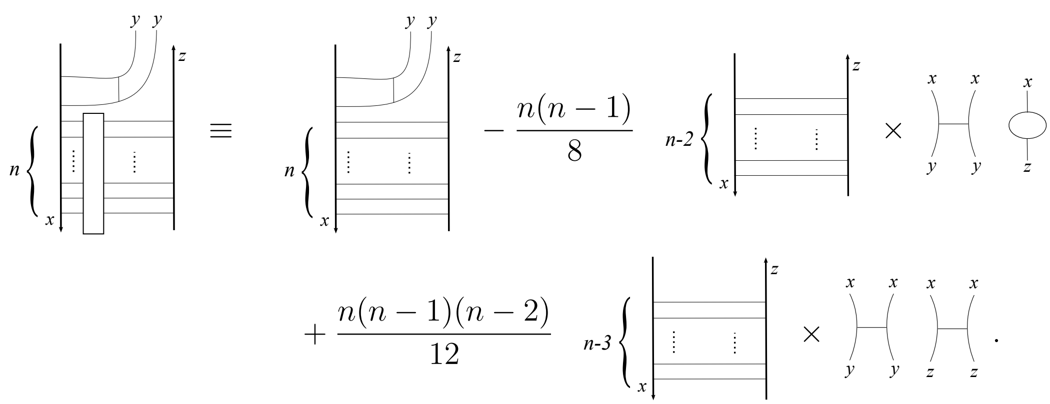
(57)

By (43), the first term of the right-hand side of (57) is calculated as follows,



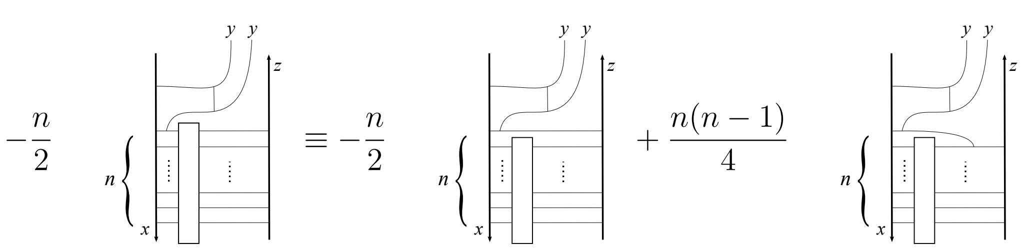
(58)

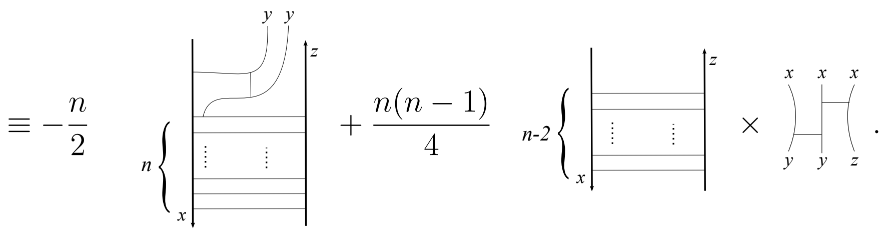
By (46), the first term of the right-hand side of (58) is calculated as follows,



(59)

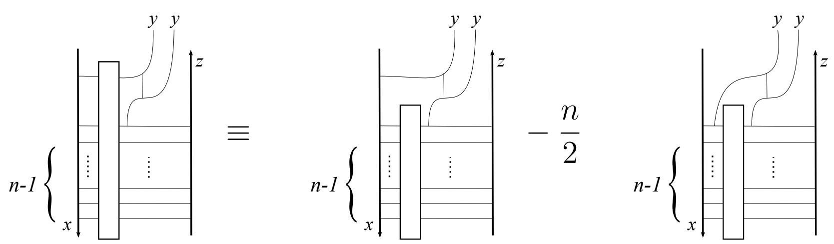
By (43) and (46), the second term of the right-hand side of (58) is calculated as follows,

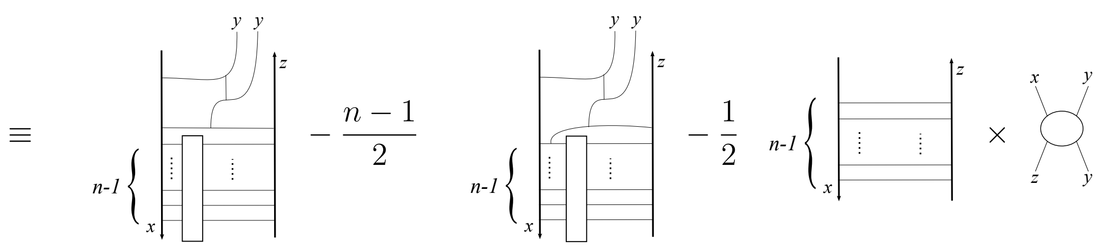


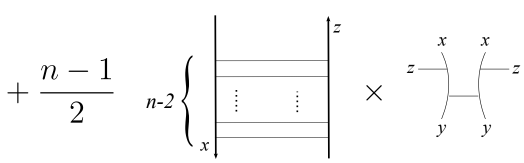


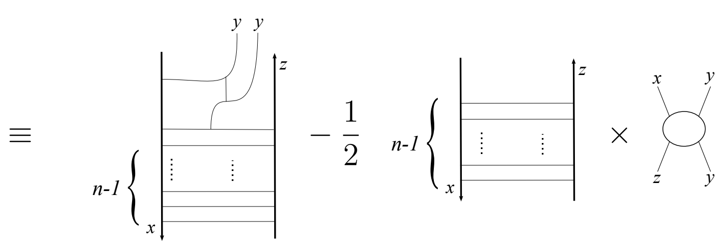
(60)

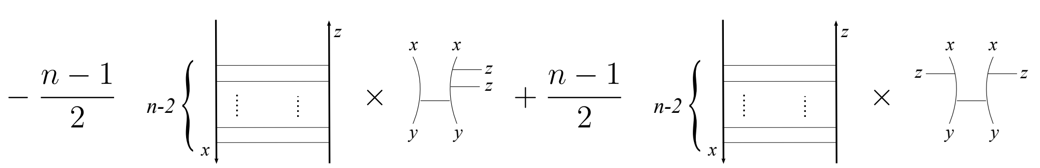
Moreover, by (43) and (46), we get



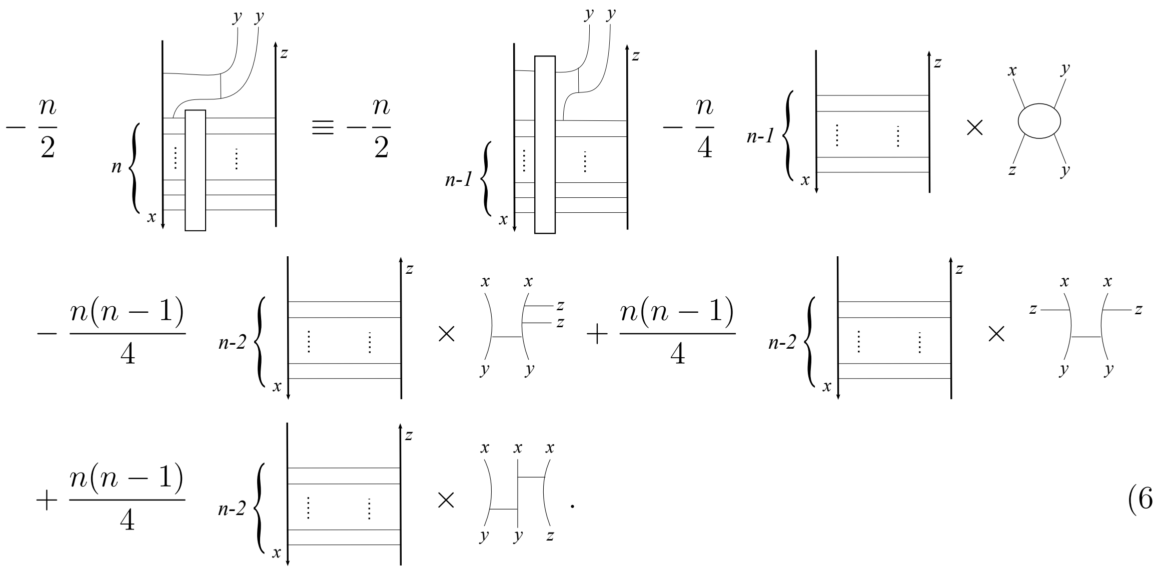






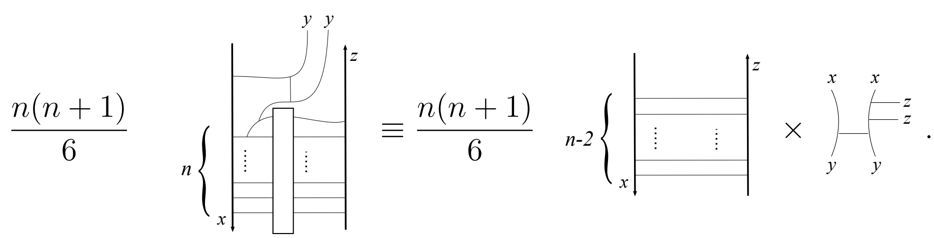


By applying this to (60), we obtain



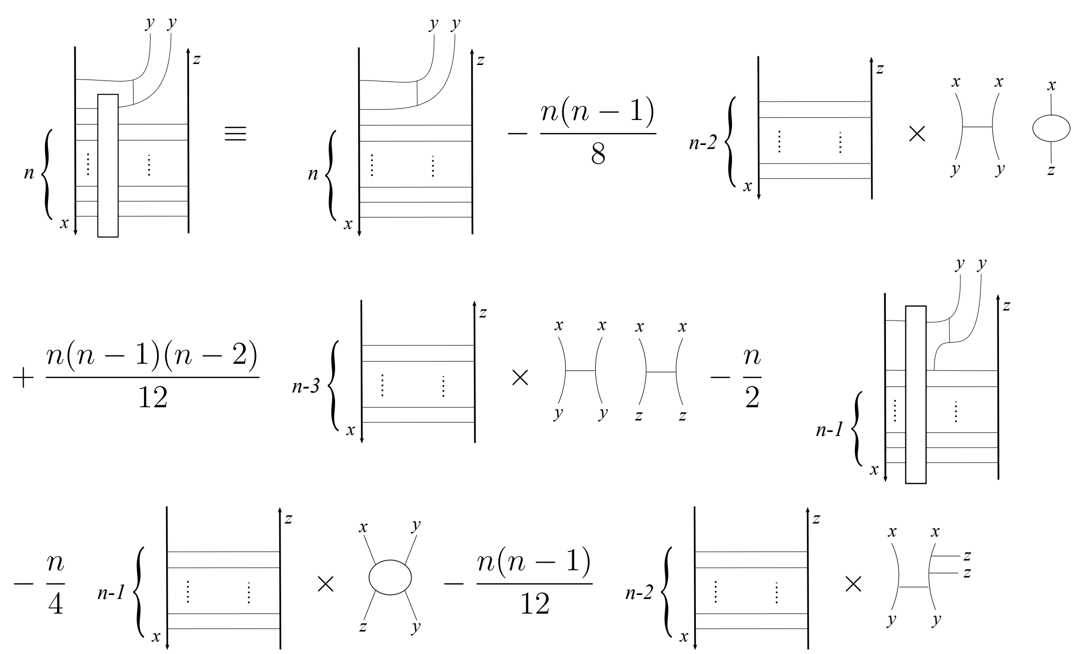
(61)

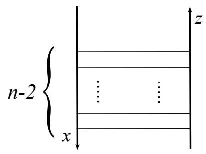
The third term of the right-hand side of (58) is calculated as follows,

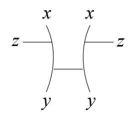


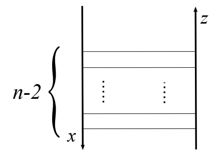
(62)

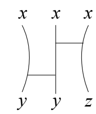
By applying (59), (61), (62) to (58), we obtain



$$+\frac{n(n-1)}{4}$$ 

$\times$ 

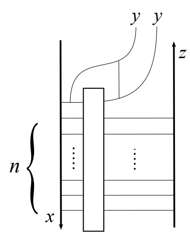
$$+\frac{n(n-1)}{4}$$ 

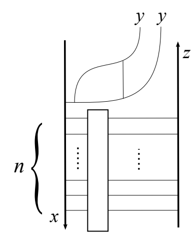
$\times$ 

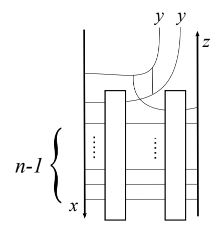
.

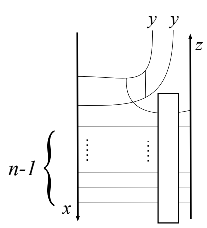
(63)

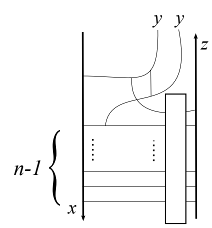
By (43) and (46), the second term of the right-hand side of (57) is calculated as follows,

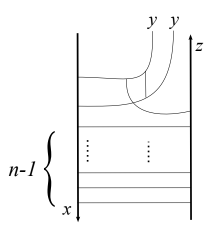
$$-\frac{n+1}{2}$$ 

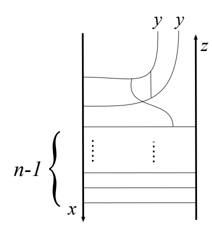
$$\equiv -\frac{1}{2}$$ 

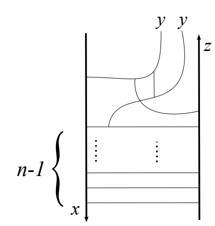
$$-\frac{n}{2}$$ 

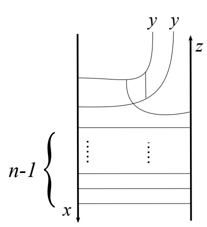
$$\equiv -\frac{n}{2}$$ 

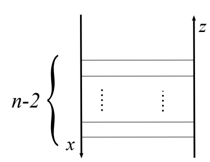
$$+\frac{n(n-1)}{4}$$ 

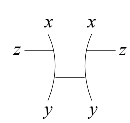
$$\equiv -\frac{n}{2}$$ 

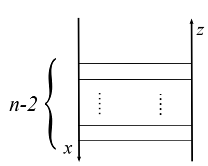
$$+\frac{n(n-1)}{4}$$ 

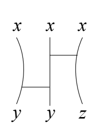
$$+\frac{n(n-1)}{4}$$ 

$$\equiv -\frac{n}{2}$$ 

$$-\frac{n(n-1)}{4}$$ 

$\times$ 

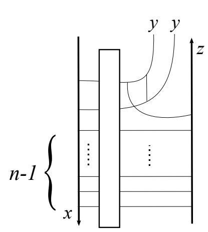
$$+\frac{n(n-1)}{4}$$ 

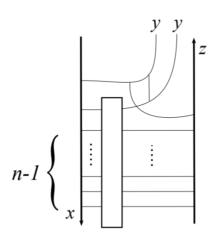
$\times$ 

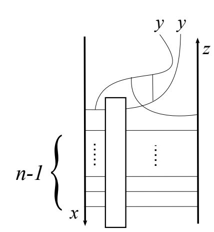
.

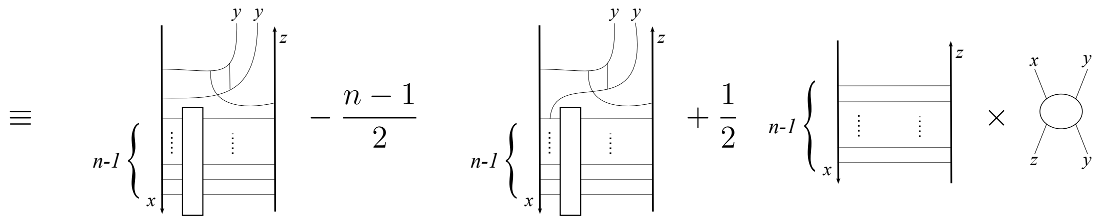
(64)

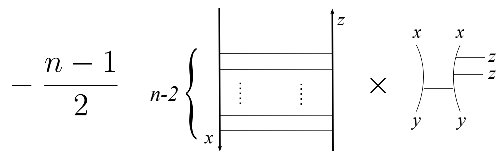
Moreover, by (43) and (46), we get

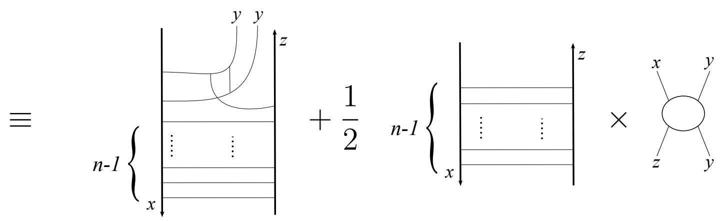


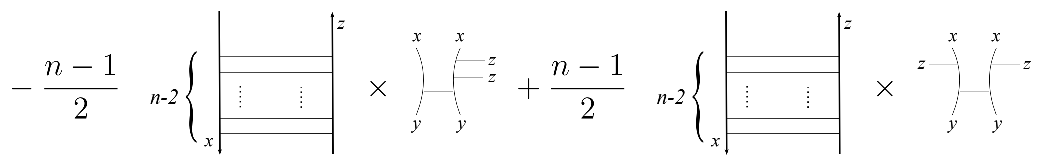
$$\equiv$$ 

$$-\frac{n}{2}$$ 

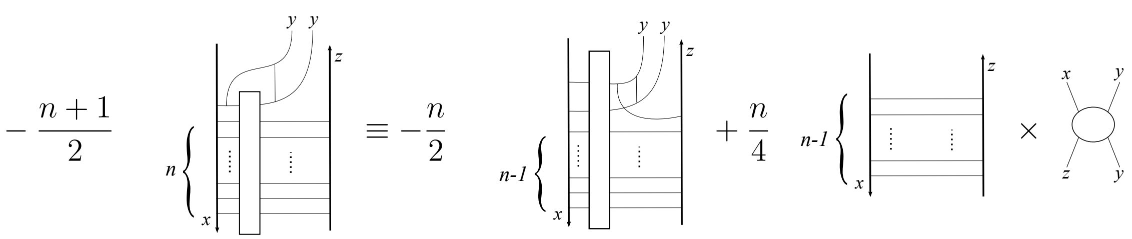


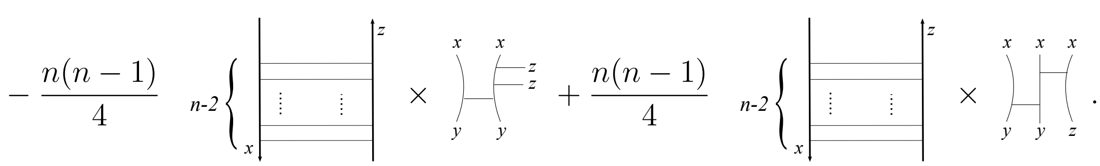






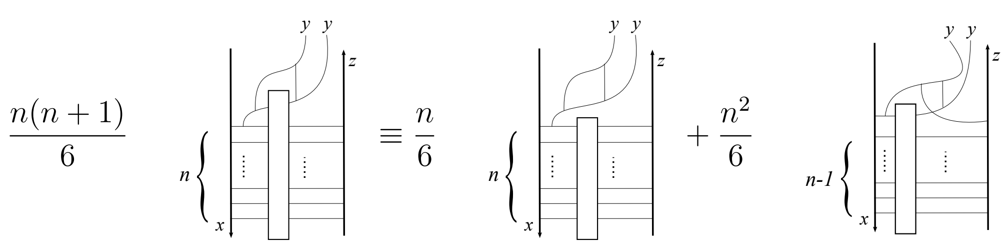
By applying to this to (64), we obtain

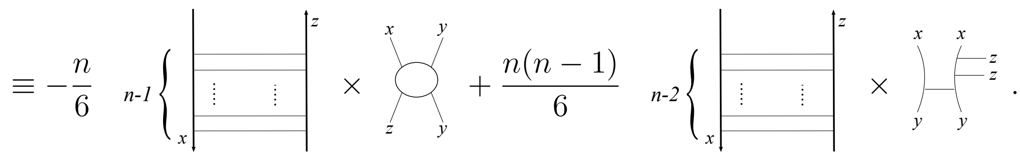




(65)

The third term of the right-hand side of (57) is calculated as follows,

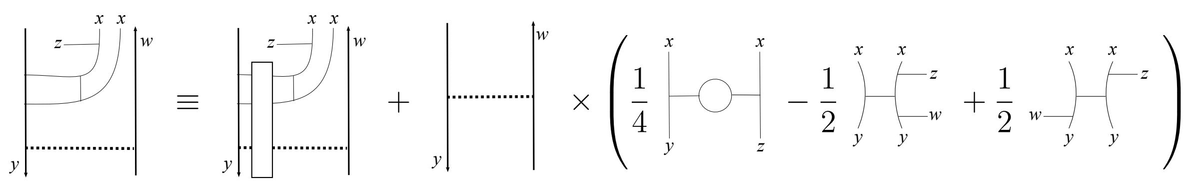




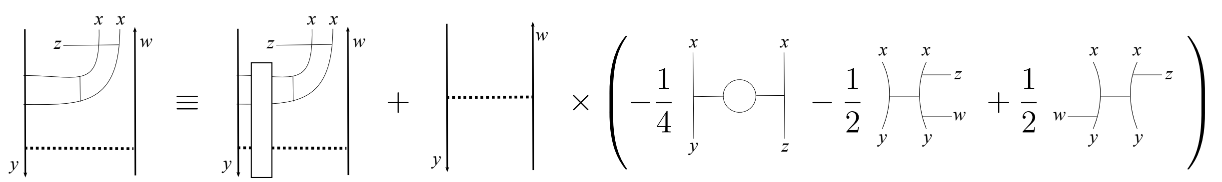
(66)

Thus, by applying (63), (65), (66) to (57), we get (56). Therefore, we obtain the required formula. □

Lemma A.7.

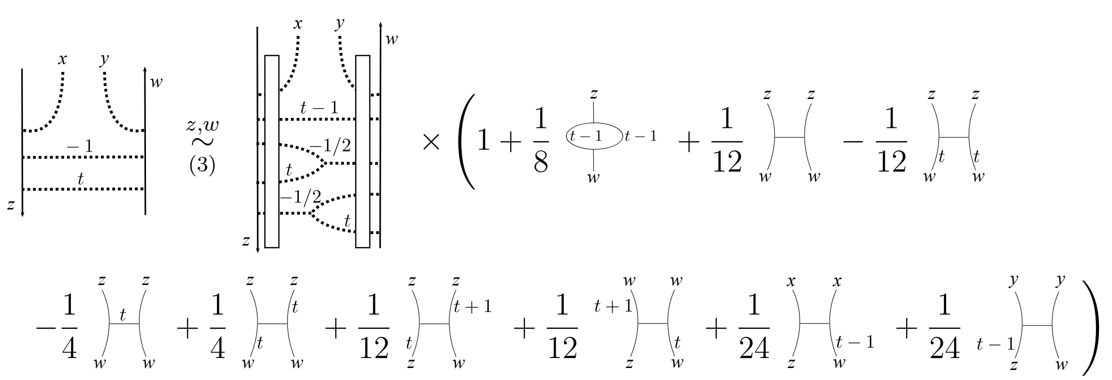


Lemma A.8.

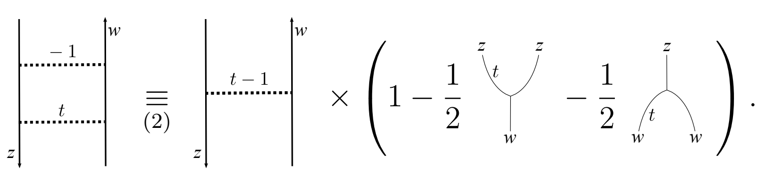


we can show Lemmas A.7 and A.8 in a same way of the proof of Lemma A.6

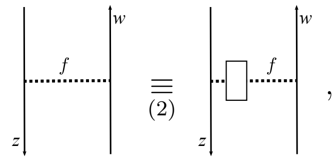
Lemma A.9. *[17, Lemma 5.17]*



Lemma A.10. *[17, Lemma 5.2, Lemma 5.7]*

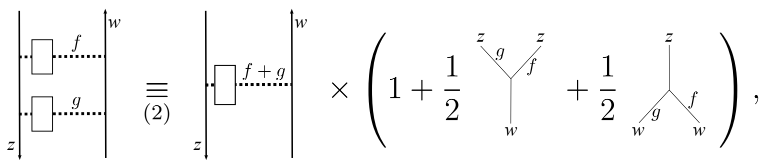


Proof. By [17, Lemma 5.2], we have



(67)

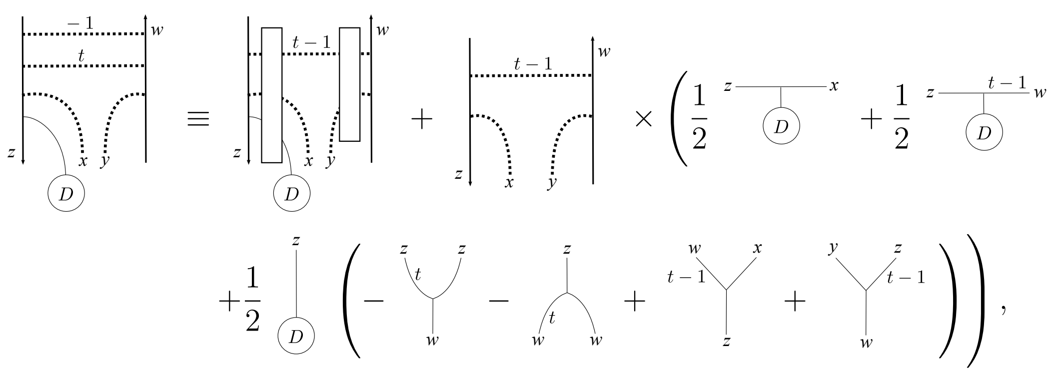
where f is a power series. Moreover, by [17, Lemma 5.7], we have



(68)

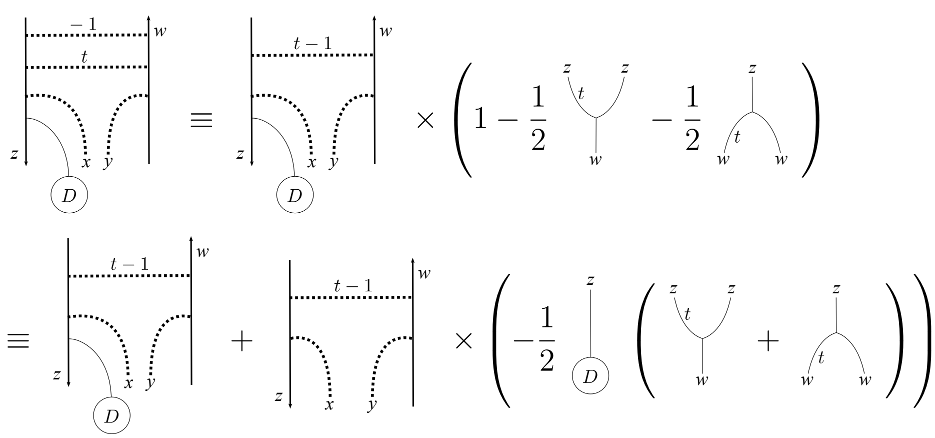
where f and g are power series. By (67) and (68), we obtain the required formula. $\square$

Lemma A.11. *We have*



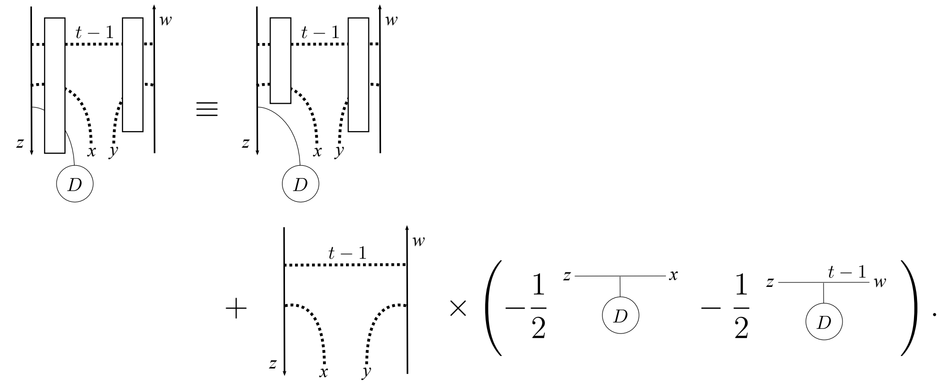
where D is a diagram with 3 trivalent vertices.

Proof. By Lemma A.10, we have



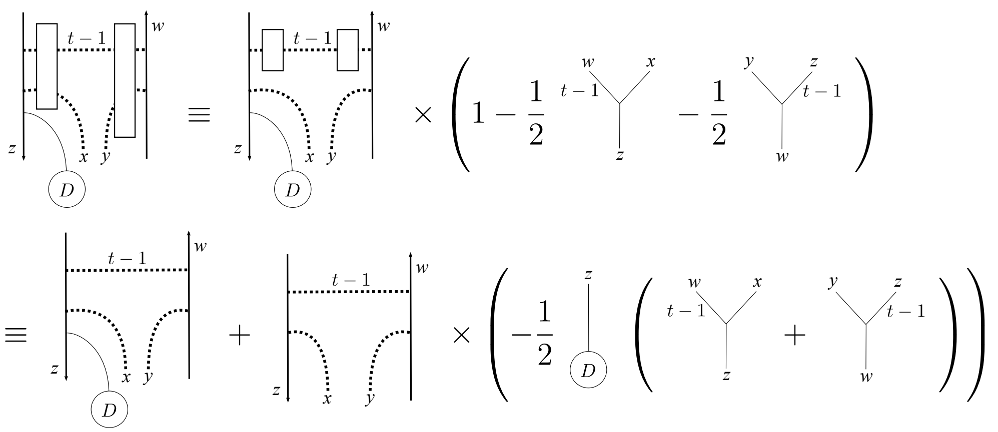
(69)

In a similar way of the proof of the Lemma A.4, we can get



(70)

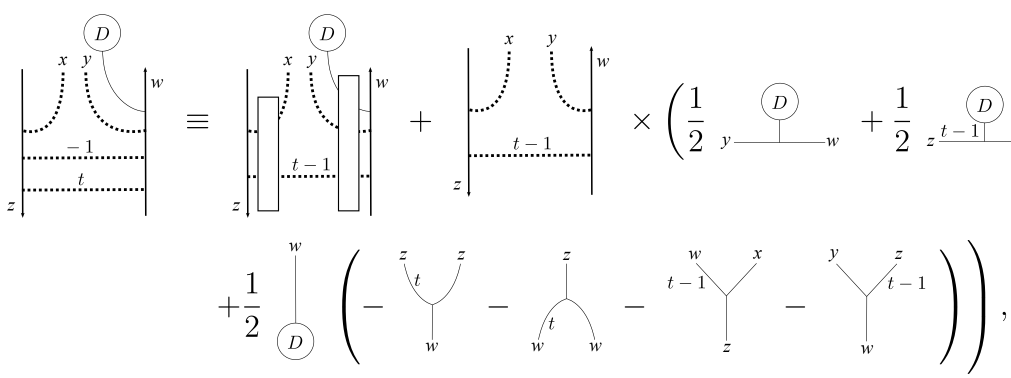
Further, in a similar way of the proof of [17, Lemma 5.4, Lemma 5.5], we can get



(71)

where we obtain the last equivalence by (67). Thus, by applying (70) and (71) to (69), we obtain the required formula. □

Lemma A.12.

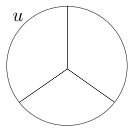


where D is a diagram with 3 trivalent vertices.

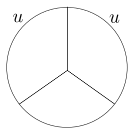
Proof. We can show this in a same way of the proof of Lemma A.11. □

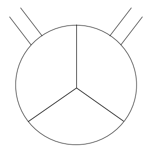
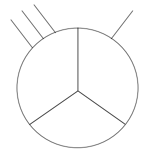
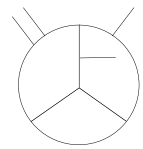
B Proof of Lemma 5.13

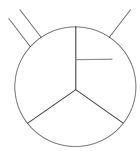
Proof. We note that $u = e^h + e^{-h} - 2 = h^2 + \frac{1}{12}h^4 + \frac{1}{360}h^6 +$ (higher terms). In the following calculations, we use the AS and IHX relations. The first, second and third diagrams are calculated as follows,

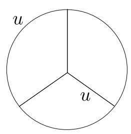


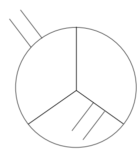
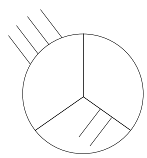
$\equiv_{d\leq 8} \lambda_1 + \frac{1}{12}\lambda_2 + \frac{1}{360}\lambda_4.$



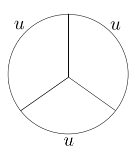
$\equiv_{d\leq 8}$  $+ \frac{1}{6}\lambda_6 =$  $-$  $+ \frac{1}{6}\lambda_6$

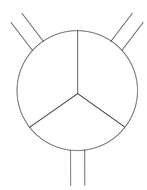
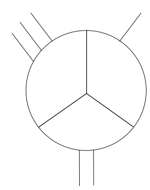
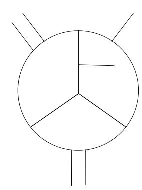
$= \frac{1}{2}\lambda_2 -$  $+ \frac{1}{6}\lambda_6 = \frac{1}{2}\lambda_2 + \frac{1}{6}\lambda_6.$



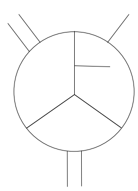
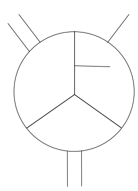
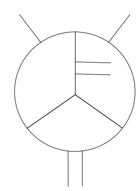
$\equiv_{d\leq 8}$  $+ \frac{1}{6}$  $= \lambda_2 - 2\lambda_3 - \frac{1}{3}\lambda_5 + \frac{1}{3}\lambda_6.$

The forth diagram is calculated as follows,

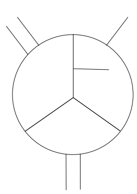
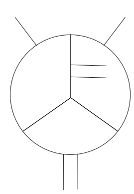
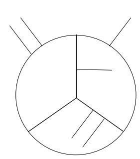


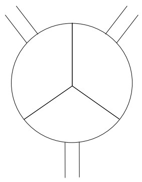
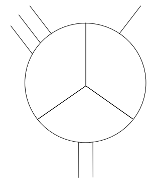
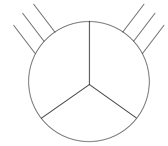
$\equiv_{d\leq 8}$  $=$  $-$  . (72)

At first, we calculate the second term of the right-hand side of (72). We have

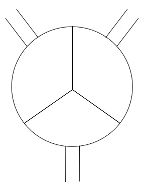
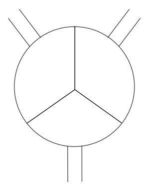
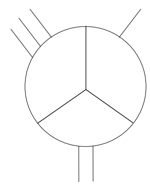
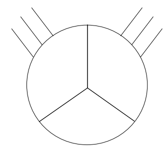
$-$  $=$  $-$  .

Thus, we get

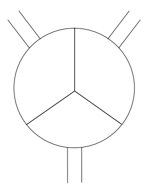
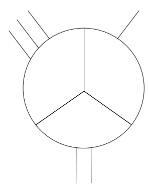
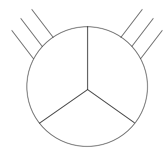
$-$  $= -\frac{1}{2}$  $= \frac{1}{2}$ 

$= -\frac{1}{2}$  $+\frac{1}{2}$  $+\frac{1}{2}$  $-\frac{1}{2}\lambda_6.$

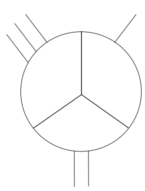
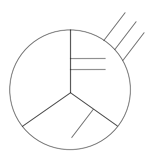
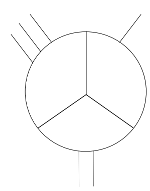
By applying this to (72), we have

 $= -\frac{1}{2}$  $+\frac{3}{2}$  $+\frac{1}{2}$  $-\frac{1}{2}\lambda_6.$

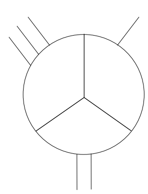
Thus, we obtain

 $=$  $+\frac{1}{3}$  $-\frac{1}{3}\lambda_6.$ (73)

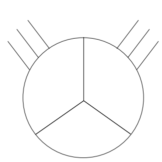
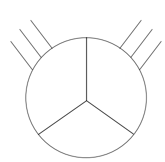
We calculate the first term of the right-hand side of (73).

 $=$  $= -$  $-\frac{1}{2}\lambda_4 + 2\lambda_5 + \lambda_6.$

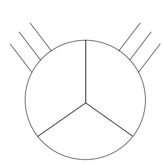
Thus, we get

 $= -\frac{1}{4}\lambda_4 + \lambda_5 + \frac{1}{2}\lambda_6.$ (74)

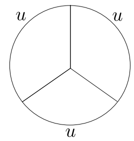
We calculate the second term of the right-hand side of (73).

 $= -$  $-\frac{1}{2}\lambda_4 + 3\lambda_6.$

Thus, we get

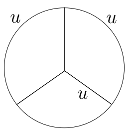
$\frac{1}{3}$  $= -\frac{1}{12}\lambda_4 + \frac{1}{2}\lambda_6.$ (75)

By applying (74), (75) to (73) and by (72), we obtain

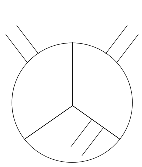


$$\equiv_{d\leq 8} -\frac{1}{3}\lambda_4 + \lambda_5 + \frac{2}{3}\lambda_6$$

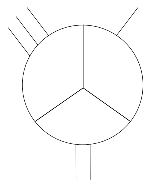
. The fifth diagram is calculated as follows,



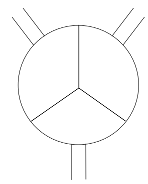
$\equiv_{d\leq 8}$



$= \lambda_6 - 2$



$+$



$$= \frac{1}{6}\lambda_4 - \lambda_5 + \frac{2}{3}\lambda_6.$$

□

References

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