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Box and ball system with numbered boxes and balls

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Abstract

Box and ball systems (BBSs) are known as discrete dynamical systems in which motions of balls among successive infinite boxes are governed by an ultradiscrete integrable system. The equation of motion in the simplest BBS is the ultradiscrete version of the discrete Toda equation, which is one of famous discrete integrable systems. The discrete Toda equation is extended to two types of discrete hungry Toda (dhToda) equations, and their ultradiscretizations are shown to be the equations of motion in the BBSs in which either boxes or balls are numbered. In this paper, we propose a new box-ball system in which both boxes and balls are numbered, and show that its equation of motion is the ultradiscretization of a variant of the dhToda equations. With the help of a combinatorial technique, we describe conserved quantities of our new numbered BBS (nBBS). We also clarify its relationship to the hungry ε-BBS, which is derived from the ultradiscretization of another extension of the discrete Toda equation.

§ 1. Introduction

The box and ball system (BBS) that was first proposed by Takahashi and Satsuma [8] is a cellular automaton in which each ball, in order from left, moves to the nearest

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empty box on the right among an infinite number of boxes arranged in a straight line. The interactions of groups of successive balls can be regarded as those of the simplified solitons. Under discrete time evolution from n to $n+1$, the motion of m ball groups is described using the equation:

$$(1.1)\qquad \begin{cases} Q_k^{(n+1)} = \min\left(\displaystyle\sum_{i=1}^{k} Q_i^{(n)} - \sum_{i=1}^{k-1} Q_i^{(n+1)}, E_k^{(n)}\right), \quad k = 1,2,\ldots,m, \\ Q_k^{(n+1)} + E_k^{(n+1)} = Q_{k+1}^{(n)} + E_k^{(n)}, \quad k = 1,2,\ldots,m-1, \\ E_0^{(n)} \equiv +\infty, \quad E_m^{(n)} \equiv +\infty, \end{cases}$$

where $Q_k^{(n)}$ and $E_k^{(n)}$ respectively correspond to the number of successive balls in the kth ball group from the left and of successive boxes in the box array between the kth and $(k+1)$th ball groups at discrete time n. Equation (1.1) is derived from the ultradiscretization of the famous discrete Toda (dToda) equation:

$$(1.2)\qquad \begin{cases} q_k^{(n+1)} + e_{k-1}^{(n+1)} = q_k^{(n)} + e_k^{(n)}, \quad k = 1,2,\ldots,m, \\ q_k^{(n+1)} e_k^{(n+1)} = q_{k+1}^{(n)} e_k^{(n)}, \quad k = 1,2,\ldots,m-1, \\ e_0^{(n)} \equiv 0, \quad e_m^{(0)} \equiv 0. \end{cases}$$

Thus, (1.1) is called the ultradiscrete Toda (udToda) equation. By the way, the dToda equation (1.2) can generate LR transformations of tridiagonal matrices where one step first decomposes a tridiagonal matrix into the product of lower and upper bidiagonal matrices, and next inverse the product. In fact, the dToda equation (1.2) is just equal to the recursion formula of the well-known quotient-difference (qd) algorithm for computing tridiagonal eigenvalues [7].

As an extension of the simple BBS, Tokihiro et al. [9] proposed a numbered BBS (nBBS) in which each ball is given one of the numbers $1,2,\ldots,M$ where M is a positive integer. Concerning the order of ball motion, the assigned number has priority over the ball position. In this case, the equation of motion is given by an extension of the udToda equation (1.1) as:

$$(1.3)\qquad \begin{cases} Q_k^{(n+M)} = \min\left(\displaystyle\sum_{i=1}^{k} Q_i^{(n)} - \sum_{i=1}^{k-1} Q_i^{(n+M)}, E_k^{(n)}\right), \quad k = 1,2,\ldots,m, \\ Q_k^{(n+M)} + E_k^{(n+1)} = Q_{k+1}^{(n)} + E_k^{(n)}, \quad k = 1,2,\ldots,m-1, \\ E_0^{(n)} \equiv +\infty, \quad E_m^{(n)} \equiv +\infty. \end{cases}$$

The inverse ultradiscretization of (1.3) leads to the discrete hungry Toda (dhToda) equation:

$$
(1.4)\qquad \begin{cases} q_k^{(n+M)} + e_{k-1}^{(n+1)} = q_k^{(n)} + e_k^{(n)}, \quad k = 1, 2, \ldots, m, \\ q_k^{(n+M)} e_k^{(n+1)} = q_{k+1}^{(n)} e_k^{(n)}, \quad k = 1, 2, \ldots, m-1, \\ e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0. \end{cases}
$$

In [3], we related the dhToda equation (1.4) to LR transformations of totally nonnegative (TN) Hessenberg matrices, and proposed its application to computing the eigenvalues of TN Hessenberg matrices. Moreover, we developed similar studies on a variant of the dhToda equation given as:

$$
(1.5)\qquad \begin{cases} q_k^{(n+1)} + e_{k-1}^{(n+N)} = q_k^{(n)} + e_k^{(n)}, \quad k = 1, 2, \ldots, m, \\ q_k^{(n+1)} e_k^{(n+N)} = q_{k+1}^{(n)} e_k^{(n)}, \quad k = 1, 2, \ldots, m-1, \\ e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, \end{cases}
$$

and then designed another nBBS in which boxes are numbered, where the value of N corresponds to the number of box types [4]. For simplicity, in this paper, we call the nBBSs with numbered balls and with numbered boxes *nBBS-I* and *nBBS-II*, respectively. We also distinguish (1.4) and (1.5) by referring to them as the dhToda-I and dhToda-II equations, respectively. The motion of nBBS-II is expressed by using the ultradiscretization of the dhToda-II (1.5) given as:

$$
(1.6)\qquad \begin{cases} Q_k^{(n+1)} = \min\left(\sum_{i=1}^{k} Q_i^{(n)} - \sum_{i=1}^{k-1} Q_i^{(n+1)}, E_k^{(n)}\right), \quad k = 1, 2, \ldots, m, \\ Q_k^{(n+1)} + E_k^{(n+N)} = Q_{k+1}^{(n)} + E_k^{(n)}, \quad k = 1, 2, \ldots, m-1, \\ E_0^{(n)} \equiv +\infty, \quad E_m^{(n)} \equiv +\infty. \end{cases}
$$

The dhToda-I equation (1.4) and the dhToda-II equation (1.5) can be unified as the following equation [1].

$$
(1.7)\qquad \begin{cases} q_{i,k}^{(n,j+1)} + e_{j,k-1}^{(n,i+1)} = q_{i,k}^{(n,j)} + e_{j,k}^{(n,i)}, \quad k = 1, 2, \ldots, m, \\ q_{i,k}^{(n,j+1)} e_{j,k}^{(n,i+1)} = q_{i,k+1}^{(n,j)} e_{j,k}^{(n,i)}, \quad k = 1, 2, \ldots, m-1, \\ e_{j,0}^{(n,i)} \equiv 0, \quad e_{j,m}^{(n,i)} \equiv 0. \end{cases}
$$

Here, there are M sets of q variables, $\{q_{i,k}\}_{k=1}^{m}$ for $i = 0, 1, \ldots, M-1$ and N sets of e variables, $\{e_{j,k}\}_{k=1}^{m-1}$ for $j = 0, 1, \ldots, N-1$. The discrete time index of $q_{i,k}$ (resp. $e_{j,k}$) consists of the main index n and the sub index j (resp. i), which takes the value between 0 and $N-1$ (resp. $M-1$). At the beginning of step n, j (resp. i) is set to zero and it is incremented by one when $q_{i,k}^{(n,j)}$ (resp. $e_{j,k}^{(n,i)}$) "interacts" with $e_{j,k}^{(n,i)}$ (resp. $q_{i,k}^{(n,j)}$) through (1.7). After it interacts with $e_{N-1,k}^{(n,i)}$ (resp. $q_{M-1,k}^{(n,j)}$), n is incremented by one and j (resp. i) is set to zero. Thus, we have

$$\begin{cases} q_{i,k}^{(n,N)} = q_{i,k}^{(n+1,0)}, & i = 0,1,\ldots,M-1, \quad k = 1,2,\ldots,m, \\ e_{j,k}^{(n,M)} = e_{j,k}^{(n+1,0)}, & j = 0,1,\ldots,N-1, \quad k = 1,2,\ldots,m-1. \end{cases} \tag{1.8}$$

To see that (1.7) with (1.8) is a generalization of (1.4) and (1.5), let us rewrite $q_{i,k}^{(n,j)}$ and $e_{j,k}^{(n,i)}$ as:

$$\begin{cases} q_{i,k}^{(n,j)} = q_k^{(MNn+jM+i)}, \\ e_{j,k}^{(n,i)} = e_k^{(MNn+iN+j)}. \end{cases} \tag{1.9}$$

Note that this is consistent with (1.8). Now, consider the case of $N = 1$. Substituting (1.9) into (1.7) and letting $N = 1$ and $j = 0$ gives

$$\begin{cases} q_k^{(Mn+M+i)} + e_{k-1}^{(Mn+i+1)} = q_k^{(Mn+i)} + e_k^{(Mn+i)}, & k = 1,2,\ldots,m, \\ q_k^{(Mn+M+i)} e_k^{(Mn+i+1)} = q_{k+1}^{(Mn+i)} e_k^{(Mn+i)}, & k = 1,2,\ldots,m-1, \\ e_0^{(Mn+i)} \equiv 0, \quad e_m^{(Mn+i)} \equiv 0. \end{cases} \tag{1.10}$$

By rewriting $Mn + i$ as n, we recover (1.4). Equation (1.5) can also be obtained by letting $M = 1$. Note that (1.7) (written in the so-called differential form) is referred to as the multiple dqd algorithm in [10]. It can also be derived from a reduction of the two-dimensional discrete Toda equation [5].

Equation (1.7) is similar to the two dhToda equations (1.4) and (1.5), however (1.7) differs from them in that it involves two types of arbitrary parameters instead of one. Obviously, (1.7) with $N = 1$ and with $M = 1$ are respectively equal to the dhToda-I equation (1.4) and the dhToda-II equation (1.5). Thus, we can regard (1.7) as a generalization of the two dhToda equations (1.4) and (1.5). We hereinafter call (1.7) the dhToda-III equation to distinguish it from the two dhToda equations (1.4) and (1.5). Note here that the first equation of the dhToda-III equation (1.7) can be rewritten using the second equation repeatedly as $q_k^{(n+M)} = (\prod_{j=1}^{k} q_j^{(n)} / \prod_{j=1}^{k-1} q_j^{(n+M)}) + e_k^{(n)}$. Thus, by replacing $q_k^{(n)}$ and $e_k^{(n)}$ with $\exp(-Q_k^{(n)}/\varepsilon)$ and $\exp(-E_k^{(n)}/\varepsilon)$, respectively, taking the logarithm of both sides, multiplying them by ε, and taking the limit $\varepsilon \to +0$, we obtain the ultra-discretization of the dhToda-III (udhToda-III) equation:

$$\begin{cases} Q_{i,k}^{(n,j+1)} = \min\left(\sum_{k'=1}^{k} Q_{i,k'}^{(n,j)} - \sum_{k'=1}^{k-1} Q_{i,k'}^{(n,j+1)}, E_{j,k}^{(n,i)} \right), & k = 1,2,\ldots,m, \\ E_{j,k}^{(n,i+1)} = Q_{i,k+1}^{(n,j)} + E_{j,k}^{(n,i)} - Q_{i,k}^{(n,j+1)}, & k = 1,2,\ldots,m-1, \\ E_{j,0}^{(n,i)} := +\infty, \quad E_{j,m}^{(n,i)} := +\infty. \end{cases} \tag{1.11}$$

In this paper, we propose a new nBBS associated with the udhToda-III equation (1.11), and then derive the conserved quantities of the resulting nBBS. Moreover, we clarify the relationship of the resulting nBBS to the hungry ε-BBS presented in Kobayashi and Tsujimoto [6].

The remainder of this paper is organized as follows. In Section 2, we first design a new nBBS with numbered both boxes and balls, and then associate it with the udhToda-III equation (1.11). In Section 3, based on the relationship to eigenvalue problem and the correspondence to combinatorial representation, we next derive conserved quantities of the resulting nBBS. In Section 4, we show that discrete-time evolutions and conserved quantities of the hungry ε-BBS can be grasped from the viewpoint of the resulting nBBS. Finally, we give concluding remarks.

§ 2. Box and ball system with numbered both boxes and balls

In this section, we propose a box-ball system with numbered boxes and balls that has the udhToda-III equation (1.11) as its equation of motion. As in the basic BBS, we assume that an infinite number of boxes are arranged in a straight line. We assume that only one ball can be put in one box, and the number of balls is finite. The types of balls and boxes are distinguished by identification numbers, which take a value between 0 and $M-1$ for balls and between 0 and $N-1$ for boxes. We here emphasize that M and N are arbitrary parameters corresponding to those in the udhToda-III equation (1.11). In the following, we refer to a set of balls in consecutive boxes as a ball group and a set of consecutive empty boxes between two ball groups as a box array.

At discrete time n, we assign identification numbers to boxes and balls so that the following four conditions hold:

(a) Every ball has an identification number, which is one of $0, 1, \dots, M-1$.

(b) For each i, $0 \le i \le M-1$, each ball group contains one or more balls with identification number i, and the balls in each ball group are lined up so that their identification numbers are in ascending order from the left.

(c) Every box between ball groups has an identification number, which is one of $0, 1, \dots, N-1$. Only boxes between the leftmost and rightmost ball groups are numbered.

(d) For each j, $0 \le j \le N-1$, each box array between ball groups contains one or more boxes with identification number j, and the boxes in each box array are lined up so that their numbers are in ascending order from the left.

We hereinafter omit "from the left" in describing the order of each box array and each ball group from the left. Now, we define the rules for discrete time evolution of our new nBBS with numbered balls and boxes. The time evolution of the nBBS from n to $n+1$ consists of MN substeps. At the $(jM+i)$th substep $(0 \le i \le M-1,\ 0 \le j \le N-1)$, we move the boxes and balls as follows:

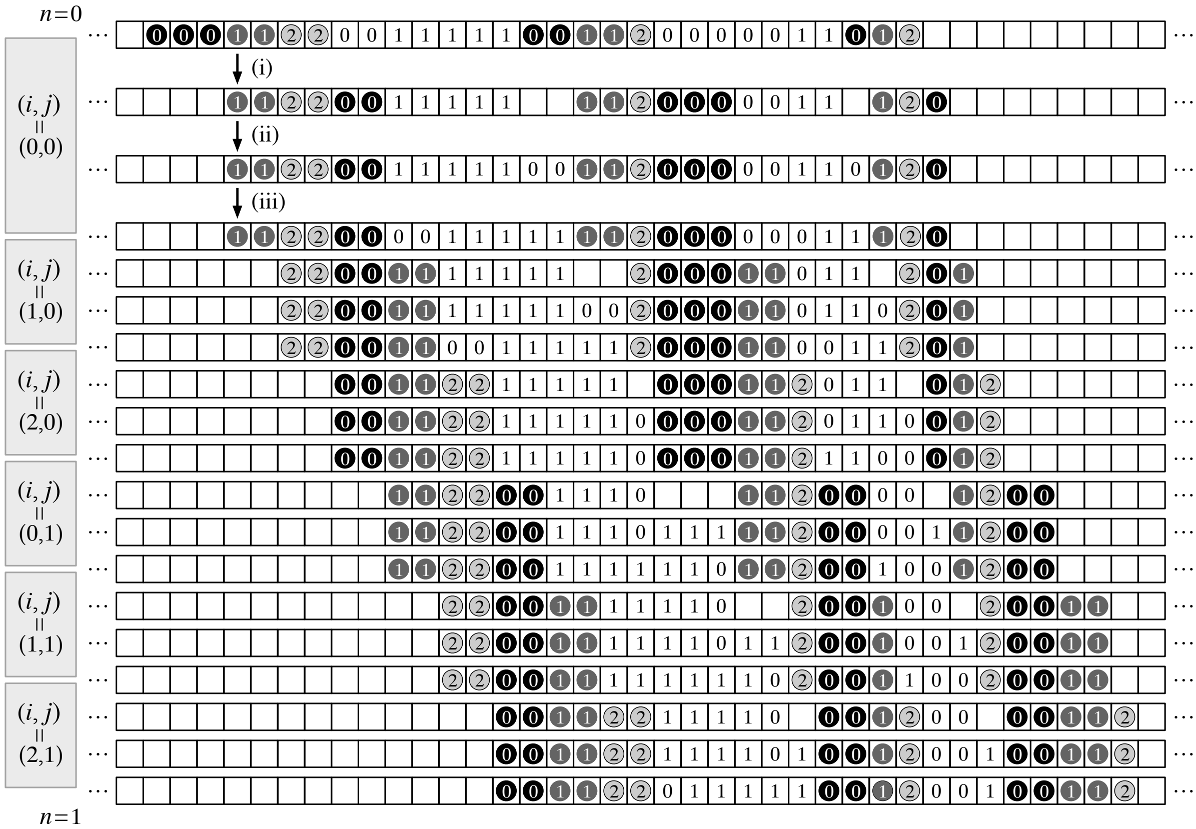

Figure 1. An example of discrete time evolution from $n = 0$ to $n = 1$ in the case where $M = 3$ and $N = 2$.

(i) Starting from the leftmost ball with number i, move each ball with number i, one by one, to the nearest right empty box numbered j or without an identification number.

(ii) After moving the balls numbered i, delete identification number j from boxes newly filled with a ball. Moreover, assign the identification number j to boxes that become empty, except for boxes to the left of the leftmost ball group. (Boxes to the left of the leftmost ball group do not have a box identification number.)

(iii) If $i < M - 1$, in each box array, gather all boxes with number j to the left end of the box array. If $i = M - 1$, in each box array, gather all boxes with number j to the right end of the box array.

We refer to the nBBS defined by the conditions (a)–(d) and the rules (i)–(iii) as *nBBS-III*. Figure 1 shows an example of discrete time evolution from $n = 0$ to $n = 1$ of nBBS-III. Now, we show a lemma concerning discrete time evolution of nBBS-III. To this end, we define new conditions $(\mathrm{b})_i$ and $(\mathrm{d})_j$, which are slight generalizations of the conditions (b) and (d), respectively.

$(\mathrm{b})_i$ For each p, $0 \leq p \leq M-1$, each ball group contains one or more balls with identification number p, and the balls in each ball group are lined up so that their identification numbers are in the order of $i, i+1, \ldots, M-1, 0, 1, \ldots, i-1$.

$(\mathrm{d})_j$ For each q, $0 \leq q \leq N-1$, each box array between ball groups contains one or more boxes with identification number q, and the boxes in each box array are lined up so that their numbers are in the order of $j, j+1, \ldots, N-1, 0, 1, \ldots, j-1$.

Lemma 2.1. *Consider the $(jM+i)$th substep during discrete time evolution from n to $n+1$, where $0 \leq i \leq M-1$ and $0 \leq j \leq N-1$. Let $i' = \mathrm{mod}(i+1, M)$ and $j' = \lfloor (jM+i+1)/M \rfloor$ where $\lfloor \cdot \rfloor$ denotes the greatest integer part of a real number. If the conditions (a), $(b)_i$, (c) and $(d)_j$ hold and the number of ball groups is m at the beginning of the substep, then the conditions (a), $(b)_{i'}$, (c) and $(d)_{j'}$ hold at the end of the substep and the number of ball groups remains unchanged.*

Proof. It is clear that (a) holds because the identification number of each ball does not change throughout discrete time evolution. It is also clear that (c) holds because the boxes that become empty are given number j unless it lies to the left of the leftmost ball group, the boxes newly filled with a ball are deprived of the numbers, and the numbers of other boxes are unchanged.

Now, we show that m remains unchanged by induction. By assumption, the balls to be moved, those with number i, are at the left end of each group. Among them, the leftmost one is moved first, the second leftmost one next, and so on. The boxes that become empty due to these movements are not filled with another ball at this substep. Also, balls with numbers other than j exist in each group and they are not moved. From these facts, it is clear that no ball groups vanish nor split into two or more groups due to the removal of the balls. On the other hand, the removed balls are then attached at the right end of some ball group, because the boxes with number j lie there. Thus, the number of ball groups does not increase. Also, since there are boxes with numbers other than j between any two ball groups and they remain empty at this substep, it does not occur that two ball groups merge due to this attachment. From these facts, we can conclude that the number of ball groups remains unchanged after the substep.

Next, we show that $(\mathrm{b})_{i'}$ holds at the end of the substep. Consider the kth ball group. The empty box with number j just at the right of this ball group is filled with a ball with number i, which comes either from this ball group or from one of the preceding ball groups. Since the balls with number other than i do not move, the kth ball group still has one or more balls with number p for $0 \leq p \leq M-1$. Also, it is clear that the balls are lined up so that their identification numbers are in the order of $i+1, i+2, \ldots, M-1, 0, 1, \ldots, i$. This shows that $(\mathrm{b})_{i'}$ holds.

Finally, we show $(\mathrm{d})_{j'}$. Let $2 \leq k \leq m$. When the balls with number i belonging to the kth ball group are moved, the boxes that stored them become empty, are given number j, and become part of the $(k-1)$th box array. Since the boxes with numbers other than j remain unchanged, the $(k-1)$th box array still has one or more boxes with number q for $0 \leq q \leq N-1$. Also, due to rule (iii), the boxes are lined up so that their identification numbers are in the order of $j, j+1, \ldots, N-1, 0, 1, \ldots, j-1$ when $0 \leq i \leq M-2$ and in the order of $j+1, j+2, \ldots, N-1, 0, 1, \ldots, j$ when $i = M-1$. Thus, $(\mathrm{d})_{j'}$ holds. □

By using Lemma 2.1 repeatedly, we obtain the following theorem.

Theorem 2.2. *Suppose that the conditions (a)–(d) hold at discrete time n and the number of ball groups is m. Then, these conditions hold also at discrete time $n+1$ and the number of ball groups remains unchanged.*

Since the number of ball groups (and therefore that of box arrays) stays constant, we can describe the state of nBBS-III by specifying the number of balls of each identification number in each ball group and the number of boxes of each identification number in each box array. Noting that the number of balls with identification number i changes only at the ith, $(M+i)$th, …, $((N-1)M+i)$th substeps, we denote the number of balls with number i in the kth ball group at the beginning of the $(jM+i)$th substep the transition under discrete time evolution from n to $n+1$ by $Q_{i,k}^{(n,j)}$. Similarly, noting that the number of boxes with identification number j changes only at the (Mj)th, $(Mj+1)$th, …, $(Mj+M-1)$th substeps, we denote the number of boxes with number j in the kth box array at the beginning of the $(jM+i)$th substep of the transition under discrete time evolution from n to $n+1$ by $E_{j,k}^{(n,i)}$.

Now, let us consider the $(jM+i)$th substep. In this substep, only balls with number i and boxes with number j are used. If we focus on these balls and boxes, their movements are exactly the same as those of balls and boxes in the standard BBS in which neither balls nor boxes are numbered. Thus, the numbers of the balls and boxes before and after the $(jM+i)$th substep, that is, $\{Q_{i,k}^{(n,j)}\}_{k=1}^{m}$, $\{E_{j,k}^{(n,i)}\}_{k=1}^{m-1}$ and $\{Q_{i,k}^{(n,j+1)}\}_{k=1}^{m}$, $\{E_{j,k}^{(n,i+1)}\}_{k=1}^{m-1}$, should satisfy the same equation of motion as that of the standard BBS. The equation is exactly (1.11). Therefore, we arrive at the following theorem.

Theorem 2.3. *The equation of motion of nBBS-III is the udhToda-III equation (1.11).*

§ 3. Conserved quantity

Fukuda [2] derived a conserved quantity of nBBS-I with the help of a combinatorial technique. In this section, along the same line, we derive combinatorial representation of a conserved quantity of nBBS-III.

§ 3.1. Conserved quantity of nBBS-I

We begin by reviewing the main result of [2]. Let us consider nBBS-I in which each ball has an identification number (color) between 0 and $M-1$. We assign index 0 to one of the boxes and then assign indices $1, 2, \ldots$ to the boxes to the right of it, starting from the nearest box. Also, assign indices $-1, -2, \ldots$ to the boxes to the left of it, staring from the nearest box. These indices denote the position of each box and do not change over time. They should not be confused with the identification numbers of the boxes defined in nBBS-III, which change during time evolution.

Now, consider the status of this nBBS-I at discrete time n. Let $L = \sum_{i=0}^{M-1} Q^{(n+i)}$ be the total number of balls and let the indices of boxes storing a ball be denoted by $i_1, i_2, \ldots, i_L$. Also, let the color (identification number) of the ball stored in a box with index i_j be a_j. Then, we can uniquely represent the status of the nBBS-I using the so-called bi-word:

$$w = \begin{pmatrix} i_1 \; i_2 \; \cdots \; i_L \\ a_1 \; a_2 \; \cdots \; a_L \end{pmatrix}.$$

Using the Robinson-Schensted-Knuth correspondence, we can construct one-to-one correspondence between w and a pair (P, Q) of the semi-standard Young tableaux. To construct the P symbol, we start from an empty Young tableaux and repeat, for $j = 1, 2, \ldots, L$, adding a new box with element a_j at the right end of the top row and reconstructing the whole Young tableaux according to an algorithm called *row bumping*. To construct the Q symbol, we prepare another empty Young tableau and repeat adding a new box at the same position where a new box of the P symbol appeared as a result of row bumping and inserting i_j into the box. Thus, P and Q become Young tableaux of the same shape. It is important to note that the P symbol can be constructed solely from the bottom row of w, whereas construction of the Q symbol requires both the top and bottom rows of w.

The P and Q symbols are defined for each discrete time n. Here, we consider discrete time evolution from n to $n+M$, because this is a period during which all the balls are moved exactly once. Then, the following theorem holds.

Theorem 3.1 (Fukuda [2]). *Consider discrete time evolution of P and Q constructed from that of nBBS-I. Then, the following two holds:*

(i) The P symbol remains unchanged.

(ii) The Q symbol evolves independently of the P symbol.

Thus, the P symbol constructed from the bi-word w gives a conserved quantity of nBBS-I. It is to be noted that the bi-word w is determined uniquely if $\{Q_k^{(n)}\}_{k=1}^m$, $\{Q_k^{(n+1)}\}_{k=1}^m$, ..., $\{Q_k^{(n+M-1)}\}_{k=1}^m$, $\{E_k^{(n)}\}_{k=1}^{m-1}$ and the index i_1 of the box containing the leftmost ball at discrete time n are given. But, $\{E_k^{(n)}\}_{k=1}^{m-1}$ and i_1 are used only to determine the top row of w. Thus, the bottom row of w, and hence the P symbol, are determined solely by $\{Q_k^{(n)}\}_{k=1}^m$, $\{Q_k^{(n+1)}\}_{k=1}^m$, ..., $\{Q_k^{(n+M-1)}\}_{k=1}^m$.

§ 3.2. Conserved quantity of nBBS-III

We now turn to the case of the nBBS-III. Consider the (jM)th, $(jM+1)$th, ..., $(jM+M-1)$th substeps of the transition under discrete time evolution from n to $n+1$, where $0 \le j \le N-1$. At these substeps, balls with identification numbers $0, 1, \ldots, M-1$ are moved using only boxes with identification number j. Thus, if we focus only on these balls and boxes, the dynamics of the system is exactly the same as that of nBBS-I with M kinds of balls. Then, it follows from Theorem 3.1 that the P symbol determined by $\{Q_{1,k}^{(n,j)}\}_{k=1}^m$, $\{Q_{2,k}^{(n,j)}\}_{k=1}^m$, ..., $\{Q_{M,k}^{(n,j)}\}_{k=1}^m$ and that determined by $\{Q_{1,k}^{(n,j+1)}\}_{k=1}^m$, $\{Q_{2,k}^{(n,j+1)}\}_{k=1}^m$, ..., $\{Q_{M,k}^{(n,j+1)}\}_{k=1}^m$ are the same. Since this holds for $j = 0, 1, \ldots, N-1$ and for any n, we obtain the following theorem concerning the P symbol in nBBS-III.

Theorem 3.2. *The P symbol determined by the variables $\{Q_{1,k}^{(n,j)}\}_{k=1}^m$, $\{Q_{2,k}^{(n,j)}\}_{k=1}^m$, ..., $\{Q_{M,k}^{(n,j)}\}_{k=1}^m$ of nBBS-III remains the same regardless of the value of j and n.*

This gives a combinatorial conserved quantity of nBBS-III. In the example shown in Figure 1, the bi-words at discrete time $n = 0$ and at discrete time $n = 1$ are respectively given as:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 15 & 16 & 17 & 18 & 19 & 27 & 28 & 29 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 14 & 15 & 16 & 17 & 18 & 19 & 26 & 27 & 28 & 29 & 33 & 34 & 35 & 36 & 37 \\ 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

Thus, we can easily check that the P symbols at discrete time $n = 0$ and at discrete time $n = 1$ are the same, and are both expressed as:

0	0	0	0	0	0	1	1	2
1	1	1	2	2				
2								

§ 4. Relationship to the hungry ϵ-BBS

In this section, we relate the hungry ε-BBS to the nBBS-III, and then derive a conserved quantity of the hungry ε-BBS from the viewpoint of the nBBS-III.

According to Kobayashi-Tsujimoto [6], the hungry ε-BBS is designed based on ultra-discretization of a discrete integrable system which can be represented in matrix form as:

$$(\mathcal{L}_1^{(n+1)})^{-1}\mathcal{L}_2^{(n+1)}\mathcal{R}^{(n+M)} = \mathcal{R}^{(n)}(\mathcal{L}_1^{(n)})^{-1}\mathcal{L}_2^{(n)}, \tag{4.1}$$

$$\mathcal{L}_1^{(n)} := \begin{pmatrix} 1 & & & & \\ -e_{\varepsilon,1}^{(n)} & 1 & & & \\ & -e_{\varepsilon,2}^{(n)} & 1 & & \\ & & \ddots & \ddots & \\ & & & -e_{\varepsilon,m-1}^{(n)} & 1 \end{pmatrix}, \quad \mathcal{L}_2^{(n)} := \begin{pmatrix} 1 & & & & \\ \bar{e}_{\varepsilon,1}^{(n)} & 1 & & & \\ & \bar{e}_{\varepsilon,2}^{(n)} & 1 & & \\ & & \ddots & \ddots & \\ & & & \bar{e}_{\varepsilon,m-1}^{(n)} & 1 \end{pmatrix}, \tag{4.2}$$

$$\mathcal{R}^{(n)} := \begin{pmatrix} q_1^{(n)} & 1 & & & \\ & q_2^{(n)} & 1 & & \\ & & q_3^{(n)} & \ddots & \\ & & & \ddots & 1 \\ & & & & q_m^{(n)} \end{pmatrix}, \tag{4.3}$$

where $e_{\varepsilon,\ell}^{(n)} := \varepsilon_\ell e_\ell^{(n)}$, $\bar{e}_{\varepsilon,\ell}^{(n)} := (1-\varepsilon_\ell)e_\ell^{(n)}$, and ε_ℓ is a constant whose value is 0 or 1. Let $\mathcal{L}_{1,\ell}^{(n)}$ be a lower bidiagonal matrix whose subdiagonal entries are 0 except for the $(\ell+1,\ell)$ entry:

$$\mathcal{L}_{1,\ell}^{(n)} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & e_{\varepsilon,\ell}^{(n)} & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Then, we can decompose the inverse matrix $(\mathcal{L}_1^{(n)})^{-1}$ in product form as:

$$(\mathcal{L}_1^{(n)})^{-1} = \begin{pmatrix} 1 & & & & \\ -e_{\varepsilon,1}^{(n)} & 1 & & & \\ & -e_{\varepsilon,2}^{(n)} & 1 & & \\ & & \ddots & \ddots & \\ & & & -e_{\varepsilon,m-1}^{(n)} & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & -e_{\varepsilon,m-1}^{(n)} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -e_{\varepsilon,m-2}^{(n)} & 1 & \\ & & & & 1 \end{pmatrix}^{-1}$$

$$\times \cdots \times \begin{pmatrix} 1 & & & & \\ -e_{\varepsilon,1}^{(n)} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}^{-1}$$

$$= \mathcal{L}_{1,m-1}^{(n)} \mathcal{L}_{1,m-2}^{(n)} \cdots \mathcal{L}_{1,1}^{(n)}. \tag{4.4}$$

We now assume that $\varepsilon_\ell = 1$ for $\ell = \ell_K, \ell_{K-1}, \ldots, \ell_1$, where $\ell_K > \ell_{K-1} > \cdots > \ell_1$, and $\varepsilon_\ell = 0$ otherwise. Then, $\mathcal{L}_{1,\ell}^{(n)}$ with $\ell \neq \ell_K, \ell_{K-1}, \ldots, \ell_1$ are the identity matrices. Thus, using (4.4), we can simplify (4.1) to a tractable form that does not involve the inverse as:

$$\mathcal{L}_{1,\ell_K}^{(n+1)} \mathcal{L}_{1,\ell_{K-1}}^{(n+1)} \cdots \mathcal{L}_{1,\ell_1}^{(n+1)} \mathcal{L}_2^{(n+1)} \mathcal{R}^{(n+M)} = \mathcal{R}^{(n)} \mathcal{L}_{1,\ell_K}^{(n)} \mathcal{L}_{1,\ell_{K-1}}^{(n)} \cdots \mathcal{L}_{1,\ell_1}^{(n)} \mathcal{L}_2^{(n)}. \tag{4.5}$$

By defining $\mathcal{R}^{(n,0)} = \mathcal{R}^{(n)}$ and introducing intermediate variables $\mathcal{R}^{(n,1)}, \ldots, \mathcal{R}^{(n,K)}$, we can rewrite the LR transformation (4.5) as a sequence of LR transformations of bidiagonal matrices:

$$\begin{cases} \mathcal{L}_{1,\ell_K}^{(n+1)} \mathcal{R}^{(n,1)} = \mathcal{R}^{(n,0)} \mathcal{L}_{1,\ell_K}^{(n)}, \\ \mathcal{L}_{1,\ell_{K-1}}^{(n+1)} \mathcal{R}^{(n,2)} = \mathcal{R}^{(n,1)} \mathcal{L}_{1,\ell_{K-1}}^{(n)}, \\ \quad \vdots \\ \mathcal{L}_{1,\ell_1}^{(n+1)} \mathcal{R}^{(n,K)} = \mathcal{R}^{(n,K-1)} \mathcal{L}_{1,\ell_1}^{(n)}, \\ \mathcal{L}_2^{(n+1)} \mathcal{R}^{(n+M,0)} = \mathcal{R}^{(n,K)} \mathcal{L}_2^{(n)}. \end{cases} \tag{4.6}$$

Let n be a multiple of M and consider (4.6) for n, $n+1$, …, $n+M-1$. These constitute one period of the hungry ε-BBS, in which all M kinds of balls are moved once. Now, we write $n = n'M$ and apply subscript-superscript swapping as follows:

$$\begin{cases} L_j^{(n',i)} := \mathcal{L}_{1,\ell_{K-j}}^{(n+i)}, \quad i = 0, 1, \ldots, M-1, \quad j = 0, 1, \ldots, K-1, \\ L_K^{(n',i)} := \mathcal{L}_2^{(n+i)}, \quad i = 0, 1, \ldots, M-1, \\ R_i^{(n',j)} := \mathcal{R}^{(n+i,j)}, \quad i = 0, 1, \ldots, M-1, \quad j = 0, 1, \ldots, K. \end{cases}$$

Using these variables, (4.6) for n, $n+1$, …, $n+M-1$ can be written succinctly as:

$$L_j^{(n',i+1)} R_i^{(n',j+1)} = R_i^{(n',j)} L_j^{(n',i)}, \quad i = 0, 1, \ldots, M-1, \quad j = 0, 1, \ldots, K, \tag{4.7}$$

with the conditions:

$$\begin{cases} L_j^{(n',M)} = L_j^{(n'+1,0)}, & j = 0, 1, \ldots, K, \\ R_i^{(n',K+1)} = R_i^{(n'+1,0)}, & i = 0, 1, \ldots, M-1. \end{cases} \tag{4.8}$$

By writing (4.7) and (4.8) entry-by-entry, we obtain (1.7) and (1.8) with $M = K + 1$ and $n = n'$. Note that (4.7) (or (1.7)) constitutes a doubly nested loop over i and j, and in the original ε-BBS, the loop over j was the inner loop, as can be seen from (4.5). But the order of computation of (4.7) is arbitrary as long as $L_j^{(n',i)}$ and $R_i^{(n',j)}$ are computed before $L_j^{(n',i+1)}$ and $R_i^{(n',j+1)}$ for all i and j. Thus, we can exchange the loops and make the loop over i (ball identification numbers) the inner one. This allows us to regard the hungry ε-BBS, which is derived from the matrix representation (4.1), as a special case of the nBBS-III with $K + 1$ kinds of boxes and M kinds of balls.

Below, we rewrite n' as n. Note that each $L_j^{(n,i)}$ ($i = 0, 1, \ldots, M - 1$, $j = 0, 1, \ldots, K - 1$) has a nonzero entry only in the $(\ell_{K-j} + 1, \ell_{K-j})$ entry on the sub-diagonal. From the viewpoint of the nBBS-III, we see that, for each $j = 0, 1, \ldots, K-1$, there are a finite number of boxes numbered j in the ℓ_{K-j}th box array and an infinite number of boxes numbered j in other box arrays. This means that when the balls are moved using boxes numbered j, the ball groups other than the ℓ_{K-j}th and $(\ell_{K-j}+1)$th ones simply move to the empty boxes to the right of them, without changing their lengths. Thus, effectively, we only need to consider the interaction between the ℓ_{K-j}th ball group and the ℓ_{K-j}th box array numbered j, which results in changes in the lengths of the ℓ_{K-j}th and $(\ell_{K-j} + 1)$th ball groups and the ℓ_{K-j}th box array. On the other hand, there are a finite number of boxes numbered K in the ℓth box array when $\ell \neq \ell_{K-j}$, $j = 0, 1, \ldots, K-1$ and an infinite number of boxes numbered K in the ℓth box array otherwise. Let $\{\bar{\ell}_1, \bar{\ell}_2, \ldots, \bar{\ell}_{m-1-K}\} = \{1, 2, \ldots, m-1\} \backslash \{\ell_1, \ell_2, \ldots, \ell_K\}$ and $\bar{\ell}_1 < \bar{\ell}_2 < \cdots < \bar{\ell}_{m-1-K}$. Then, when the balls are moved using boxes numbered K, we need to care about only the interactions between the ℓth ball group and the ℓth box array for $\ell = \bar{\ell}_1, \bar{\ell}_2, \ldots, \bar{\ell}_{m-1-K}$.

Until now, we have assumed that there are $K + 1$ kinds of boxes. However, as is clear from the explanation above, for each j, $0 \leq j \leq K-1$, there is only one box array such that the number of boxes numbered j in it is finite. Hence, for $0 \leq j \leq K - 1$, instead of saying "use boxes numbered j", we can say "use the finite number of boxes in the ℓ_{K-j}th box array". After this has been done for $0 \leq j \leq K - 1$, we use the finite number of boxes in the ℓth box array for $\ell = \bar{\ell}_1, \bar{\ell}_2, \ldots, \bar{\ell}_{m-1-K}$ in this order. Thus, we can do without the box identification numbers and instead consider a variant of nBBS-I in which the order of box arrays to be used is changed from the natural order. In summary, we arrive at the following alternative discrete-time evolution rule of the hungry ε-BBS.

Algorithm 1 Alternative discrete-time evolution rule of the hungry ε-BBS.

```
for i = 0, M − 1 do
    for j = 0, K − 1 do
        Move the balls numbered i according to the standard BBS rule assuming that
        only the ℓ_{K−j}th box array has finite length and the other box arrays have infinite
        lengths.
    end for
    Move the balls numbered i according to the standard BBS rule assuming that
    only the ℓ̄_1th, ℓ̄_2th, ..., ℓ̄_{m−1−K}th box arrays have finite length and the other box
    arrays have infinite lengths.
end for
```

Note that the loops over i and j can be interchanged, so that when a box identification number is specified, all M kinds of balls are moved successively using those boxes.

Since the hungry ε-BBS can be interpreted as a special case of nBBS-III, we immediately obtain its conserved quantity from Theorem 3.2 as follows.

Theorem 4.1. *Under discrete-time evolution according to Algorithm 1, the P symbol computed from ball sequences before discrete-time evolution coincides with that computed from ball sequences after discrete-time evolution.*

Theorem 4.1 is equivalent to Proposition 4.1 in Kobayashi and Tsujimoto [6]. Our proof above is an alternative one based on the results of [2] and an interpretation of the hungry ε-BBS as a special case of nBBS-III.

Example To show that the discrete-time evolution rule given as Algorithm 1 is equivalent to the rule given in [6], we show an example of time evolution below. Here, there are three kinds of balls and four ball groups ($M = 3, m = 4$) and $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $\varepsilon_3 = 0$. These settings, as well as the initial state of the system, are the same as those of the first example of Example 3.2 in [6]. In this example, we use a modified version of Algorithm 1 in which the loops over i and j are interchanged. Since only ε_2 is nonzero, we have $K = 1$ and $\ell_1 = 2$ and therefore $\bar{\ell}_1 = 1$ and $\bar{\ell}_2 = 3$. In the transition from n to $n + 1/2$, the three kinds of balls are moved using the 2nd box array. The 1st and 3rd box arrays are treated as having infinite lengths (denoted by double line) and their original lengths do not change after the transition. In the transition from $n + 1/2$ to $n+1$, the three kinds of balls are moved using the 1st and 3rd box arrays. The 2nd box array is treated as having an infinite length (denoted by double line) and its original length does not change after the transition. By comparing the states at $n = 1$, 2 and 3 with those of [6], we see that our rule gives identical results.

```
n=0:       _1111222===112233____133=====11223________________
n=1/2:     _1111222___11223==1333_____11223__________________
n=1:       ________11122===1112223__1333======11223__________
n=1+1/2:   ________11122___112===11223333______11223_________
n=2:       _____________11122=112___11223333===11223_________
n=2+1/2:   _____________11122_112========11223333___11223____
n=3:       __________________122=11112________1122333=112233_
```

§ 5. Concluding remarks

In this paper, we proposed a new numbered box and ball system (nBBS) in which both boxes and balls are numbered. We first designed rules of discrete time evolutions of the new nBBS, and showed that its dynamics is described by an extension of the ultradiscrete hungry Toda (udhToda) equations corresponding to the nBBSs in which either boxes or balls are numbered. We next focused on a pair of semi-standard Young tableaux which represent the status of our nBBS, and showed that one of the tableaux constitutes a conserved quantity of our nBBS under discrete-time evolutions, by slightly extending the approach for the simplest nBBS. We also showed that the matrix LR transformation associated with the hungry ε-BBS is a specialization of that associated with our nBBS, and thereby derived an already known conserved quantity of the hungry ε-BBS from the viewpoint of our nBBS.

In fact, there is another version of nBBS with numbered boxes and balls, and we have described its elementary properties in our previous paper. Our future work is thus to enrich the study of this nBBS by, for example, deriving its conserved quantities using the Young tableaux approach, and finding its relationships to other BBSs.

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