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A q-analogue of the matrix fifth Painlevé system

By

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Abstract

We consider a degeneration of the q-matrix sixth Painlevé system. As a result, we obtain a system of non-linear q-difference equations, which describes a deformation of a certain "non-Fuchsian" linear q-difference system. We define the spectral type for non-Fuchsian q-difference systems and characterize the associated linear problem in terms of the spectral type. We also consider a continuous limit of the non-linear q-difference system and show that the resulting system of non-linear differential equations coincides with the matrix fifth Painlevé system.

§ 1. Introduction

The Painlevé equations are non-linear second order ordinary differential equations that define novel transcendental functions. Historically, the Painlevé equations were classified into six equations. We refer to them as $P_{\rm I}, P_{\rm II}, \ldots, P_{\rm VI}$. The sixth Painlevé equation $P_{\rm VI}$ serves as the "source" from which all the other Painlevé equations can be derived through degeneration processes.

Since the 1990s, various generalizations of the Painlevé equations have been proposed in the literature, such as discretizations, higher-dimensional analogues, quantizations, and so on.

Recently, Painlevé-type differential equations with four-dimensional phase space have been classified from the perspective of isomonodromic deformations of linear differential equations [4, 9, 10, 11]. This series of studies shows that, in the four-dimensional case, there exist four "sources" as extensions of the sixth Painlevé equation. Namely, they are

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- the Garnier system [3], which is a classically known multivariate extension of P_{VI},
- the Fuji-Suzuki-Tsuda system [2, 22], which is an extension of P_{VI} with the affine Weyl group symmetry of type A,
- the Sasano system [19], which is an extension of P_{VI} with the affine Weyl group symmetry of type D,
- the matrix sixth Painlevé system [1, 4], which is a non-abelian extension of P_{VI}.

Note that each of the four equations has its extensions defined in arbitrary even dimensions. These four families are expected to have an impact on fields such as integrable systems, special functions, and so on.

On the other hand, Sakai [16] established an algebro-geometric theory which provides a comprehensive understanding of two-dimensional (or second order) Painlevé equations. According to Sakai's theory, when the phase spaces are two-dimensional, the discrete Painlevé equations are more fundamental. Roughly speaking, by classifying a certain kind of rational surfaces, 22 different surfaces are obtained. From the discrete symmetry of each surface, a discrete dynamical system (a system of difference equations) is generated. The theory classifies these discrete Painlevé equations into three types: additive difference, multiplicative difference (q-difference), and elliptic difference equations. The Painlevé (differential) equations are understood through the continuous limit of these discrete Painlevé equations. In this sense, we can say that the discrete Painlevé equations are more fundamental than the Painlevé differential equations.

Our aim is, inspired by the two-dimensional case, to construct a unified framework for discrete Painlevé-type equations in higher dimensions. However, it is difficult to classify algebraic varieties when the phase spaces have four or more dimensions. Instead, from the standpoint of the classification theory (by Katz [7] and Oshima [15]) of linear differential equations and the isomonodromic/connection-preserving deformation theory, we would like to develop a framework for higher-dimensional Painlevé-type equations that involves discrete Painlevé-type equations.

In [13], we have defined an equivalence relation between spectral types of linear differential equations (that is, those can be transformed into each other by Möbius transformations, the Harnad dual, or certain scalar gauge transformations are equivalent) and shown that there is a tree structure among the equivalence classes including differential equations without continuous deformations. The tree structure of equivalence classes corresponds to the additive surfaces in Sakai's list. As a next step, we investigate multiplicative difference Painlevé-type equations in higher dimensions.

Among the four families mentioned above, q-analogues of the Garnier systems, the Fuji-Suzuki-Tsuda systems, and the Sasano systems have been obtained and studied by several authors [17, 20, 21, 14]. Recently, a q-analogue of the matrix sixth Painlevé system, which we call the q-matrix P_{VI}, has been obtained [12].

In this paper we investigate a degeneration of the q-matrix P_{VI} with the aim of constructing a degeneration scheme for higher dimensional q-difference Painlevé-type equations. As a result, a system of non-linear q-difference equations is obtained, which corresponds to the matrix fifth Painlevé system in the continuous limit. We tentatively denote the non-linear system by the q-matrix P_{V}. The q-matrix P_{V} is expressed as a deformation equation of a non-Fuchsian linear q-difference equation.

This paper is organized as follows. In Section 2 we describe how to construct formal solutions to linear q-difference systems. We also give the definition of spectral types for non-Fuchsian linear q-difference systems. In Section 3 we review the q-matrix P_{VI}. In Section 4 we consider a degeneration of the q-matrix P_{VI}. In Section 5 we consider a continuous limit of the q-matrix P_{V} obtained in Section 4. The appendix is devoted to a brief description of the matrix fifth Painlevé system.

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§ 2. Linear q-difference systems

In this section, we collect some facts about linear q-difference systems. The formal normal form is used to define the spectral type for non-Fuchsian systems.

§ 2.1. Formal normal form: Fuchsian case

Let q be a complex number satisfying $0 < |q| < 1$. Consider a linear q-difference system with polynomial coefficients

$$Y(qx) = A(x)Y(x), \quad A(x) = A_0 + A_1 x + \cdots + A_N x^N \tag{2.1}$$

where $A_j \in M_m(\mathbb{C})$ and $A_0, A_N \neq O$. If A_0 and A_N are both invertible, then the system (2.1) is said to be *Fuchsian*. For simplicity we assume that A_0 and A_N are diagonalizable. In this subsection we outline the procedure to transform the given Fuchsian system into its formal normal forms at $x = 0$ and $x = \infty$.

We use the following well-known fact from linear algebra. We denote the set of all eigenvalues of a matrix A by $\mathrm{Sp}(A)$.

Proposition 2.1. *Let $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$. Then the linear map*

$$\varphi : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C}), \quad \varphi(X) = AX - XB \tag{2.2}$$

is an isomorphism of vector spaces if and only if $\mathrm{Sp}(A) \cap \mathrm{Sp}(B) = \emptyset$.

The linear q-difference systems that will be treated mainly in this paper are those with polynomial coefficients, while the process of constructing the formal normal form involves systems with infinite series coefficients. Therefore, in Section 2.1 and 2.2, we will discuss the linear q-difference systems of infinite series coefficients.

First we explain the formal normal form at $x = 0$. For the convenience of later discussion, we consider a system of the following form:

$$Y(qx) = A(x)Y(x), \quad A(x) = x^r(A_0 + A_1x + A_2x^2 + \cdots) \tag{2.3}$$

where $A(x)$ is an $m \times m$ formal power series. Here A_0 is invertible and diagonalizable, and r is a non-negative integer. Let the eigenvalues of A_0 be $\theta_1, \ldots, \theta_m$. We also assume that the system is *non-resonant*, that is, for any i, j

$$\theta_j/\theta_i \notin q^{\mathbb{Z}_{\geq 1}} = \{q^n \mid n \in \mathbb{Z}_{\geq 1}\}. \tag{2.4}$$

Let $P(x) = \sum_{n=0}^{\infty} P_n x^n$ be an $m \times m$ formal power series with $P_0 = I_m$. The substitution $Y(x) = P(x)Z(x)$ yields

$$Z(qx) = P(qx)^{-1}A(x)P(x)Z(x). \tag{2.5}$$

We can choose the matrix $P(x)$ so that

$$P(qx)^{-1}A(x)P(x) = x^r A_0. \tag{2.6}$$

The matrix $P(x)$ can be constructed as follows. The equation (2.6) can be written as

$$(A_0 + A_1x + A_2x^2 + \cdots)(I_m + P_1x + P_2x^2 + \cdots) = (I_m + qP_1x + q^2P_2x^2 + \cdots)A_0. \tag{2.7}$$

Equating the coefficients of x^n $(n \geq 1)$ on both sides, we obtain

$$A_0P_n - P_n(q^nA_0) = -\sum_{k=0}^{n-1} A_{n-k}P_k. \tag{2.8}$$

If the coefficient matrices $P_1, \ldots, P_{n-1}$ are determined, then the equation (2.8) uniquely determines P_n by Proposition 2.1 and non-resonant condition. In this way, the matrix $P(x)$ is constructed inductively. Then the matrix x^rA_0 is the formal normal form of (2.3) in this case.

The construction at $x = \infty$ is almost the same. Assume that

$$A(x) = x^r(A_0 + A_1x^{-1} + A_2x^{-2} + \cdots) \tag{2.9}$$

where A_0 is invertible, $r \in \mathbb{Z}_{\geq 0}$. We can construct the transformation matrix $P(x) = \sum_{n=0}^{\infty} P_n x^{-n}$ at $x = \infty$ such that $P(qx)^{-1}A(x)P(x) = x^rA_0$ holds in a similar way.

§ 2.2. Formal normal form: non-Fuchsian case

In the case that A_0 or A_N of (2.1) is not invertible, the construction of the formal normal form can be modified as follows. Consider at $x = 0$

$$Y(qx) = A(x)Y(x), \quad A(x) = x^r(A_0 + A_1x + A_2x^2 + \cdots). \tag{2.10}$$

Here we assume that

$$A_0 = \mathrm{diag}(\theta_1, \dots, \theta_s, 0, \dots, 0) = \Theta \oplus O_{m-s}, \tag{2.11}$$

where for any i, j

$$\theta_i \neq 0, \quad \theta_j/\theta_i \notin q^{\mathbb{Z}_{\geq 1}}. \tag{2.12}$$

Let $P(x) = \sum_{n=0}^{\infty} P_n x^n$ be an $m \times m$ formal power series with $P_0 = I_m$. We have $Z(qx) = P(qx)^{-1}A(x)P(x)Z(x)$ by the substitution $Y(x) = P(x)Z(x)$. Set

$$B(x) := P(qx)^{-1}A(x)P(x) = x^r \sum_{n=0}^{\infty} B_n x^n. \tag{2.13}$$

From the coefficients of x^0 in $A(x)P(x) = P(qx)B(x)$, we have $B_0 = A_0$. Then the coefficients of x^n $(n \geq 1)$ gives the following relation:

$$B_n = A_n + \sum_{j=1}^{n-1}(A_{n-j}P_j - q^j P_j B_{n-j}) + A_0 P_n - P_n(q^n A_0). \tag{2.14}$$

Set

$$C_n = \begin{pmatrix} C_{11}^{(n)} & C_{12}^{(n)} \\ C_{21}^{(n)} & C_{22}^{(n)} \end{pmatrix} := A_n + \sum_{j=1}^{n-1}(A_{n-j}P_j - q^j P_j B_{n-j}) \tag{2.15}$$

for simplicity. Here $C_{11}^{(n)}$ is $s \times s$, $C_{12}^{(n)}$ is $s \times (m-s)$, $C_{21}^{(n)}$ is $(m-s) \times s$, and $C_{22}^{(n)}$ is $(m-s) \times (m-s)$. Then we have

$$A_0 P_n - P_n(q^n A_0) = B_n - C_n. \tag{2.16}$$

Unlike the Fuchsian case, $\mathrm{Sp}(A_0) \cap \mathrm{Sp}(q^n A_0) \neq \emptyset$. Thus the equation (2.16) with respect to P_n does not necessarily have a solution. Instead, we partition P_n conformably with C_n

$$P_n = \begin{pmatrix} P_{11}^{(n)} & P_{12}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} \end{pmatrix} \tag{2.17}$$

and choose P_n so that $P_{11}^{(n)} = O,\ P_{22}^{(n)} = O$, and

$$B_n = C_n + \begin{pmatrix} O & \Theta P_{12}^{(n)} \\ -P_{21}^{(n)}(q^n\Theta) & O \end{pmatrix} \tag{2.18}$$

to be block-diagonal. More specifically, we set

$$P_{12}^{(n)} = -\Theta^{-1} C_{12}^{(n)}, \quad P_{21}^{(n)} = q^{-n} C_{21}^{(n)} \Theta^{-1}. \tag{2.19}$$

Thus we have the following proposition.

Proposition 2.2. *For any system* (2.10) *with* (2.11) *and* (2.12) *there exists a formal power series with matrix coefficients* $P(x) = \sum_{n=0}^{\infty} P_n x^n$ *where* $P_{11}^{(n)} = O$ *and* $P_{22}^{(n)} = O$ *such that the gauge transformation by* $P(x)$ *is block-diagonal:*

$$Z(qx) = B(x)Z(x), \quad B(x) = P(qx)^{-1}A(x)P(x) = \begin{pmatrix} B_1(x) & O \\ O & B_2(x) \end{pmatrix}. \tag{2.20}$$

Now we apply the above construction to a polynomial coefficient system

$$Y(qx) = A(x)Y(x), \quad A(x) = A_0 + A_1 x + \cdots + A_N x^N. \tag{2.21}$$

If A_0 is of the form (2.11), then the constant term of $B_1(x)$ of (2.20) is Θ. Thus the formal normal form of $B_1(x)$ at $x = 0$ is Θ. On the other hand, $B_2(x)$ is of the following form:

$$B_2(x) = x^{r_2} B_0' + x^{r_2+1} B_1' + \cdots \tag{2.22}$$

where r_2 is a positive integer. If B_0' is similar to $\Theta' \oplus O$ where Θ' is diagonal, invertible, and non-resonant (in particular we assume that B_0' is diagonalizable), then $B_2(x)$ can be block-diagonalized into the following form

$$\begin{pmatrix} x^{r_2}\Theta' & O \\ O & x^{r_3} C_2(x) \end{pmatrix}. \tag{2.23}$$

To summarize the above, a linear q-difference system (2.21) satisfying diagonalizability (of the first term of each direct summand) and the non-resonant condition can be transformed into the following block diagonal form:

$$Z(qx) = B(x)Z(x), \quad B(x) = P(qx)^{-1}A(x)P(x) = \begin{pmatrix} x^{r_1} B_1 & & \\ & \ddots & \\ & & x^{r_k} B_k \end{pmatrix} \tag{2.24}$$

where $B_j \in \mathrm{GL}_{m_j}(\mathbb{C})$. r_i's are non-negative integers satisfying $r_1 = 0 < r_2 < \cdots < r_k$. Here the numbers r_i's and m_j's are uniquely determined only by the original system (2.21). Moreover, if we require that any eigenvalue λ of B_j satisfies $|q| < |\lambda| \leq 1$, then the conjugacy class of B_j is uniquely determined (for example see [5]). Then (2.24) is the formal normal form of (2.21) at $x = 0$.

Similarly, the formal normal form of (2.21) at $x = \infty$ has the following form:

$$\begin{pmatrix} x^{N-s_1}B'_1 & & \\ & \ddots & \\ & & x^{N-s_\ell}B'_\ell \end{pmatrix} \tag{2.25}$$

where $B'_j \in \mathrm{GL}_{m'_j}(\mathbb{C})$ and $s_1 = 0 < s_2 < \cdots < s_\ell$. The formal normal form at $x = \infty$ is also unique in the same sense as above.

§ 2.3. Spectral types of linear q-difference systems

First we recall the notion of spectral type of Fuchsian linear q-difference systems [18]. Consider the following Fuchsian linear q-difference system of rank m:

$$Y(qx) = A(x)Y(x), \quad A(x) = A_0 + A_1x + \cdots + A_Nx^N \tag{2.26}$$

where A_0 and A_N are invertible. We assume that, for any $a \in \mathbb{C}$, $A(a) \neq O$. In addition, we assume that A_0 and A_N are diagonalizable for simplicity. Let the eigenvalues of A_0 be θ_j $(j = 1, \ldots, k)$, and let their multiplicities be m_j $(j = 1, \ldots, k)$. Also, let the eigenvalues of A_N be κ_j $(j = 1, \ldots, \ell)$, and let their multiplicities be n_j $(j = 1, \ldots, \ell)$:

$$A_0 \sim \theta_1 I_{m_1} \oplus \cdots \oplus \theta_k I_{m_k}, \quad A_N \sim \kappa_1 I_{n_1} \oplus \cdots \oplus \kappa_\ell I_{n_\ell}. \tag{2.27}$$

Then we define partitions S_0 and S_∞ of m as

$$S_0 = m_1, \ldots, m_k, \quad S_\infty = n_1, \ldots, n_\ell. \tag{2.28}$$

Let Z_A be the set of the zeros of $\det A(x)$:

$$Z_A = \{a \in \mathbb{C} \mid \det A(a) = 0\} = \{\alpha_1, \ldots, \alpha_p\}. \tag{2.29}$$

We denote by d_i $(i = 1, \ldots, m)$ the elementary divisors of $A(x)$. Here we assume that $d_{i+1}|d_i$. For any $\alpha_i \in Z_A$, we denote by $\tilde{n}^i_k$ the order of α_i in d_k. For each i, let $\{n^i_j\}_j$ be the partition conjugate to $\{\tilde{n}^i_k\}_k$. Then we define S_{div} as

$$S_{\mathrm{div}} = n^1_1 \ldots n^1_{k_1}, \ldots, n^p_1 \ldots n^p_{k_p}. \tag{2.30}$$

We call the triple $[S_0; S_\infty; S_{\mathrm{div}}]$ the *spectral type* of the Fuchsian system (2.26).

Example 2.3. Consider a linear q-difference system $Y(qx) = A(x)Y(x)$ with

$$A(x) = A_0 + A_1 x + A_2 x^2,$$

where A_0 is similar to $\mathrm{diag}(\theta_1, \theta_1, \theta_1, \theta_2)$, A_2 is similar to $\mathrm{diag}(\kappa_1, \kappa_1, \kappa_2, \kappa_2)$, and the Smith normal form of $A(x)$ is

$$\begin{pmatrix} (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)^2(x-\alpha_4)(x-\alpha_5) & & & \\ & (x-\alpha_1)(x-\alpha_2) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Then the spectral type of the system is $[3, 1; 2, 2; 2, 2, 11, 1, 1]$.

Spectral types can also be defined for non-Fuchsian systems. Taking the formal normal form (2.24) into account, we can define S_0 as $S_0 = \overbrace{(\cdots(\lambda_1)\cdots)}^{r_1\text{-tuple}}, \ldots, \overbrace{(\cdots(\lambda_k)\cdots)}^{r_k\text{-tuple}}$ where λ_j is the partition of m_j determined by the multiplicities of the eigenvalues of B_j. For example, if the normal form of $A(x)$ around $x = 0$ is $(B_1) \oplus (xB_2) \oplus (x^3 B_3)$ and the partitions corresponding to B_j's are

$$B_1 : 3, 1 \quad B_2 : 2, 1 \quad B_3 : 2, 2, 2 \tag{2.31}$$

then $S_0 = 3, 1, (2, 1), (((2, 2, 2)))$.

Similarly, taking (2.25) into account, we define S_∞ as $S_\infty = \overbrace{(\cdots(\lambda'_1)\cdots)}^{s_1\text{-tuple}}, \ldots, \overbrace{(\cdots(\lambda'_\ell)\cdots)}^{s_\ell\text{-tuple}}$ where λ'_j is the partition of m'_j corresponding to B'_j.

S_{div} is the same as in the Fuchsian case. Then the triple $[S_0; S_\infty; S_{\mathrm{div}}]$ is the spectral type.

§ 3. q-matrix P_{VI}

In this section we review the q-matrix P_{VI} [12], which describes a connection-preserving deformation of the Fuchsian linear q-difference system of spectral type $[m, m; m, m-1, 1; m, m, m, m]$ (see [6, 12] for the connection-preserving deformation).

Consider a linear q-difference system of the following form:

$$Y(qx) = A(x)Y(x), \quad A(x) = A_0 + A_1 x + A_2 x^2, \quad A_j \in M_{2m}(\mathbb{C}), \tag{3.1}$$

where

$$A_2 = \begin{pmatrix} \kappa_1 I_m & O \\ O & K \end{pmatrix}, \quad K = \mathrm{diag}(\overbrace{\kappa_2, \ldots, \kappa_2}^{m-1}, \kappa_3), \quad A_0 \sim \begin{pmatrix} \theta_1 t I_m & O \\ O & \theta_2 t I_m \end{pmatrix}. \tag{3.2}$$

Since $S_{\mathrm{div}} = m, m, m, m$, the Smith normal form of the polynomial matrix $A(x)$ is of the following form:

$$\begin{pmatrix} I_m & O \\ O & \prod_{i=1}^{4}(x-\alpha_i)I_m \end{pmatrix}. \tag{3.3}$$

That is, $d_1 = \cdots = d_m = \prod_{i=1}^{4}(x-\alpha_i),\ d_{m+1} = \cdots = d_{2m} = 1$, so we have $\tilde{n}^i_k = 1\,(i = 1, 2, 3, 4,\ k = 1, \ldots, m)$.

We assume that α_j's depend on t as follows:

$$\alpha_j = \begin{cases} a_j t & (j = 1, 2), \\ a_j & (j = 3, 4). \end{cases} \tag{3.4}$$

We also assume $q\alpha_i \neq \alpha_j\ (i \neq j)$.

The linear q-difference systems satisfying the above conditions can be parametrized as follows:

(3.5)
$$A(x) = \begin{pmatrix} WK\{\kappa_1(xI_m - F)(xI_m - \boldsymbol{\alpha}) + \kappa_1 G_1\}K^{-1}W^{-1} & WK(xI_m - F) \\ \kappa_1(\boldsymbol{\gamma}x + \boldsymbol{\delta})W^{-1} & K(xI_m - \boldsymbol{\beta})(xI_m - F) + KG_2 \end{pmatrix}$$

where

(3.6)
$$\boldsymbol{\alpha} = (\kappa_1 - K)^{-1}\left\{(\theta_1 + \theta_2)tF^{-1} - \kappa_1 F^{-1}G_1 - KG_2F^{-1} + K(F + G_1^{-1}FG_1 + \beta_1)\right\},$$

(3.7)
$$\boldsymbol{\beta} = (\kappa_1 - K)^{-1}\left\{-(\theta_1 + \theta_2)tF^{-1} + \kappa_1 F^{-1}G_1 + KG_2F^{-1} - \kappa_1(F + G_1^{-1}FG_1 + \beta_1)\right\},$$

(3.8)
$$\boldsymbol{\gamma} = K\{G_1 + G_2 + F\boldsymbol{\alpha} + \boldsymbol{\beta}F + \boldsymbol{\beta}\boldsymbol{\alpha} - G_1^{-1}(F^2 + \beta_1 F + \beta_2)G_1\}K^{-1},$$

(3.9)
$$\boldsymbol{\delta} = \kappa_1^{-1}\{t^2\theta_1\theta_2F^{-1} - \kappa_1 K(G_2 + \boldsymbol{\beta}F)F^{-1}(G_1 + F\boldsymbol{\alpha})\}K^{-1}.$$

Here the auxiliary parameters β_j's are defined by

$$\sum_{j=0}^{4} \beta_{4-j} z^j := \prod_{j=1}^{4}(z - \alpha_j). \tag{3.10}$$

Also, the matrices G_1 and G_2 satisfy

$$G_1 G_2 = (F - \alpha_1 I_m)(F - \alpha_2 I_m)(F - \alpha_3 I_m)(F - \alpha_4 I_m). \tag{3.11}$$

The relation (3.11) allows us to introduce a new variable G by

$$G_1 = q^{-1}\kappa_1^{-1}(F - \alpha_1)(F - \alpha_2)G^{-1}, \quad G_2 = q\kappa_1 G(F - \alpha_3)(F - \alpha_4). \tag{3.12}$$

Then, from the assumption about the Smith normal form (3.3), F and G must satisfy the following commutation relation:

$$F^{-1}GFG^{-1}=\rho K,\quad \rho=\frac{a_1a_2a_3a_4\kappa_1}{\theta_1\theta_2}. \tag{3.13}$$

Since

$$\det A(x)=\kappa_1^m\kappa_2^{m-1}\kappa_3\prod_{i=1}^{4}(x-\alpha_i)^m, \tag{3.14}$$

we have

$$\kappa_1{}^m\kappa_2{}^{m-1}\kappa_3\prod_{i=1}^{4}a_i{}^m=\theta_1{}^m\theta_2{}^m. \tag{3.15}$$

Let us consider the connection-preserving deformation of the system (3.1). We choose t as a deformation parameter. The parameters θ_j, κ_j, and a_j's are independent of t. In the following we write $A(x,t)$ instead of $A(x)$ when it is necessary to emphasize that $A(x)$ depends on t.

The connection-preserving deformation of (3.1) is given by

$$Y(x,qt)=B(x,t)Y(x,t) \tag{3.16}$$

where

$$B(x,t)=\frac{x(xI_{2m}+B_0)}{(x-qa_1t)(x-qa_2t)},\quad B_0=\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}. \tag{3.17}$$

Here B_{ij}'s are $m\times m$ matrices and given as follows:

$$B_{11}=qWK(I_m-\overline{G}K)^{-1}\overline{G}K\left[K^{-1}\overline{G}^{-1}\{F-(a_1+a_2)t\}+\boldsymbol{\beta}\right]K^{-1}W^{-1}, \tag{3.18}$$

$$B_{12}=qWK(I_m-\overline{G}K)^{-1}\overline{G}, \tag{3.19}$$

$$\begin{aligned}B_{21}&=q\kappa_1\left\{q^{-1}\kappa_1^{-1}(\overline{F}-qa_2t)\overline{G}^{-1}-qa_1t+\overline{\boldsymbol{\alpha}}\right\}(I_m-q\kappa_1\overline{G})^{-1}\\ &\quad\times\overline{G}K\left\{K^{-1}\overline{G}^{-1}(F-a_2t)-a_1t+\boldsymbol{\beta}\right\}K^{-1}W^{-1}\end{aligned} \tag{3.20}$$

$$\begin{aligned}&=q\kappa_1\left\{q^{-1}\kappa_1^{-1}(\overline{F}-qa_1t)\overline{G}^{-1}-qa_2t+\overline{\boldsymbol{\alpha}}\right\}(I_m-q\kappa_1\overline{G})^{-1}\\ &\quad\times\overline{G}K\left\{K^{-1}\overline{G}^{-1}(F-a_1t)-a_2t+\boldsymbol{\beta}\right\}K^{-1}W^{-1},\end{aligned} \tag{3.21}$$

$$B_{22}=\left[q^{-1}\kappa_1^{-1}\{\overline{F}-q(a_1+a_2)t\}\overline{G}^{-1}+\overline{\boldsymbol{\alpha}}\right]q\kappa_1\overline{G}(I_m-q\kappa_1\overline{G})^{-1}. \tag{3.22}$$

Here the overline denotes the q-shift with respect to t: $\overline{f}=f(qt)$ for $f=f(t)$.

Now we have the pair of linear q-difference systems:

$$\begin{cases} Y(qx,t) = A(x,t)Y(x,t), \\ Y(x,qt) = B(x,t)Y(x,t). \end{cases} \tag{3.23}$$

Then the compatibility condition of (3.23)

$$A(x,qt)B(x,t) = B(qx,t)A(x,t) \tag{3.24}$$

reduces to a system of non-linear q-difference equations satisfied by F, G, and W.

Theorem 3.1 ([12]). *The compatibility condition $A(x,qt)B(x,t) = B(qx,t)A(x,t)$ is equivalent to*

$$\overline{G}KG = \frac{1}{q\kappa_1}(F-a_1t)(F-a_2t)(F-a_3)^{-1}(F-a_4)^{-1}, \tag{3.25}$$

$$\overline{F}KF = \frac{\theta_1\theta_2}{\kappa_1 a_1 a_2}\left(\overline{G} - t\frac{a_1a_2}{\theta_1}\right)\left(\overline{G} - t\frac{a_1a_2}{\theta_2}\right)\left(\overline{G} - \frac{1}{q\kappa_1}\right)^{-1}\left(\overline{G}-\rho\right)^{-1}, \tag{3.26}$$

$$W^{-1}\overline{W} = q\kappa_1(\overline{G} - K^{-1})^{-1}\left(\overline{G} - \frac{1}{q\kappa_1}\right)K^{-1}. \tag{3.27}$$

We call the system (3.25) and (3.26) (with (3.13)) the q-matrix P_{VI}. Although this system appears to have eight parameters (θ_i's, κ_i's, and a_i's with a single relation (3.15)), the number of parameters can be reduced to five by rescaling F, G, and t.

§ 4. Degeneration of q-matrix P_{VI}

Now we consider a degeneration of the q-matrix P_{VI} which corresponds to the limit κ_1 to 0.

§ 4.1. From q-matrix P_{VI} to q-matrix P_{V}

Consider the following transformation:

$$\begin{aligned} &t = \varepsilon\tilde{t}, \quad F = \varepsilon\tilde{F}, \quad G = \varepsilon\tilde{G}, \quad W = \varepsilon\tilde{W}, \\ &a_3 = -\varepsilon\tilde{a}_3, \quad a_4 = -\varepsilon^{-1}\tilde{\kappa}_1, \quad \kappa_1 = \varepsilon, \quad \kappa_2 = \varepsilon^{-1}\tilde{\kappa}_2, \quad \kappa_3 = \varepsilon^{-1}\tilde{\kappa}_3. \end{aligned} \tag{4.1}$$

We set $\tilde{K} = \mathrm{diag}(\overbrace{\tilde{\kappa}_2, \dots, \tilde{\kappa}_2}^{m-1}, \tilde{\kappa}_3)$ so that we have $K = \varepsilon^{-1}\tilde{K}$. The other parameters a_1, a_2 and θ_1, θ_2 are not changed. This transformation is compatible with the commutation relation (3.13), that is, $\tilde{F}^{-1}\tilde{G}\tilde{F}\tilde{G}^{-1} = \tilde{\rho}\tilde{K}$ holds where $\tilde{\rho} = \frac{a_1a_2\tilde{a}_3\tilde{\kappa}_1}{\theta_1\theta_2}$. From the relation (3.15), we have

$${\tilde{\kappa}_1}^m{\tilde{\kappa}_2}^{m-1}\tilde{\kappa}_3{a_1}^m{a_2}^m{\tilde{a}_3}^m = {\theta_1}^m{\theta_2}^m. \tag{4.2}$$

Substituting (4.1) into (3.25), we have

$$\varepsilon\overline{\tilde{G}}\tilde{K}\tilde{G} = \frac{\varepsilon}{q}(\tilde{F} - a_1\tilde{t})(\tilde{F} - a_2\tilde{t})(\tilde{F} + \tilde{a}_3)^{-1}(\varepsilon^2\tilde{F} + \tilde{\kappa}_1)^{-1}. \tag{4.3}$$

Letting $\varepsilon \to 0$, we obtain

$$\overline{\tilde{G}}\tilde{K}\tilde{G} = \frac{1}{q\tilde{\kappa}_1}(\tilde{F} - a_1\tilde{t})(\tilde{F} - a_2\tilde{t})(\tilde{F} + \tilde{a}_3)^{-1}. \tag{4.4}$$

Similarly, from the equation (3.26) we have

$$\varepsilon\overline{\tilde{F}}\tilde{K}\tilde{F} = \varepsilon\frac{\theta_1\theta_2}{a_1a_2}\left(\overline{\tilde{G}} - \tilde{t}\frac{a_1a_2}{\theta_1}\right)\left(\overline{\tilde{G}} - \tilde{t}\frac{a_1a_2}{\theta_2}\right)\left(\varepsilon^2\overline{\tilde{G}} - \frac{1}{q}\right)^{-1}\left(\overline{\tilde{G}} - \tilde{\rho}\right)^{-1}. \tag{4.5}$$

Letting $\varepsilon \to 0$, we obtain

$$\overline{\tilde{F}}\tilde{K}\tilde{F} = -q\frac{\theta_1\theta_2}{a_1a_2}\left(\overline{\tilde{G}} - \tilde{t}\frac{a_1a_2}{\theta_1}\right)\left(\overline{\tilde{G}} - \tilde{t}\frac{a_1a_2}{\theta_2}\right)\left(\overline{\tilde{G}} - \tilde{\rho}\right)^{-1}. \tag{4.6}$$

From the equation (3.27) we have

$$\tilde{W}^{-1}\overline{\tilde{W}} = q(\overline{\tilde{G}} - \tilde{K}^{-1})^{-1}\left(\varepsilon^2\overline{\tilde{G}} - \frac{1}{q}\right)\tilde{K}^{-1}. \tag{4.7}$$

Letting $\varepsilon \to 0$, we obtain

$$\tilde{W}^{-1}\overline{\tilde{W}} = -(\tilde{K}\overline{\tilde{G}} - I_m)^{-1}. \tag{4.8}$$

Omitting the tilde, we obtain the following system of non-linear q-difference equations

$$\overline{G}KG = \frac{1}{q\kappa_1}(F - a_1t)(F - a_2t)(F + a_3)^{-1}, \tag{4.9}$$

$$\overline{F}KF = -q\frac{\theta_1\theta_2}{a_1a_2}\left(\overline{G} - t\frac{a_1a_2}{\theta_1}\right)\left(\overline{G} - t\frac{a_1a_2}{\theta_2}\right)\left(\overline{G} - \rho\right)^{-1}, \tag{4.10}$$

$$W^{-1}\overline{W} = (I_m - K\overline{G})^{-1}. \tag{4.11}$$

The associated linear system (3.1) can also be degenerated in the same manner as above. Set $x = \varepsilon\tilde{x}$. Notice that

$$\boldsymbol{\alpha} = -\tilde{\kappa}_1\varepsilon^{-1} + O(1), \quad \boldsymbol{\beta} = O(\varepsilon), \quad \boldsymbol{\gamma} = O(1), \quad \boldsymbol{\delta} = O(\varepsilon), \quad G_1 = O(1), \quad G_2 = O(\varepsilon^2). \tag{4.12}$$

We set

$$\tilde{\boldsymbol{\beta}} := \lim_{\varepsilon\to 0}\frac{\boldsymbol{\beta}}{\varepsilon}, \quad \tilde{\boldsymbol{\gamma}} := \lim_{\varepsilon\to 0}\boldsymbol{\gamma}, \quad \tilde{\boldsymbol{\delta}} := \lim_{\varepsilon\to 0}\frac{\boldsymbol{\delta}}{\varepsilon}, \quad \tilde{G}_1 := \lim_{\varepsilon\to 0}G_1, \quad \tilde{G}_2 := \lim_{\varepsilon\to 0}\frac{G_2}{\varepsilon^2}. \tag{4.13}$$

Then it is easy to see that

$$\tilde{A}(\tilde{x}) := \lim_{\varepsilon\to 0}\varepsilon^{-1}A(x) = \begin{pmatrix} \tilde{W}\tilde{K}\{\tilde{\kappa}_1(\tilde{x}I_m - \tilde{F}) + \tilde{G}_1\}\tilde{K}^{-1}\tilde{W}^{-1} & \tilde{W}\tilde{K}(\tilde{x}I_m - \tilde{F}) \\ (\tilde{\boldsymbol{\gamma}}\tilde{x} + \tilde{\boldsymbol{\delta}})\tilde{W}^{-1} & \tilde{K}(\tilde{x}I_m - \tilde{\boldsymbol{\beta}})(\tilde{x}I_m - \tilde{F}) + \tilde{K}\tilde{G}_2 \end{pmatrix}. \tag{4.14}$$

Remark. The multiplication of $A(x)$ by ε^{-1} can be realized by a simple gauge transformation of the linear system. For example, consider the transformation $Y = x^{\log\varepsilon/\log q}\tilde{Y}$ or use the ratio of theta functions (5.12) instead of $x^{\log\varepsilon/\log q}$. Then we have $\tilde{Y}(qx) = \varepsilon^{-1}A(x)\tilde{Y}(x)$.

Thus we obtain (by omitting the tilde)

$$\begin{aligned} A(x) &= \begin{pmatrix} WK\{\kappa_1(xI_m - F) + G_1\}K^{-1}W^{-1} & WK(xI_m - F) \\ (\boldsymbol{\gamma}x + \boldsymbol{\delta})W^{-1} & K(xI_m - \boldsymbol{\beta})(xI_m - F) + KG_2 \end{pmatrix} \\ &=: A_0 + A_1 x + A_2 x^2, \end{aligned} \tag{4.15}$$

where

$$\boldsymbol{\beta} = K^{-1}\{(\theta_1 + \theta_2)tF^{-1} - F^{-1}G_1 - KG_2F^{-1} + \kappa_1\}, \tag{4.16}$$

$$\boldsymbol{\gamma} = K\{G_1 - \kappa_1(F + \boldsymbol{\beta} + GFG^{-1} - (a_1 + a_2)t + a_3)\}K^{-1}, \tag{4.17}$$

$$\boldsymbol{\delta} = F^{-1}(G_1 - \kappa_1 F - \theta_1 t)(G_1 - \kappa_1 F - \theta_2 t)K^{-1}. \tag{4.18}$$

From the determinant of (4.15) we have

$${\kappa_1}^m {\kappa_2}^{m-1}\kappa_3 \prod_{i=1}^{3} {a_i}^m = {\theta_1}^m {\theta_2}^m. \tag{4.19}$$

The matrices G_1 and G_2 are given by

$$G_1 = q^{-1}(F - a_1 t)(F - a_2 t)G^{-1}, \quad G_2 = q\kappa_1 G(F + a_3) \tag{4.20}$$

and satisfy

$$G_1 G_2 = \kappa_1(F - a_1 t)(F - a_2 t)(F + a_3). \tag{4.21}$$

The matrices F and G satisfy the following commutation relation:

$$F^{-1}GFG^{-1} = \rho K, \quad \rho = \frac{a_1 a_2 a_3 \kappa_1}{\theta_1 \theta_2}. \tag{4.22}$$

The system in t-direction (3.16) can also be degenerated in the same manner. As a result, we have (by omitting the tilde)

$$B(x,t) = \frac{x(xI_{2m} + B_0)}{(x - qa_1 t)(x - qa_2 t)}, \quad B_0 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{4.23}$$

where B_{ij}'s are $m\times m$ matrices given by

$$B_{11}=qWK(I_m-\overline{G}K)^{-1}\left\{F-(a_1+a_2)t+\overline{G}K\boldsymbol{\beta}\right\}K^{-1}W^{-1}, \tag{4.24}$$

$$B_{12}=qWK(I_m-\overline{G}K)^{-1}\overline{G}, \tag{4.25}$$

$$B_{21}=\left\{(\overline{F}-qa_2t)\overline{G}^{-1}-q\kappa_1\right\}\left(F-a_2t-a_1t\overline{G}K+\overline{G}K\boldsymbol{\beta}\right)K^{-1}W^{-1} \tag{4.26}$$

$$=\left\{(\overline{F}-qa_1t)\overline{G}^{-1}-q\kappa_1\right\}\left(F-a_1t-a_2t\overline{G}K+\overline{G}K\boldsymbol{\beta}\right)K^{-1}W^{-1}, \tag{4.27}$$

$$B_{22}=\overline{F}-q(a_1+a_2)t-q\kappa_1\overline{G}. \tag{4.28}$$

We obtain the following theorem by a direct calculation.

Theorem 4.1. *The compatibility condition $A(x,qt)B(x,t)=B(qx,t)A(x,t)$ with (4.15) and (4.23) is equivalent to*

$$\overline{G}KG=\frac{1}{q\kappa_1}(F-a_1t)(F-a_2t)(F+a_3)^{-1}, \tag{4.29}$$

$$\overline{F}KF=-q\frac{\theta_1\theta_2}{a_1a_2}\left(\overline{G}-t\frac{a_1a_2}{\theta_1}\right)\left(\overline{G}-t\frac{a_1a_2}{\theta_2}\right)\left(\overline{G}-\rho\right)^{-1}, \tag{4.30}$$

$$W^{-1}\overline{W}=-(K\overline{G}-I_m)^{-1}. \tag{4.31}$$

We call the system (4.29) and (4.30) (with (4.22)) the q-matrix fifth Painlevé system (q-matrix P_{V}). Although this system appears to have seven parameters (θ_i's, κ_i's, and a_i's with a single relation (4.19)), the number of parameters can be reduced to four by rescaling F, G, and t.

§ 4.2. Characterization of the linear system

The matrix (4.15) satisfies
(C1): A_0 is similar to $\theta_1tI_m\oplus\theta_2tI_m$.
(C2): The formal normal form of $A(x)$ at $x=\infty$ is

$$\begin{pmatrix} x^2K & O \\ O & x(\kappa_1I_m)\end{pmatrix}. \tag{4.32}$$

(C3): The Smith normal form of $A(x)$ is

$$\begin{pmatrix} I_m & O \\ O & \prod_{j=1}^3(x-\alpha_j)I_m\end{pmatrix}\quad(\alpha_1=a_1t,\ \alpha_2=a_2t,\ \alpha_3=-a_3). \tag{4.33}$$

Conversely, it can be shown that a polynomial matrix $A(x)$ satisfying the above three conditions can be written (generically) in the form (4.15). Thus the linear system associated with the q-matrix P_{V} is characterized by the conditions (C1), (C2), and (C3).

From the definition given in Section 2.3, the spectral type of the system is written as $[m, m;\ m-1, 1, (m);\ m, m, m]$.

§ 5. Continuous limit of q-matrix P_{V}

The system (4.29) and (4.30) can be viewed as a q-analogue of the matrix P_{V} (Appendix A.6). That is, taking the limit $q \to 1$, one can obtain (Appendix A.6) from (4.29) and (4.30). In fact, let us define the parameter ε by $q = 1-\varepsilon$. We set

$$\begin{aligned} &\theta_i = 1-\sigma_i\varepsilon\ (i=1,2), \quad \kappa_1 = -1-\mu_1\varepsilon, \quad \kappa_i = \varepsilon(1+\mu_i\varepsilon)\ (i=2,3), \\ &a_i = 1+\zeta_i\varepsilon\ (i=1,2), \quad a_3 = \varepsilon^{-1}, \end{aligned} \tag{5.1}$$

and $M = \mathrm{diag}(\overbrace{\mu_2, \ldots, \mu_2}^{m-1}, \mu_3)$. Moreover, we introduce new dependent variables Q and P which are related to F and G by

$$F = -(\tilde{P}+\varepsilon^{-1}t)(\phi_1-\phi_2\tilde{Q})^{-1}\tilde{Q}, \quad G = (\tilde{P}+\varepsilon^{-1}t)(\phi_1-\phi_2\tilde{Q})^{-1}, \tag{5.2}$$

$$\phi_1 = \varepsilon^{-1}-1-\zeta_1-\zeta_2-\frac{\sigma_1+\sigma_2}{2}, \quad \phi_2 = \varepsilon^{-1}-\frac{\zeta_1+\zeta_2}{2}, \tag{5.3}$$

$$\tilde{Q} = I_m-\hat{Q}^{-1}, \quad \tilde{P} = t\left\{(\hat{Q}-I_m)\hat{P}\hat{Q}+\frac{\zeta_2-\zeta_1+\sigma_1-\sigma_2}{2}\hat{Q}+\frac{\zeta_1-\zeta_2}{2}\right\}, \tag{5.4}$$

$$\hat{Q} = g^{-1}Qg, \quad \hat{P} = g^{-1}Pg. \tag{5.5}$$

Here $g = t^M$, which is a solution to $\frac{dg}{dt}g^{-1} = \frac{1}{t}M$.

Then, taking the limit $\varepsilon \to 0$, we find that Q and P satisfy the following equations:

$$t\frac{dQ}{dt} = Q(Q-1)(P+t)+P(Q-1)Q-(\zeta_1-\zeta_2)(Q-1)+(\sigma_1-\sigma_2)Q, \tag{5.6}$$

$$t\frac{dP}{dt} = -(Q-1)P(P+t)-(P+t)PQ-(\zeta_2-\zeta_1+\sigma_1-\sigma_2)P-(\zeta_2+\zeta_4+\sigma_1)t. \tag{5.7}$$

These equations coincide with (Appendix A.6) by the following correspondence of the parameters:

$$\sigma_1-\sigma_2 = \theta^0, \quad \zeta_1-\zeta_2 = \theta^1, \quad \mu_i+\zeta_2+\sigma_2 = \theta_i^\infty\ (i=1,2,3). \tag{5.8}$$

Expanding (4.22) with respect to the small parameter ε and taking the coefficient of ε^1, we have the commutation relation between P and Q:

$$PQ-QP = (\mu_1+\zeta_1+\zeta_2+\sigma_1+\sigma_2)I_m+M. \tag{5.9}$$

The linear system (4.15) also admits the continuous limit in a similar way. To see this, we first change the dependent variable Y to Z: $Y(x) = f(x)Z(x)$, where $f(x)$ is a solution of the following q-difference equation

$$f(qx) = -(x-t)f(x). \tag{5.10}$$

For example, we can take

$$f(x) = \frac{\vartheta_q(x/t)}{(x/t;q)_\infty \vartheta_q(x)}, \tag{5.11}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad \vartheta_q(x) = \prod_{n=0}^{\infty}(1-q^{n+1})(1+xq^n)(1+x^{-1}q^{n+1}). \tag{5.12}$$

Then we have

$$\frac{Z(x)-Z(qx)}{(1-q)x} = \frac{1}{\varepsilon x}\left\{ I_{2m} - \frac{1}{-(x-t)}A(x) \right\} Z(x). \tag{5.13}$$

Set $W = tU^{-1}$. Define matrices A_0, A_1, and A_∞ by

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon x}\left\{ I_{2m} - \frac{1}{-(x-t)}A(x) \right\} = \frac{\mathsf{A}_0}{x} + \frac{\mathsf{A}_1}{x-t} + \mathsf{A}_\infty. \tag{5.14}$$

It can be shown that the matrices A_0, A_1, and A_∞ (almost) coincide with (Appendix A.2). More precisely, performing suitable scalar gauge transformations (in other words, adding suitable scalar matrices to A_0, A_1, and A_∞), performing the gauge transformation by

$$h = \begin{pmatrix} O & I_m \\ I_m & O \end{pmatrix}, \tag{5.15}$$

and setting $x = t\tilde{x}$, we have

(5.16)
$$h^{-1}(\mathsf{A}_0 - \sigma_2 I_{2m})h = \mathcal{A}_0, \quad h^{-1}(\mathsf{A}_1 - \zeta_2 I_{2m})h = \mathcal{A}_1, \quad h^{-1}(t\mathsf{A}_\infty - tI_{2m})h = \mathcal{A}_\infty.$$

Thus the resulting system of linear differential equations

$$\frac{d\tilde{Z}}{d\tilde{x}} = \left(\frac{\mathcal{A}_0}{\tilde{x}} + \frac{\mathcal{A}_1}{\tilde{x}-1} + \mathcal{A}_\infty \right) \tilde{Z} \tag{5.17}$$

coincides with the x-direction of (Appendix A.1).

§ Appendix A. The matrix fifth Painlevé system

In this appendix, we review the matrix fifth Painlevé system (matrix P_{V}) [8, 11]. The matrix P_{V} is derived from the isomonodromic deformation of a certain linear differential system. There are several Lax pairs for the matrix P_{V}, one of them is the following:

(Appendix A.1)
$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{\mathcal{A}_0}{x} + \dfrac{\mathcal{A}_1}{x-1} + \mathcal{A}_\infty\right) Y, \\ \dfrac{\partial Y}{\partial t} = (-E_2 \otimes I_m x + \mathcal{B}_1) Y, \end{cases}$$

where

(Appendix A.2)
$$\mathcal{A}_\xi = (I_m \oplus U)^{-1} \hat{\mathcal{A}}_\xi (I_m \oplus U) \quad (\xi = 0, 1),$$
$$\hat{\mathcal{A}}_0 = \begin{pmatrix} QP + \theta^0 + \theta^\infty_1 \\ tI_m \end{pmatrix} \left(I_m - Q,\ \frac{1}{t}\{(Q - I_m)QP + (\theta^0 + \theta^\infty_1)Q - \theta^\infty_1\} \right),$$
$$\hat{\mathcal{A}}_1 = \begin{pmatrix} (Q - I_m)PQ + (\theta^0 + \theta^\infty_1)Q + \theta^1 \\ tQ \end{pmatrix} \left(I_m,\ \frac{1}{t}\{(I_m - Q)P - \theta^0 - \theta^\infty_1\} \right),$$
$$\mathcal{A}_\infty = \begin{pmatrix} O_m & O_m \\ O_m & -tI_m \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta^\infty_2 I_{m-1} & \\ & \theta^\infty_3 \end{pmatrix}.$$

Furthermore, the matrix $\mathcal{B}_1$ is given by

(Appendix A.3)
$$\mathcal{B}_1 = (I_m \oplus U)^{-1} \begin{pmatrix} O_m & \frac{[\hat{\mathcal{A}}_0 + \hat{\mathcal{A}}_1]_{12}}{t} \\ \frac{[\hat{\mathcal{A}}_0 + \hat{\mathcal{A}}_1]_{21}}{t} & O_m \end{pmatrix} (I_m \oplus U),$$

where $[\hat{\mathcal{A}}_0 + \hat{\mathcal{A}}_1]_{ij}$ is the (i, j)-block of the matrix $\hat{\mathcal{A}}_0 + \hat{\mathcal{A}}_1$. The Fuchs-Hukuhara relation is written as $m(\theta^0 + \theta^1 + \theta^\infty_1) + (m-1)\theta^\infty_2 + \theta^\infty_3 = 0$. P and Q satisfy $[P, Q] = (\theta^0 + \theta^1 + \theta^\infty_1)I_m + \Theta$. The system in x-direction of (Appendix A.1) is characterized by the spectral type $(m)(m-1\,1), mm, mm$.

The compatibility condition (in other words, isomonodromic deformation equation) for (Appendix A.1) has two descriptions, which are mutually equivalent. One is the Hamiltonian form and the other is the "non-abelian" form. The Hamiltonian is given by

(Appendix A.4)
$$tH^{\mathrm{Mat},m}_{\mathrm{V}} \begin{pmatrix} -\theta^0 - \theta^1 - \theta^\infty_1, \theta^0 - \theta^1 \\ \theta^1, \theta^0 + \theta^1 + \theta^\infty_1 + \theta^\infty_2 \end{pmatrix}; t; Q, P \Bigg)$$
$$= \mathrm{tr}[P(P + t)Q(Q - 1) + (\theta^0 - \theta^1)PQ + \theta^1 P + (\theta^0 + \theta^\infty_1)tQ].$$

Then the compatibility condition can be written as follows:

(Appendix A.5)
$$\frac{dq_{ij}}{dt} = \frac{\partial H^{\mathrm{Mat},m}_{\mathrm{V}}}{\partial p_{ji}}, \quad \frac{dp_{ij}}{dt} = -\frac{\partial H^{\mathrm{Mat},m}_{\mathrm{V}}}{\partial q_{ji}}.$$

On the other hand, the non-abelian description is given as follows [11]:
(Appendix A.6)

$$\begin{cases} t\dfrac{dQ}{dt} = Q(Q-1)(P+t) + PQ(Q-1) + (\theta^0 - \theta^1)Q + \theta^1, \\ t\dfrac{dP}{dt} = -(Q-1)P(P+t) - P(P+t)Q - (\theta^0 - \theta^1)P - (\theta^0 + \theta^\infty_1)t. \end{cases}$$

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