Quasi L_2/L_2 Hankel Norms and L_2/L_2 Hankel Norm/Operator of Sampled-Data Systems

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***Abstract*—This paper is relevant to appropriately defining the L_2/L_2 Hankel norm of sampled-data systems through setting a general time instant Θ at which past and future are to be separated and introducing the associated quasi L_2/L_2 Hankel operator/norm at Θ. We first provide a method for computing the quasi L_2/L_2 Hankel norm for each Θ, which is carried out by introducing a shifted variant of the standard lifting technique for sampled-data systems. In particular, it is shown that the quasi L_2/L_2 Hankel norm can be represented as the l_2/l_2 Hankel norm of a Θ-dependent discrete-time system. It is further shown that an equivalent discretization of the generalized plant exists, which means that the aforementioned discrete-time system can be represented as the feedback connection of the discretized plant and the same discrete-time controller as the one in the sampled-data system. It is also shown that the supremum of the quasi L_2/L_2 Hankel norms at Θ belonging to a sampling interval is actually attained as the maximum, which means that what is called a critical instant always exists and *the* L_2/L_2 Hankel operator is always definable (as the quasi L_2/L_2 Hankel operator at the critical instant). Finally, we illustrate those theoretical developments through a numerical example.**

***Index Terms*—dynamical systems, Hankel norm, Hankel operator, sampled-data systems, shifted lifting.**

I. Introduction

There are numerous systems whose output depends not only on the current input but also on the past input. They are called dynamical systems, for which it is important from the control perspective to evaluate how the past input affects the future output. In general, the mapping from the past input to the future output is called the Hankel operator, and evaluating the worst influence of the past input on the future output amounts to computing its norm called the Hankel norm. The Hankel operator and the Hankel norm play a significant role in model reduction for high-order models of large-scale systems and many results are obtained, e.g., in [1]–[3].

Sampled-data systems are dynamical control systems in which the plant is a continuous-time system and the controller is a discrete-time system. Since most control systems nowadays use digital and thus discrete-time controllers, we are interested in studying the Hankel operator and Hankel norm of sampled-data systems in this paper. A pioneering study in this direction was carried out in [4], but it lacked a very important and essential viewpoint, unfortunately. In sampled-data systems, the continuous-time plant is connected to the discrete-time controller through a zero-order hold and an ideal sampler, and they all work periodically. Hence, when we aim at studying the Hankel operator of sampled-data systems associated with continuous-time disturbances and continuous-time controlled variables, sampled-data systems should be viewed as periodically time-varying systems, even when the continuous-time plant and the discrete-time controller are both time-invariant. Thus, it crucially matters when to take the time instant at which past and future are separated in defining the Hankel norm of sampled-data systems. In spite of this crucial fact, the study in [4] simply takes the separating instant of past and future at a sampling instant. This issue was pointed out in [5], where a general instant $\Theta \in [0, h)$ was introduced under the situation where 0 and h are two consecutive sampling instants, and past and future were separated at Θ. More specifically, the operator mapping the input up to time Θ to the output after Θ was called the quasi Hankel operator at Θ, and its norm was called the quasi Hankel norm at Θ. This viewpoint naturally led us to studying whether the supremum of the quasi Hankel norms over $\Theta \in [0, h)$, which is to be defined as the Hankel norm, is attained, and if it is, whether or not it is at $\Theta = 0$.

Now, the relationship among the pioneering study [4], the aforementioned amended study [5] with the introduction of Θ, the present study, and an earlier conference version of the present study [6] is as follows.

a) L_∞/L_2 vs. L_2/L_2 relevant to the treatment of Θ: First of all, it was a standpoint of the study in [5] that taking a slightly modified handling of the output compared with that in [4] was more tractable in connection with the introduction of Θ. That is, even though these two studies both consider taking the past input from the L_2 space, the future output was regarded as an element in the L_∞ space in [5], while it was regarded as an element in L_2 in the pioneering study [4]. Such treatment indeed led to interesting and fruitful studies [5], [7], but the present study aims at reverting to the same treatment as in [4] in the sense that the future output is also regarded as an element in L_2. Hence, the corresponding quasi Hankel operator at Θ is called the quasi L_2/L_2 Hankel operator and its norm is called the quasi L_2/L_2 Hankel norm (as opposed to the corresponding terms of the quasi L_∞/L_2 Hankel operator/norm for the study in [5], [7]). The reason why regarding the output as an element of L_∞ was considered more tractable is deeply related to the use of the quite powerful technique of lifting for sampled-data systems [8]–[10]. Indeed, the general instant Θ lies in the midst of the so-called lifting interval $[0, h)$, and this makes the description of the quasi Hankel operator at nonzero Θ somewhat inconsistent with the standard lifting technique. Yet, if we switch our focus to the L_∞/L_2 setting, the associated difficulty of directly dealing

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with the quasi L_∞/L_2 Hankel operator/norm at a nonzero Θ was successfully circumvented by working instead on what is called the overlap L_∞/L_2 Hankel operator [5], which can be described readily in an entirely compatible fashion with the standard lifting technique.

b) Standard lifting vs. shifted lifting relevant to the treatment of Θ: On the other hand, the L_2/L_2 setting with Θ studied in the present paper was actually tackled also in an earlier conference version of our study [6]. Regarding the aforementioned difficulty in the treatment of nonzero Θ arising in the standard lifting technique, two directions can be considered. The first approach would be introducing a variant of the standard lifting technique, where the lifting interval is shifted from $[0,h)$ to $[\Theta,\Theta+h)$ in accordance with Θ. In our earlier study [6], however, this direction was not adopted because the control input (via the zero-order hold) then changes its value over a single lifting interval in this treatment, and it was thought to lead to a different, rather annoying and less transparent lifted description of sampled-data systems, compared with the standard (unshifted) lifting technique. Instead, our earlier study [6] adopted an alternative second approach, which directly (i.e., without resorting to the use of lifting) deals with the continuous-time behavior of sampled-data systems as far as the fractioned intervals $[0,\Theta)$ and $[\Theta,h)$ are concerned. This second approach was indeed successful to an extent reported in the conference paper [6], but the standpoint of the present paper is to pursue the first approach to have a breakthrough to be described shortly.

c) Existence of a critical instant ensured in the L_2/L_2 setting as opposed to the L_∞/L_2 setting: We are first interested (as in our earlier study [6] in the L_2/L_2 setting) in whether the supremum of the quasi L_2/L_2 Hankel norms over $\Theta\in[0,h)$ is attained. The supremum is called the L_2/L_2 Hankel norm, and if it is attained as the quasi L_2/L_2 Hankel norm at $\Theta=\Theta^\star$, then $\Theta^\star$ is called a critical instant and the quasi L_2/L_2 Hankel operator at $\Theta^\star$ will be called *the* L_2/L_2 Hankel operator[1]. With respect to the pioneering study [4], we will be further interested in whether $\Theta^\star=0$ or not. As opposed to the L_∞/L_2 setting [5], [7] in which a critical instant does not necessarily exist, it will be shown (in Theorem 3) that a critical instant always exists in the L_2/L_2 setting. Nevertheless, our study leads to a negative answer to the question of $\Theta^\star$ being zero for every sampled-data system. That is, the quasi Hankel norm at Θ can exceed that at 0 for some $\Theta\in(0,h)$, as we will see in the numerical example in Section VII, and studying these and relevant issues definitely provides us with much deeper understanding about sampled-data systems than the study in [4] without Θ.

d) Equivalent plant discretization in the sense of the quasi L_2/L_2 Hankel norm at Θ: Our interest covers not only the above theoretical issues but also developing a computation method for the quasi L_2/L_2 Hankel norm for each Θ. Our earlier study [6] was successful in this respect, but was not entirely satisfactory in the sense that the developed method was related to no discretization process of the continuous-time plant that is "equivalent" in the sense of the quasi L_2/L_2 Hankel norm. The most important feature of the present paper utilizing the shifted lifting technique lies in the breakthrough in this respect, whereby we succeed in deriving for the first time an equivalent discretized plant in the sense of the quasi L_2/L_2 Hankel norm. It is such a discretized generalized plant that yields, together with the discrete-time controller, a closed-loop system whose discrete-time l_2/l_2 Hankel norm coincides with the quasi L_2/L_2 Hankel norm of the sampled-data system. This kind of equivalently discretized plant plays a crucial role, e.g., in the H_2 control problem of sampled-data systems because it enables us to reduce the controller synthesis problem for sampled-data systems to that for a discrete-time plant (and similarly for the H_∞ control problem). Without such an equivalent discretized plant, it is almost impossible to tackle the controller synthesis problem in sampled-data systems. The breakthrough in the present study over our earlier study [6] in deriving an equivalently discretized plant is thus expected to shed a new light on the study of the L_2/L_2 Hankel norm for sampled-data systems.

[1]We slightly abuse the term in the sense that $\Theta^\star$ may not be unique.

The organization of this paper is as follows. In Section II, we first introduce an adequate definition of the L_2/L_2 Hankel norm by first introducing the quasi L_2/L_2 Hankel norm at Θ, where we also introduce the notion of a critical instant. Section III introduces the framework of the shifted lifting treatment of sampled-data systems as a key technique for the arguments in this paper. Section IV applies the shifted lifting treatment to show that the quasi L_2/L_2 Hankel norm at each Θ can be represented as the l_2/l_2 Hankel norm of a Θ-dependent discrete-time system. Section V then shows that the aforementioned discrete-time system can be represented as the feedback connection of an equivalently discretized Θ-dependent generalized plant and the same discrete-time controller as the one in the sampled-data system. Section VI then shows that a critical instant always exists, meaning that the supremum of the quasi L_2/L_2 Hankel norms over Θ belonging to a sampling interval (which is actually the definition of the L_2/L_2 Hankel norm) is actually attained as the maximum and thus *the* L_2/L_2 Hankel operator is always definable. Finally, Section VII gives a numerical example to illustrate the arguments in this paper.

The notation in this paper is as follows. The symbol $\mathbb{R}^m$ denotes the set of real m-vectors, and $\mathbb{R}^{l\times m}$ denotes the set of real $l\times m$-matrices. The set of positive integers is denoted by $\mathbb{N}$, and $\mathbb{N}_0$ implies $\mathbb{N}\cup\{0\}$. The symbol $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix whose eigenvalues are all real. The adjoint operator of a bounded linear operator $\mathbf{T}$ on a Hilbert space is denoted by $\mathbf{T}^*$.

II. Sampled-Data Systems, Quasi L_2/L_2 Hankel Norms and L_2/L_2 Hankel Norm/Operator

This section first defines the quasi L_2/L_2 Hankel norms of sampled-data systems by taking account of the periodicity of their input-output relation. This further leads to the definition of the L_2/L_2 Hankel norm/operator of sampled-data system.

Let us consider the internally stable sampled-data system Σ_{SD} shown in Fig. 1 (see Remark 1 for the details of this stability notion), where P denotes the continuous-time linear time-invariant (LTI) generalized plant, and Ψ, $\mathcal{S}$ and $\mathcal{H}$ denote the discrete-time LTI controller, the ideal sampler and the zero-order hold, respectively, operating with sampling period h in a synchronous fashion. We assume that P and Ψ are described by

$$P: \begin{cases} \dfrac{dx}{dt} = Ax + B_1 w + B_2 u & \text{(1a)} \\ z = C_1 x + D_{12} u & \text{(1b)} \\ y = C_2 x & \text{(1c)} \end{cases}$$

$$\Psi: \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k & \text{(2a)} \\ u_k = C_\Psi \psi_k + D_\Psi y_k & \text{(2b)} \end{cases}$$

respectively, where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, $y(t) \in \mathbb{R}^{n_y}$, $\psi_k \in \mathbb{R}^{n_\Psi}$, $y_k = y(kh)$ and $u(t) = u_k$ $(kh \le t < (k+1)h)$, assuming that sampling instants are given by integer multiples of h, and the matrices A, B_1, B_2, C_1, C_2 and D_{12} in P and the matrices A_Ψ, B_Ψ, C_Ψ and D_Ψ in Ψ are given constant matrices with appropriate dimensions.

The sampled-data system $\Sigma_{\rm SD}$ is viewed as a continuous-time mapping between w and z, which is periodically time-varying with period h. Hence, to consider the past input, we introduce the general time instant $\Theta \in [0, h)$ at which past and future are separated, and take the past input from $w \in L_2(-\infty, \Theta)$; what is precisely meant by this is that w is actually defined on $(-\infty, \infty)$ but $w(t) = 0$ for $t \ge \Theta$ and belongs to the relevant L_2 space if it is restricted to $(-\infty, \Theta)$. We then consider the corresponding future output $z \in L_2[\Theta, \infty)$ for $x(-\infty) = 0$ and $\psi_{-\infty} = 0$, and the associated mapping from the past input to the future output called the quasi L_2/L_2 Hankel operator at Θ is denoted by $\mathbf{H}^{[\Theta]}$. We further call the norm of $\mathbf{H}^{[\Theta]}$ the quasi L_2/L_2 Hankel norm at Θ:

$$\|\mathbf{H}^{[\Theta]}\| := \sup_{w \in L_2(-\infty,\Theta)} \frac{\|z\|_{L_2[\Theta,\infty)}}{\|w\|_{L_2(-\infty,\Theta)}}. \tag{3}$$

Then, taking a standpoint that the term Hankel norm should represent how much the past input could affect the future output in dynamical systems in the worst case, we define the L_2/L_2 Hankel norm of the sampled-data system $\Sigma_{\rm SD}$ by

$$\|\Sigma_{\rm SD}\|_{\rm H} := \sup_{\Theta \in [0,h)} \|\mathbf{H}^{[\Theta]}\|. \tag{4}$$

Furthermore, if $\|\Sigma_{\rm SD}\|_{\rm H}$ is attained as the maximum at some $\Theta = \Theta^\star$, we call $\Theta^\star$ a critical instant and call the operator $\mathbf{H}^{[\Theta^\star]}$ *the* L_2/L_2 Hankel operator of $\Sigma_{\rm SD}$.

Given the fact that a critical instant does not necessarily exist in the L_∞/L_2 setting [5], [7], this paper is interested in whether a critical instant (and thus *the* L_2/L_2 Hankel operator) always exists, as well as how to compute the quasi L_2/L_2 Hankel norm for each $\Theta \in [0, h)$. As it turns out, the study for the latter issue is useful in tackling the former issue, for which we will eventually give a positive answer.

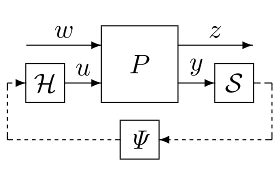

Fig. 1. Sampled-data system $\Sigma_{\rm SD}$

III. Shifted Lifting of Sampled-Data Systems

It is well known that the lifting technique [8]–[10] is quite useful in describing the input-output behavior of $\Sigma_{\rm SD}$ with respect to the continuous-time input w and the continuous-time output z in a simple fashion. For studying the L_2/L_2 Hankel norm/operator of $\Sigma_{\rm SD}$, however, we need to introduce $\Theta \in [0, h)$ at which past and future are to be separated, and it makes the use of the standard lifting inconvenient, as stated in detail in Introduction. A key idea in the present paper is thus to introduce a shifted variant of the standard lifting of signals in accordance with this time instant Θ, which converts the function $f(t)$ into the sequence $\{\widehat{f}_{\Theta,k}\}$ in the integer k, where

$$\widehat{f}_{\Theta,k}(\theta) := f(kh + \Theta + \theta)\ (0 \le \theta < h). \tag{5}$$

The associated lifting representation of the sampled-data system $\Sigma_{\rm SD}$ becomes more involved, compared with the case when the standard lifting is adopted (i.e., when $\Theta = 0$). This is why the use of the shifted version was not considered in the L_∞/L_2 setting [5], [7] and an alternative way around was studied instead, but its use will turn out to be significant in the L_2/L_2 setting, particularly in the context of deriving an equivalent discretized plant with respect to the quasi L_2/L_2 Hankel norm at each Θ.

Once we determine which variables to take as the state for the shifted lifting representation of $\Sigma_{\rm SD}$, the derivation of such a representation is (tedious but) essentially straightforward through the solution formula of the state equation (1a), as in the standard lifting case. By taking

$$\chi_{\Theta,k} := \begin{bmatrix} x_{\Theta,k}^T & u_k^T & \psi_{k+1}^T \end{bmatrix}^T \tag{6}$$

as the state (see Remark 2 at the end of this section for the rationale for taking this state), where $x_{\Theta,k} := x(kh + \Theta)$, we are led to the shifted lifting representation of $\Sigma_{\rm SD}$ in the form of

$$\begin{cases} \chi_{\Theta,k+1} = \mathcal{A}_\Theta \chi_{\Theta,k} + \mathcal{B}_\Theta \widehat{w}_{\Theta,k} & \text{(7a)} \\ \widehat{z}_{\Theta,k} = \mathcal{C}_\Theta \chi_{\Theta,k} + \mathcal{D}_\Theta \widehat{w}_{\Theta,k}. & \text{(7b)} \end{cases}$$

A brief sketch of the derivation of this form and the definitions of the matrix $\mathcal{A}_\Theta$ and the operators $\mathcal{B}_\Theta$, $\mathcal{C}_\Theta$ and $\mathcal{D}_\Theta$ are as described in the next paragraph, once we introduce the matrices and operators

$$A_{d,\theta} := e^{A\theta},\ B_{2d,\theta} := \int_0^\theta e^{A\sigma} B_2 d\sigma,\ C_{2d} := C_2 \tag{8}$$

$$C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} \in \mathbb{R}^{(n+n_u)\times(n+n_\Psi)} \tag{9}$$

$$J_\Sigma := \begin{bmatrix} I & 0 \end{bmatrix}^T \in \mathbb{R}^{(n+n_\Psi)\times n} \tag{10}$$

$$\mathbf{B}_{1,[\theta_1,\theta_2)} \widehat{w}_k := \int_{\theta_1}^{\theta_2} e^{A(\theta_2-\sigma)} B_1 \widehat{w}_k(\sigma) d\sigma \tag{11}$$

$$\left(\mathbf{M}_{1,[\theta_1,\theta_2)} \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) := M_1 e^{A_2(\theta-\theta_1)} \begin{bmatrix} x \\ u \end{bmatrix} \tag{12}$$

$$(\mathbf{D}_{11,[\theta_1,\theta_2)} \widehat{w}_k)(\theta) := \int_{\theta_1}^{\theta} C_1 e^{A(\theta-\sigma)} B_1 \widehat{w}_k(\sigma) d\sigma \tag{13}$$

where the subscript d means "discretized" and $\theta_1 \le \theta < \theta_2$ and

$$M_1 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix},\ A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+n_u)\times(n+n_u)} \tag{14}$$

together with the matrices

$$\mathcal{A}_{\Theta,-} := \begin{bmatrix} A_{d,h-\Theta} & B_{2d,h-\Theta} & 0 \\ 0 & 0 & I_{n_\Psi} \end{bmatrix} \tag{15}$$
$$\mathcal{A}_{\Theta,+} := \begin{bmatrix} A_{d,\Theta} + B_{2d,\Theta} D_\Psi C_{2d} & B_{2d,\Theta} C_\Psi \\ D_\Psi C_{2d} & C_\Psi \\ B_\Psi C_{2d} & A_\Psi \end{bmatrix} \tag{16}$$
$$J_{\Sigma,\Theta,-} := \mathcal{A}_{\Theta,+} J_\Sigma \tag{17}$$
$$J_{\Sigma,\Theta,+} := \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T \in \mathbb{R}^{(n+n_u+n_\Psi)\times n} \tag{18}$$
$$C_{\Sigma,\Theta,-} := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \in \mathbb{R}^{(n+n_u)\times(n+n_u+n_\Psi)} \tag{19}$$
$$C_{\Sigma,\Theta,+} := C_\Sigma \mathcal{A}_{\Theta,-}. \tag{20}$$

Towards the derivation of (7), let us first define $\xi_k := [x_k^T, \psi_k^T]^T$ where $x_k := x(kh)$. Then, it readily follows that

$$\chi_{\Theta,k+1} = \mathcal{A}_{\Theta,+}\xi_{k+1} + J_{\Sigma,\Theta,+}\mathbf{B}_{1,[h-\Theta,h)}\widehat{w}_{\Theta,k} \tag{21}$$
$$\xi_{k+1} = \mathcal{A}_{\Theta,-}\chi_{\Theta,k} + J_\Sigma \mathbf{B}_{1,[0,h-\Theta)}\widehat{w}_{\Theta,k}. \tag{22}$$

Substituting (22) into (21) and rearranging the results leads to the claim (7a), where

$$\mathcal{A}_\Theta = \mathcal{A}_{\Theta,+}\mathcal{A}_{\Theta,-} \tag{23}$$
$$\mathcal{B}_\Theta = J_{\Sigma,\Theta,-}\mathbf{B}_{1,[0,h-\Theta)} + J_{\Sigma,\Theta,+}\mathbf{B}_{1,[h-\Theta,h)} : L_2[0,h) \to \mathbb{R}^{n+n_u+n_\Psi}. \tag{24}$$

On the other hand, we readily see that

$$\widehat{z}_{\Theta,k}(\theta) = \begin{cases} \left(\mathbf{M}_{1,[0,h-\Theta)} \begin{bmatrix} x_{\Theta,k} \\ u_k \end{bmatrix}\right)(\theta) + (\mathbf{D}_{11,[0,h-\Theta)}\widehat{w}_{\Theta,k})(\theta) \\ \quad (0 \le \theta < h-\Theta) \\ \left(\mathbf{M}_{1,[h-\Theta,h)} \begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix}\right)(\theta) + \left(\mathbf{D}_{11,[h-\Theta,h)}\widehat{w}_{\Theta,k}\right)(\theta) \\ \quad (h-\Theta \le \theta < h) \end{cases}$$

where we also readily see that

$$\begin{bmatrix} x_{\Theta,k}^T & u_k^T \end{bmatrix}^T = C_{\Sigma,\Theta,-}\chi_{\Theta,k} \tag{25}$$
$$\begin{bmatrix} x_{k+1}^T & u_{k+1}^T \end{bmatrix}^T = C_\Sigma \xi_{k+1}. \tag{26}$$

These observations together with (22) lead to (7b), where

$$(\mathcal{C}_\Theta \chi_{\Theta,k})(\theta) = \begin{cases} \left(\mathbf{M}_{1,[0,h-\Theta)} C_{\Sigma,\Theta,-}\chi_{\Theta,k}\right)(\theta) \\ \quad (0 \le \theta < h-\Theta) \\ \left(\mathbf{M}_{1,[h-\Theta,h)} C_{\Sigma,\Theta,+}\chi_{\Theta,k}\right)(\theta) \\ \quad (h-\Theta \le \theta < h) \end{cases}$$
$$\mathcal{C}_\Theta : \mathbb{R}^{n+n_u+n_\Psi} \to L_2[0,h) \tag{27}$$

$$(\mathcal{D}_\Theta \widehat{w}_{k,\Theta})(\theta) = \begin{cases} \left(\mathbf{D}_{11,[0,h-\Theta)}\widehat{w}_{k,\Theta}\right)(\theta) \\ \quad (0 \le \theta < h-\Theta) \\ \left(\left(\mathbf{M}_{1,[h-\Theta,h)} C_\Sigma J_\Sigma \mathbf{B}_{1,[0,h-\Theta)} + \mathbf{D}_{11,[h-\Theta,h)}\right)\widehat{w}_{k,\Theta}\right)(\theta) \\ \quad (h-\Theta \le \theta < h) \end{cases}$$
$$\mathcal{D}_\Theta : L_2[0,h) \to L_2[0,h) \tag{28}$$

For the sake of later arguments, we note that

$$J_\Sigma A_{d,h-\Theta} = \mathcal{A}_{\Theta,-} J_{\Sigma,\Theta,+} \tag{29}$$
$$e^{A_2\Theta} C_\Sigma = C_{\Sigma,\Theta,-}\mathcal{A}_{\Theta,+}. \tag{30}$$

Remark 1: Although shifted lifting reduces to the standard one when $\Theta = 0$, the shifted lifting representation (7) does not reduce to the standard form of

$$\begin{cases} \xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k & (31a) \\ \widehat{z}_k = \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k & (31b) \end{cases}$$

with $\widehat{f}_k := \widehat{f}_{\Theta,k}|_{\Theta=0}$, because of the size difference between ξ_k and $\chi_{\Theta,k}$. However, it is easily seen that

$$\mathcal{A} = \mathcal{A}_{\Theta,-}\mathcal{A}_{\Theta,+} \tag{32}$$

for each $\Theta \in [0,h)$, which obviously shares nonzero eigenvalues with $\mathcal{A}_\Theta$ in (23), and thus $\mathcal{A}_\Theta$ is Schur stable (i.e., each eigenvalue of $\mathcal{A}_\Theta$ has modulus less than 1) by the internal stability assumption of Σ_{SD} (because this stability notion is defined by the Schur stability of $\mathcal{A}$).

Remark 2: One might consider taking $[x_{\Theta,k}^T, y_k^T, \psi_k^T]^T$ as the state for the shifted lifting representation of Σ_{SD} instead of (6), and it might look more natural. It, however, leads to difficulties (precisely the same ones as those encountered in our earlier study [6] without the use of shifted lifting) in deriving an equivalent discretized plant in the sense of the quasi L_2/L_2 Hankel norm at Θ in Section V (see Remark 4).

IV. Shifted Lifting Treatment of the Quasi L_2/L_2 Hankel Norm at Θ

This section is devoted to the shifted lifting treatment of the quasi L_2/L_2 Hankel norm at each $\Theta \in [0,h)$. We begin by introducing the notation for the shifted lifting representations of $w \in L_2(-\infty,\Theta)$ and $z \in L_2[0,\infty)$, i.e.,

$$\widehat{w}_\Theta := [\widehat{w}_{\Theta,-1}^T, \widehat{w}_{\Theta,-2}^T, \dots]^T \in l^2_{L_2[0,h),-} \tag{33}$$
$$\widehat{z}_\Theta := [\widehat{z}_{\Theta,0}^T, \widehat{z}_{\Theta,1}^T, \dots]^T \in l^2_{L_2[0,h),0+} \tag{34}$$

where $l^2_{L_2[0,h),-}$ denotes the set of $\widehat{f} := [\widehat{f}_{-1}^T, \widehat{f}_{-2}^T, \dots]^T$ endowed with the norm defined as $\|\widehat{f}\| := (\sum_{k=1}^\infty \|\widehat{f}_{-k}\|^2_{L_2[0,h)})^{1/2}$, and $l^2_{L_2[0,h),0+}$ denotes the set of $\widehat{f} := [\widehat{f}_0^T, \widehat{f}_1^T, \dots]^T$ endowed with $\|\widehat{f}\| := (\sum_{k=0}^\infty \|\widehat{f}_k\|^2_{L_2[0,h)})^{1/2}$. It is obvious from the relevant norm definitions that shifted lifting is norm-preserving, i.e., $\|w\|_{L_2(-\infty,\Theta)} = \|\widehat{w}_\Theta\|$ and $\|z\|_{L_2[\Theta,\infty)} = \|\widehat{z}_\Theta\|$. Furthermore, it readily follows from (7a) that

$$\chi_{\Theta,0} = \mathbf{V}_\Theta \widehat{w}_\Theta,\ \mathbf{V}_\Theta := [\mathcal{B}_\Theta, \mathcal{A}_\Theta\mathcal{B}_\Theta, \mathcal{A}_\Theta^2\mathcal{B}_\Theta, \dots] \tag{35}$$

and it also follows readily from (7b) that

$$\widehat{z}_\Theta = \mathbf{W}_\Theta \chi_{\Theta,0},\ \mathbf{W}_\Theta := [\mathcal{C}_\Theta^T, (\mathcal{C}_\Theta \mathcal{A}_\Theta)^T, (\mathcal{C}_\Theta \mathcal{A}_\Theta^2)^T, \dots]^T. \tag{36}$$

Hence, the quasi L_2/L_2 Hankel norm at Θ given by (3) admits the alternative representation

$$\|\mathbf{H}^{[\Theta]}\| = \|\mathbf{W}_\Theta \mathbf{V}_\Theta\| \tag{37}$$

where the right-hand side denotes the norm of the operator $\mathbf{W}_\Theta \mathbf{V}_\Theta : l^2_{L_2[0,h),-} \to l^2_{L_2[0,h),0+}$.

Now, for each $\theta \in [0,h)$, let us introduce the matrices F_θ and G_θ such that

$$F_\theta F_\theta^T = \int_0^\theta e^{A\sigma} B_1 B_1^T e^{A^T \sigma} d\sigma \tag{38}$$

$$G_\theta^T G_\theta = \int_0^\theta e^{A_2^T \sigma} M_1^T M_1 e^{A_2 \sigma} d\sigma \tag{39}$$

where it is obvious that F_θ can be made to have a common size regardless of θ and can be made continuous with respect to $\theta \in [0,h)$, and similarly for G_θ. With these matrices, we are led to the following key result.

Lemma 1: For each $\Theta \in [0,h)$, the operators $\mathbf{V}_\Theta$ and $\mathbf{W}_\Theta$ satisfy

$$\mathbf{V}_\Theta \mathbf{V}_\Theta^* = \sum_{k=0}^{\infty} \mathscr{A}_\Theta^k \mathscr{B}_\Theta \mathscr{B}_\Theta^T (\mathscr{A}_\Theta^T)^k \tag{40}$$

$$\mathbf{W}_\Theta^* \mathbf{W}_\Theta = \sum_{k=0}^{\infty} (\mathscr{A}_\Theta^T)^k \mathscr{C}_\Theta^T \mathscr{C}_\Theta \mathscr{A}_\Theta^k \tag{41}$$

where

$$\mathscr{A}_\Theta = \mathcal{A}_\Theta,\ \mathscr{B}_\Theta = \begin{bmatrix} J_{\Sigma,\Theta,-} F_{h-\Theta} & J_{\Sigma,\Theta,+} F_\Theta \end{bmatrix}, \tag{42}$$

$$\mathscr{C}_\Theta = \begin{bmatrix} G_{h-\Theta} C_{\Sigma,\Theta,-} \\ G_\Theta C_{\Sigma,\Theta,+} \end{bmatrix}. \tag{43}$$

Proof: It suffices to show that

$$\mathcal{B}_\Theta \mathcal{B}_\Theta^* = \mathscr{B}_\Theta^T \mathscr{B}_\Theta,\ \mathcal{C}_\Theta^* \mathcal{C}_\Theta = \mathscr{C}_\Theta^T \mathscr{C}_\Theta. \tag{44}$$

To confirm the first relation, we first give an explicit representation of $\mathcal{B}_\Theta^*$. To this end, let us take arbitrary $f \in L_2[0,h)$ and $v \in \mathbb{R}^{n+n_u+n_\psi}$. By the definition of $\mathcal{B}_\Theta$ in (24) together with (11), we have

$$\begin{aligned} \langle \mathcal{B}_\Theta f, v \rangle =& v^T \Bigg(J_{\Sigma,\Theta,-} \int_0^{h-\Theta} e^{A(h-\Theta-\theta)} B_1 f(\theta) d\theta \\ & + J_{\Sigma,\Theta,+} \int_{h-\Theta}^{h} e^{A(h-\theta)} B_1 f(\theta) d\theta \Bigg) \end{aligned} \tag{45}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. Since it must coincide with the inner product $\langle f, \mathcal{B}_\Theta^* v \rangle$ on $L_2[0,h)$, we readily see that

$$(\mathcal{B}_\Theta^* v)(\theta) = \begin{cases} B_1^T e^{A^T(h-\Theta-\theta)} J_{\Sigma,\Theta,-}^T v \ (0 \le \theta < h-\Theta) \\ B_1^T e^{A^T(h-\theta)} J_{\Sigma,\Theta,+}^T v \ (h-\Theta \le \theta < h). \end{cases} \tag{46}$$

This together with the definitions of $\mathcal{B}_\Theta$ and F_θ immediately leads to

$$\begin{aligned} \mathcal{B}_\Theta \mathcal{B}_\Theta^* v = \big(& J_{\Sigma,\Theta,-} F_{\Theta-h} F_{\Theta-h}^T J_{\Sigma,\Theta,-} \\ & + J_{\Sigma,\Theta,+}^T F_\Theta F_\Theta^T J_{\Sigma,\Theta,+} \big) v. \end{aligned} \tag{47}$$

Since v is arbitrary, we readily see that this implies the first relation if we note the definition of $\mathscr{B}_\Theta$ in (42). The second relation can be confirmed in a similar fashion by (27) together with (12), and the details are omitted. ∎

It is obvious that the infinite sums in the above lemma can be computed by solving Lyapunov equations as the controllability and observability Gramians of the pairs $(\mathscr{A}_\Theta, \mathscr{B}_\Theta)$ and $(\mathscr{C}_\Theta, \mathscr{A}_\Theta)$, respectively, because of the stability assumption of $\mathscr{A}_\Theta = \mathcal{A}_\Theta$ (recall Remark 1). With this in mind, we are now in a position to characterize the quasi L_2/L_2 Hankel norm $\|\mathbf{H}^{[\Theta]}\|$ as the l_2/l_2 Hankel norm of a Θ-dependent discrete-time system. To this end, let the matrices V_Θ and W_Θ be such that $V_\Theta V_\Theta^T$ and $W_\Theta^T W_\Theta$ equal the aforementioned controllability and observability Gramians, respectively, i.e.,

$$\mathbf{V}_\Theta \mathbf{V}_\Theta^* = V_\Theta V_\Theta^T,\ \mathbf{W}_\Theta^* \mathbf{W}_\Theta = W_\Theta^T W_\Theta. \tag{48}$$

Then, we have the following theorem (in which no positive-definiteness assumption is necessary for the above two Gramians).

Theorem 1: For each $\Theta \in [0,h)$, the quasi L_2/L_2 Hankel norm $\|\mathbf{H}^{[\Theta]}\|$ equals the l_2/l_2 Hankel norm of the discrete-time system[2]

$$\mathcal{G}_{d,\Theta} := \left[\begin{array}{c|c} \mathscr{A}_\Theta & \mathscr{B}_\Theta \\ \hline \mathscr{C}_\Theta & * \end{array} \right] \tag{49}$$

and thus $\|\mathbf{H}^{[\Theta]}\| = \|W_\Theta V_\Theta\| = \lambda_{\max}^{1/2}(V_\Theta^T W_\Theta^T W_\Theta V_\Theta)$.

Proof: Note from (37) that

$$\begin{aligned} \|\mathbf{H}^{[\Theta]}\|^2 &= \|\mathbf{W}_\Theta \mathbf{V}_\Theta (\mathbf{W}_\Theta \mathbf{V}_\Theta)^*\| = \|\mathbf{W}_\Theta \mathbf{V}_\Theta \mathbf{V}_\Theta^* \mathbf{W}_\Theta^*\| \\ &= \|\mathbf{W}_\Theta V_\Theta V_\Theta^T \mathbf{W}_\Theta^*\| = \|\mathbf{W}_\Theta V_\Theta (\mathbf{W}_\Theta V_\Theta)^*\| \\ &= \|(\mathbf{W}_\Theta V_\Theta)^* \mathbf{W}_\Theta V_\Theta\| = \|V_\Theta^T \mathbf{W}_\Theta^* \mathbf{W}_\Theta V_\Theta\| \\ &= \|V_\Theta^T W_\Theta^T W_\Theta V_\Theta\| = \lambda_{\max}(V_\Theta^T W_\Theta^T W_\Theta V_\Theta) \end{aligned} \tag{50}$$

which further equals $\lambda_{\max}(V_\Theta V_\Theta^T W_\Theta^T W_\Theta)$ as well as $\|W_\Theta V_\Theta\|^2$. As stated earlier, on the other hand, $V_\Theta V_\Theta^T$ and $W_\Theta^T W_\Theta$ are the controllability and observability Gramians of $(\mathscr{A}_\Theta, \mathscr{B}_\Theta)$ and $(\mathscr{C}_\Theta, \mathscr{A}_\Theta)$, respectively, so that $\lambda_{\max}(V_\Theta V_\Theta^T W_\Theta^T W_\Theta)$ is the squared l_2/l_2 Hankel norm of $\mathcal{G}_{d,\Theta}$. This completes the proof. ∎

Remark 3: It can readily be seen from the relevant arguments in the above proof (together with the definition of the singular values for operators [11]) that we may call the square roots of the eigenvalues of the matrix $V_\Theta^T W_\Theta^T W_\Theta V_\Theta$ (and thus the eigenvalues of $W_\Theta V_\Theta$) the (quasi L_2/L_2 Hankel) singular values of $\mathbf{H}^{[\Theta]}$.

[2]Just in case, it is defined as the worst l_2 norm of the future output over the nonnegative time instants to the worst past l_2 input of unit norm over the negative time instants. Hence, taking any matrix with a compatible size as $*$ in (49) does not affect the l_2/l_2 Hankel norm.

V. Equivalent Discretized Plant in the Sense of the Quasi L_2/L_2 Hankel Norm at Θ

This section is devoted to showing that $\mathcal{G}_{d,\Theta}$ in Theorem 1 is nothing but the closed-loop system consisting of a discrete-time generalized plant $P_{d,\Theta}$ and the same controller Ψ as that in the sampled-data system $\Sigma_{\rm SD}$. More precisely, we give a state-space representation of $P_{d,\Theta}$ such that the associated closed-loop system with Ψ leads to the state-space representation in the right-hand side of (49). Note that the specific $P_{d,\Theta}$ given in this section (see Theorem 2 below) obviously leads to the associated direct feedthrough matrix "$\mathscr{D}_\Theta$" of the closed-loop system, but this matrix does not actually affect the l_2/l_2 Hankel norm of the closed-loop system. To put it conversely, the direct feedthrough matrix of $\mathcal{G}_{d,\Theta}$ is arbitrary as long as the statement of Theorem 1 is concerned, and this is why it is simply denoted by "*" in (49). In connection with this freedom, the arguments in this section can be regarded as taking a specific matrix for this arbitrary matrix so that the resulting $\mathcal{G}_{d,\Theta}$ can be described through $P_{d,\Theta}$ with a simple form of state-space representation.

The discrete-time plant $P_{d,\Theta}$ derived for this purpose can be regarded as an equivalent discretized plant in the sense of the quasi L_2/L_2 Hankel norm at Θ. This viewpoint would provide us with much clear understanding and perspective on the quasi L_2/L_2 Hankel norms of the sampled-data system $\Sigma_{\rm SD}$ for $\Theta \in [0, h)$, as well as the L_2/L_2 Hankel norm of $\Sigma_{\rm SD}$. This is because separating the controller Ψ from the matrices $\mathscr{A}_\Theta$, $\mathscr{B}_\Theta$ and $\mathscr{C}_\Theta$ and deriving the equivalent discretized plant $P_{d,\Theta}$ reveals some sort of structural aspect about how the controller Ψ would affect these norms, as opposed to the mere expression in terms of $\mathcal{G}_{d,\Theta}$ in (49). In other words, the discretized plant $P_{d,\Theta}$ could play a significant role (as the relevant discretized plant does notably in the H_2 and H_∞ problems of sampled-data systems) in future studies on the controller synthesis aiming at reducing the L_2/L_2 Hankel norm.

Theorem 2: Let $P_{d,\Theta}$ be defined as

$$P_{d,\Theta}: \begin{cases} \zeta_{k+1} = \overline{A}\zeta_k + \overline{B}_1 w_k + \overline{B}_2 u_k & \text{(51a)} \\ z_k = \overline{C}_1\zeta_k + \overline{D}_{11} w_k + \overline{D}_{12} u_k & \text{(51b)} \\ y_k = \overline{C}_2\zeta_k + \overline{D}_{21} w_k & \text{(51c)} \end{cases}$$

with

$$\overline{A} = \begin{bmatrix} A_{d,h} & A_{d,\Theta}B_{2d,h-\Theta} \\ 0 & 0 \end{bmatrix},\ \overline{B}_1 = \begin{bmatrix} A_{d,\Theta}F_{h-\Theta} & F_\Theta \\ 0 & 0 \end{bmatrix}, \tag{52}$$

$$\overline{B}_2 = \begin{bmatrix} B_{2d,\Theta} \\ I \end{bmatrix},\ \overline{C}_1 = \begin{bmatrix} G_{h-\Theta} \\ G_{\Theta,1}\begin{bmatrix} A_{d,h-\Theta} & B_{2d,h-\Theta} \end{bmatrix} \end{bmatrix}, \tag{53}$$

$$\overline{D}_{11} = 0,\ \overline{D}_{12} = \begin{bmatrix} 0 \\ G_{\Theta,2} \end{bmatrix}, \tag{54}$$

$$\overline{C}_2 = C_{2d}\begin{bmatrix} A_{d,h-\Theta} & B_{2d,h-\Theta} \end{bmatrix},\ \overline{D}_{21} = \begin{bmatrix} C_{2d}F_{h-\Theta} & 0 \end{bmatrix} \tag{55}$$

where $G_{\Theta,1}$ and $G_{\Theta,2}$ are the submatrices of G_Θ consisting of the first n columns and the remaining n_u columns, respectively. Then, $\mathcal{G}_{d,\Theta}$ in (49) (with its direct feedthrough matrix determined appropriately) coincides with the lower fractional transformation $\mathcal{F}(P_{d,\Theta}, \Psi)$ obtained by connection the discrete-time controller Ψ to $P_{d,\Theta}$ as $u = \Psi y$.

Proof: By direct computation, we have

$$\mathcal{F}(P_{d,\Theta}, \Psi) = \left[\begin{array}{c|c} \overline{\mathscr{A}}_\Theta & \overline{\mathscr{B}}_\Theta \\ \hline \overline{\mathscr{C}}_\Theta & * \end{array}\right] \tag{56}$$

where

$$\overline{\mathscr{A}}_\Theta = \begin{bmatrix} \overline{A} + \overline{B}_2 D_\Psi \overline{C}_2 & \overline{B}_2 C_\Psi \\ B_\Psi \overline{C}_2 & A_\Psi \end{bmatrix} \tag{57}$$

$$\overline{\mathscr{B}}_\Theta = \begin{bmatrix} \overline{B}_1 + \overline{B}_2 D_\Psi \overline{D}_{21} \\ B_\Psi \overline{D}_{21} \end{bmatrix} \tag{58}$$

$$\overline{\mathscr{C}}_\Theta = \begin{bmatrix} \overline{C}_1 + \overline{D}_{12} D_\Psi \overline{C}_2 & \overline{D}_{12} C_\Psi \end{bmatrix}. \tag{59}$$

Comparing these matrices with the expanded forms of $\mathscr{A}_\Theta = \mathcal{A}_\Theta$ computed through (15), (16) and (23), $\mathscr{B}_\Theta$ computed through (17), (18) and (42), and $\mathscr{C}_\Theta$ computed through (19), (20) and (43), we can confirm that $\overline{\mathscr{A}}_\Theta = \mathscr{A}_\Theta$, $\overline{\mathscr{B}}_\Theta = \mathscr{B}_\Theta$ and $\overline{\mathscr{C}}_\Theta = \mathscr{C}_\Theta$. This completes the proof. ■

Remark 4: As stated in Remark 2, the description of the sampled-data system $\Sigma_{\rm SD}$ under the shifted lifting treatment is possible also in an alternative fashion by taking a different state vector. As far as the statement of Theorem 1 is concerned, it remains as it is, provided that the definitions of the matrices $\mathscr{A}_\Theta$, $\mathscr{B}_\Theta$ and $\mathscr{C}_\Theta$ are modified accordingly. Under this different state vector, however, for example the right-upper block of $\mathscr{A}_\Theta$ becomes inconsistent with the specific form of the same block in (57), although we skip the details. This implies that extracting the matrix $\overline{B}_2$ through the comparison of $\mathscr{A}_\Theta$ and $\overline{\mathscr{A}}_\Theta$ is impossible. In other words, showing the existence of an equivalent discretized plant in the sense of the quasi L_2/L_2 Hankel norm at Θ becomes nontrivial under the alternative treatment with a different state vector suggested in Remark 2. In this sense, taking (6) as the state vector of $\Sigma_{\rm SD}$ in the shifted lifting treatment is actually much more desirable.

VI. Existence of a Critical Instant and *the* L_2/L_2 Hankel Operator

We have derived a computation method for the quasi L_2/L_2 Hankel norm $\|\mathbf{H}^{[\Theta]}\|$ at an arbitrary $\Theta \in [0, h)$ by the preceding arguments (which also involve the derivation of an equivalent discretized plant). As defined in Section II, if the supremum of $\|\mathbf{H}^{[\Theta]}\|$ over $\Theta \in [0, h)$ is attained as the maximum at $\Theta = \Theta^\star$, then $\Theta^\star$ is called a critical instant, in which case (and only in which case) we can define *the* L_2/L_2 Hankel operator for the sampled-data system $\Sigma_{\rm SD}$ as $\mathbf{H}^{[\Theta^\star]}$. Even though $\|\mathbf{H}^{[\Theta]}\|$ is continuous with respect to Θ, however, the interval $[0, h)$ for Θ is not compact, and thus it is not necessarily clear whether a critical instant always exists and *the* L_2/L_2 Hankel operator is always definable. In the relevant study under the setting of the L_∞/L_2 Hankel norm/operator [5], [7], it was shown that a critical instant defined in a similar fashion does not necessarily exist, which means that *the* L_∞/L_2 Hankel operator is not necessarily definable, either. It is thus interesting to study whether a critical instant always exists in the L_2/L_2 setting, or equivalently, whether *the* L_2/L_2 Hankel operator is always definable for sampled-data systems. We have the following theorem giving a positive answer to this question.

Theorem 3: The quasi L_2/L_2 Hankel norms satisfy

$$\lim_{\Theta \to h-0} \|\mathbf{H}^{[\Theta]}\| = \|\mathbf{H}^{[0]}\| \tag{60}$$

and the (stable) sampled-data system $\Sigma_{\rm SD}$ always has a critical instant. Hence *the* L_2/L_2 Hankel operator is always definable.

Proof: Since it is obvious that $\|\mathbf{H}^{[\Theta]}\|$ is continuous with respect to $\Theta \in [0, h)$, it suffices to show the relation (60), because it then immediately implies that $\sup_{\Theta \in [0,h)} \|\mathbf{H}^{[\Theta]}\|$ is attained as the maximum over $\Theta \in [0, h)$. To show this relation, we note Theorem 1 implying that $\|\mathbf{H}^{[\Theta]}\|$ is determined by the Markov parameters $\mathscr{C}_\Theta \mathscr{A}_\Theta^k \mathscr{B}_\Theta$ $(k \in \mathbb{N}_0)$ (because the infinite-dimensional Hankel matrix representation associated with the discrete-time system $\mathcal{G}_{d,\Theta}$ is described by the Markov parameters). Hence, we compare $\mathscr{C}_h \mathscr{A}_h^k \mathscr{B}_h$ with $\mathscr{C}_0 \mathscr{A}_0^k \mathscr{B}_0$, where $\mathscr{A}_h := \lim_{\Theta \to h-0} \mathscr{A}_\Theta$, and $\mathscr{B}_h$ and $\mathscr{C}_h$ are defined in a similar fashion. If we note that $F_\theta = 0$ and $G_\theta = 0$ for $\theta = 0$ as well as (32), we readily see from (17) and (30) that

$$\mathscr{C}_0 \mathscr{A}_0^k \mathscr{B}_0 = \begin{bmatrix} G_h C_\Sigma \mathcal{A}^k J_\Sigma F_h & 0 \\ 0 & 0 \end{bmatrix} \tag{61}$$

and also from (20) and (29) that

$$\mathscr{C}_h \mathscr{A}_h^k \mathscr{B}_h = \begin{bmatrix} 0 & 0 \\ 0 & G_h C_\Sigma \mathcal{A}^k J_\Sigma F_h \end{bmatrix}. \tag{62}$$

These matrices are the permuted versions of each other, but the corresponding permutations of the entries of the vectors w_k and z_k do not change the norms. Hence, it is obvious that $\lim_{\Theta \to h-0} \|\mathbf{H}^{[\Theta]}\| = \|\mathbf{H}^{[0]}\|$ and thus the proof is completed. ■

Remark 5: It immediately follows from the above proof that $\|\mathbf{H}^{[0]}\|$ is given by the l_2/l_2 Hankel norm of the discrete-time system

$$\mathcal{G}_d^{[0]} := \left[\begin{array}{c|c} \mathcal{A} & J_\Sigma F_h \\ \hline G_h C_\Sigma & * \end{array}\right] \tag{63}$$

and this consequence can readily be confirmed to be consistent with the main result of the pioneering study [4] confined only to the treatment of $\Theta = 0$.

VII. Numerical Example

Consider the internally stable sampled-data system with

$$\begin{aligned} &A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ &C_1 = \begin{bmatrix} 1 & 4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{12} = 0, \\ &A_\Psi = \begin{bmatrix} -0.1547 & 0.3951 \\ 0.0658 & -0.1681 \end{bmatrix}, \quad B_\Psi = \begin{bmatrix} 5.3919 \\ -1.5425 \end{bmatrix}, \\ &C_\Psi = \begin{bmatrix} 1.1027 & -2.8164 \end{bmatrix}, \quad D_\Psi = -7.3716 \end{aligned} \tag{64}$$

and $h = 0.1$, where Ψ is an optimal controller minimizing the H_∞ norm of the mapping from w to z in the closed-loop sampled-data system [12]. We computed through Theorem 1 the quasi L_2/L_2 Hankel norms $\|\mathbf{H}^{[\Theta]}\|$ for $\Theta = ih'$ $(i = 0, \ldots, M-1)$ with $M = 10{,}000$ and $h' := h/M$, as shown in Fig. 2; we have

$$\begin{aligned} &\|\mathbf{H}^{[0]}\| = 0.2928, \quad \lim_{\Theta \to h-0} \|\mathbf{H}^{[\Theta]}\| = 0.2928, \\ &\|\Sigma_{\rm SD}\|_{\rm H} = 0.3434, \quad \tau_h^+ := \Theta^\star/h = 0.6799 \end{aligned} \tag{65}$$

where the value at $\Theta = h - h'$ is regarded as the left limit at h. We can thus readily confirm that (60) is fulfilled and that

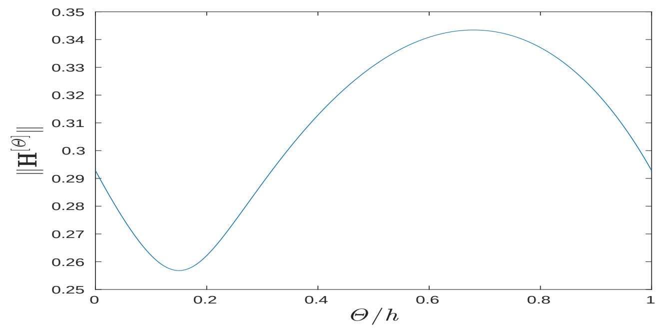

Fig. 2. Quasi L_2/L_2 Hankel norms for the numerical example

$\Theta^\star \neq 0$ and thus $\|\Sigma_{\rm SD}\|_{\rm H} > \|\mathbf{H}^{[0]}\|$ by more than 17%, since the ratio $\|\Sigma_{\rm SD}\|_{\rm H}/\|\mathbf{H}^{[0]}\|$ is 1.1730 $(=: \delta_h^+)$. Hence, when we consider defining the L_2/L_2 Hankel norm/operator, it is inappropriate to adopt an unreserved standpoint of the earlier study [4] in taking the instant separating past and future at a sampling instant. Instead, it is necessary to consider separating past and future at an arbitrary instant $\Theta \in [0, h)$ as in the present paper, by adequately taking account of the periodic nature of sampled-data systems.

For reference, the minimum of the quasi L_2/L_2 Hankel norms is attained when $\tau_h^- := \Theta/h = 0.1502$ and given by 0.2568 $(=: \delta_h^- \|\mathbf{H}^{[0]}\|$ with $\delta_h^- = 0.8771)$, by which we see that the size of the deviations of the quasi L_2/L_2 Hankel norms from the value $\|\mathbf{H}^{[0]}\|$ at $\Theta = 0$ (corresponding to $\delta_h^+ - \delta_h^-$) is about 29.6%. Interestingly, the feature summarized above for $h = 0.1$ does not change much even if we consider smaller sampling periods such as $h = 0.01$ and 0.001, for which $(\delta_h^+, \delta_h^-, \tau_h^+, \tau_h^-) = (1.1772, 0.8669, 0.6823, 0.1544)$ and $(1.1778, 0.8713, 0.6813, 0.1533)$, respectively. This situation could be interpreted as suggesting that the deviations of the quasi L_2/L_2 Hankel norms in Θ should be attributed to some intrinsic dynamics of the continuous-time generalized plant combined with the discrete-time controller. This implies that introducing the viewpoint of the quasi L_2/L_2 Hankel operator/norm at Θ is verified to be quite essential.

VIII. Conclusion

Stimulated by a pioneering study on the L_2/L_2 Hankel norm of sampled-data systems in [4], this paper introduced a modified and adequate definition of the L_2/L_2 Hankel norm by setting a general time instant Θ at which past and future are to be separated. This led to first introducing the notion of the quasi L_2/L_2 Hankel norm at Θ, whose supremum over Θ belonging to a sampling interval defines the L_2/L_2 Hankel norm, and the L_2/L_2 Hankel operator is definable if and only if a critical instant $\Theta^\star$ exists, i.e., the supremum is attained as the maximum at some $\Theta = \Theta^\star$. We provided a method for computing the quasi L_2/L_2 Hankel norm for each Θ by introducing the nonstandard and shifted lifting treatment of sampled-data systems in accordance with the introduction of Θ. In particular, we showed that the continuous-time generalized plant can be discretized in an equivalent fashion in the sense of the quasi L_2/L_2 Hankel norm at Θ, by which this norm can be alternatively represented as the l_2/l_2 Hankel norm of the associated discrete-time closed-loop system. We then showed that the L_2/L_2 Hankel operator is always definable for stable sampled-data systems, as opposed to the case with the L_∞/L_2 setting studied in [5], [7]. Finally, we verified those theoretical developments through a numerical example.

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