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**B94** (2023), 037–053

# The blow-up curve for a weakly coupled system of semilinear wave equations with nonlinearities of derivative-type

By

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**Abstract**

In this paper, we study a blow-up curve for a weakly coupled system of semilinear wave equations with nonlinearities of derivative type in one space dimension. Employing the idea of Caffarelli and Friedman [1], we prove the blow-up curve becomes Lipschitz continuous under suitable initial conditions. Moreover, we show the blow-up rates of the solution of the wave equations.

## § 1. Introduction

We consider the following weak coupled system of semilinear wave equations with nonlinearities of derivative type:

$$
\begin{cases}
\partial_t^2 u_1 - c_1^2 \partial_x^2 u_1 = 2^{p_1} (\partial_t u_2)^{p_1}, & x \in \mathbb{R}, \quad t > 0, \\
\partial_t^2 u_2 - c_2^2 \partial_x^2 u_2 = 2^{p_2} (\partial_t u_1)^{p_2}, & x \in \mathbb{R}, \quad t > 0, \\
u_1(x,0) = u_{1,1}(x), \quad \partial_t u_1(x,0) = u_{1,2}(x), & x \in \mathbb{R}, \\
u_2(x,0) = u_{2,1}(x), \quad \partial_t u_2(x,0) = u_{2,2}(x), & x \in \mathbb{R},
\end{cases}
\tag{1.1}
$$

where $c_1, c_2 > 0$, $u_{i,j}$ $(i,j = 1,2)$ are given smooth functions and

$$p_i > 1 \text{ is a constant such that the function } s^{p_i} \text{ for } s \geq 0 \text{ is of class } C^4$$

Received Feburary 17, 2023. Revised July 2, 2023.
2020 Mathematics Subject Classification(s):
*Key Words*: *system of semilinear wave equations, one dimension, classical solution, blow-up curve.*
This work was supported by JSPS Grant-in-Aid for Early-Career Scientists, 18K13447
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for $i = 1, 2$. Through this paper, we assume $c_1 \geq c_2$.

We only consider (1.1) in the following time-space region. Let $R^*$ and $T^*$ be positive constants and set

$$K_{R^*,T^*,i} = \bigcup_{x \in B_{R^*}} K_{-,i}(x, T^*) \quad (i = 1, 2)$$

where

$$B_r = \{x \in \mathbb{R} \mid |x| < r\} \ \ (r > 0),$$
$$K_{-,i}(x_0, t_0) = \big\{(x, t) \in \mathbb{R}^2 \mid |x - x_0| < c_i(t_0 - t),\ t > 0\big\} \quad (i = 1, 2).$$

We are interested $T(x)$, the maximal existence time of classical solutions of wave equations at $x \in B_{R^*}$. In the case of (1.1), $T(x)$ satisfies

$$T(x) = \sup\{t \in (0, T^*) \mid |\partial_t(u_1 + u_2)(x, t)| < \infty\} \quad \text{for} \quad x \in B_{R^*}.$$

In this paper, we call the set $\Gamma = \{(x, T(x)) \mid x \in B_{R^*}\}$ the blow-up curve. Below, we identify $\Gamma$ with $T$ itself.

Before we proceed to our problem, we recall some previous results. We begin with a single equation:

$$\partial_t^2 u - \partial_x^2 u = F(u, \partial_t u), \quad x \in \mathbb{R}, \quad t > 0.$$

The pioneering study of this topic was done by Caffarelli and Friedman[1]. They showed the blow-up curve $T$ becomes continuously differentiable under suitable initial conditions in the case of $F = |u|^p$. Godin[2] verified such results for $F = e^u$. Sasaki[3] proved the blow-up curve also becomes continuously differentiable under suitable initial conditions with $F = |\partial_t u|^p$.

On the other hand, Uesaka[4] considered systems of semilinear wave equations with nonlinearities which are not involved in the derivatives. He showed the blow-up curve becomes a Lipschitz function under suitable initial conditions.

Our result in this paper is to prove that the blow-up curve becomes Lipschitz continuous and the upper bounds of the Lipschitz constants depend on initial conditions of (1.1). Moreover, we show the blow-up rates of the solution for (1.1).

The article is organized as follows. In Section 2, we state our main results. In Section 3, we show that the monotonicity and the regularity of the solution of (1.1). In Section 4, we give a proof of the Lipschitz continuity of the blow-up curve and the blow-up rates of the solution for (1.1).

## § 2. Main results

Let $i' = 3 - i$ $(i = 1, 2)$. We rewrite (1.1) as the following:

$$\begin{cases} D_{i,-}\phi_i = (\phi_{i'} + \psi_{i'})^{p_i}, & x \in \mathbb{R}, \quad t > 0, \\ D_{i,+}\psi_i = (\phi_{i'} + \psi_{i'})^{p_i}, & x \in \mathbb{R}, \quad t > 0, \\ \phi_i(x,0) = f_i(x), \quad \psi_i(x,0) = g_i(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $\phi_i$ and $\psi_i$ are

$$\phi_i = \partial_t u_i + c_i \partial_x u_i, \quad \psi_i = \partial_t u_i - c_i \partial_x u_i \quad (i = 1, 2).$$

where $D_{i,\pm} = \partial_t \pm c_i \partial_x$ and $f_i(x) = u_{i,2}(x) + c_i \partial_x u_{i,1}(x)$, $g_i(x) = u_{i,2}(x) - c_i \partial_x u_{i,1}(x)$, $(i = 1, 2)$.

To state our main results, we give some assumptions on initial data. We consider the following system of ordinary equations:

$$\begin{cases} \dfrac{d\tilde{\phi}}{dt} = (\tilde{\phi} + \tilde{\psi})^p, & t > 0, \\ \dfrac{d\tilde{\psi}}{dt} = (\tilde{\phi} + \tilde{\psi})^p, & t > 0, \\ \tilde{\phi}(0) = \gamma_\phi, \quad \tilde{\psi}(0) = \gamma_\psi, \end{cases}$$

where $\gamma_\phi, \gamma_\psi \geq 1$ and $p = \min\{p_1, p_2\}$. We can easily see that there exists a positive constant $\tilde{T}$ such that

$$(\tilde{\phi} + \tilde{\psi})(t) \to \infty \quad \text{as} \quad t \to \tilde{T}.$$

For $i = 1, 2$, we pose the following assumptions.

**(A1)** $\tilde{T} < T^*$.

**(A2)** $f_i \geq \gamma_\phi$, $g_i \geq \gamma_\psi$ in $B_{R^* + c_1 T^*}$.

**(A3)** $f_i, g_i \in C^4(B_{R^* + c_1 T^*})$.

**(A4)** There exits a positive constant $\varepsilon_0$ such that

$$(\gamma_\phi + \gamma_\psi)^{p_i} \geq c_1 (2 + \varepsilon_0) \left( |\partial_x f_i| + |\partial_x g_i| \right) \quad \text{in} \quad B_{R^* + c_1 T^*}.$$

Our main result in this paper is as follows.

**Theorem 2.1.** *Let $R^*$ and $T^*$ be positive constants and assume* (A1)–(A4). *Then, there exists a unique Lipschitz function $T$ such that $0 < T(x) < T^*$ $(x \in B_{R^*})$ and a unique $(C^{3,1}(\Omega))^4$ solution $(\phi_1, \psi_1, \phi_2, \psi_2)$ of* (2.1) *satisfying*

$$\phi_i(x,t), \psi_i(x,t) \to \infty \quad as \quad t \to T(x) \quad (i=1,2)$$

*for any* $x \in B_{R^*+c_1T^*}$, *where* $\Omega = \{(x,t) \in \mathbb{R}^2 \mid x \in B_{R^*},\ 0 < t < T(x)\}$. *Moreover, there exist positive constants* $C_1, C_2$ *such that*

$$C_1(T(x)-t)^{-r_i} \le (\phi_i + \psi_i)(x,t) \le C_2(T(x)-t)^{-r_i} \quad in \quad \Omega$$

*where* $r_i = (p_i+1)/(p_i p_{i'} - 1)$ $(i=1,2)$.

*Remark.* We see that (2.1) is equivalent to (1.1). Let $u_i$ satisfy

$$u_i(x,t) = u_{i,1}(x) + \frac{1}{2}\int_0^t (\phi_i + \psi_i)(x,s)ds \quad (i=1,2).$$

Then, $u_i$ satisfies (1.1). This fact implies that

$$\partial_t u_i(x,t) \to \infty \quad \text{as} \quad t \to T(x) \quad (i=1,2).$$

## § 3. Preliminaries

For $i = 1,2$, let $\{(\phi_{i,n}, \psi_{i,n})\}_{n \in \mathbb{N}\cup\{0\}}$ be a sequence $\phi_{i,0} \equiv \gamma_\phi$ and $\psi_{i,0} \equiv \gamma_\psi$ and

$$\begin{cases} \phi_{i,n+1}(x,t) = f_i(x+c_it) + \displaystyle\int_0^t (\phi_{i',n} + \psi_{i',n})^{p_i}(x + c_i(t-s), s)ds \\ \psi_{i,n+1}(x,t) = g_i(x-c_it) + \displaystyle\int_0^t (\phi_{i',n} + \psi_{i',n})^{p_i}(x - c_i(t-s), s)ds \end{cases} \quad \text{for} \quad (x,t) \in K_{R^*,T^*,1},$$

$$\phi_{i,n+1}(x,0) = f_i(x), \quad \psi_{i,n+1}(x,0) = g_i(x) \quad \text{for} \quad x \in B_{R^*+c_1T^*}$$

for $n \in \mathbb{N}\cup\{0\}$. First we note the following fact.

**Lemma 3.1.** *Assume* (A2). *Then, we have*

$$\phi_{i,n+1} \ge \phi_{i,n} \ge \gamma_\phi, \quad and \quad \psi_{i,n+1} \ge \psi_{i,n} \ge \gamma_\psi \quad in \quad K_{R^*,T^*,1} \quad (i=1,2)$$

*for* $n \in \mathbb{N}\cup\{0\}$.

*Proof.* Since the proof is quite similar to the proof of Lemma 3.2, we omit the proof. □

The following Lemma plays an important role in the proof of the Lipschitz continuity of the blow-up curve $T$.

**Lemma 3.2.** *Assume* (A2)–(A4). *For* $i = 1,2$, *we have*

$$\partial_t\phi_{i,n} \ge c_1(1+\varepsilon_0)|\partial_x\phi_{i,n}|, \quad \partial_t\psi_{i,n} \ge c_1(1+\varepsilon_0)|\partial_x\psi_{i,n}| \quad in \quad K_{R^*,T^*,1} \tag{3.1}$$

*for* $n \in \mathbb{N}\cup\{0\}$.

*Proof.* Let $\lambda = c_1(1+\varepsilon_0)$ and

$$J_{i,n}^{\pm} = (\partial_t \pm \lambda\partial_x)\phi_{i,n}, \quad L_{i,n}^{\pm} = (\partial_t \pm \lambda\partial_x)\psi_{i,n} \quad (i=1,2).$$

for $n \in \mathbb{N}$. We show

$$J_{i,n}^{\pm},\ L_{i,n}^{\pm} \geq 0 \quad \text{in} \quad K_{R^*,T^*,1} \quad (i=1,2).$$

for $n \in \mathbb{N}$. First, since (A4) yields that

$$\begin{aligned} J_{i,1}^{\pm}(x,0) &= (\partial_t \pm \lambda\partial_x)\phi_{i,1}(x,0) \\ &= (c_i \pm c_1(1+\varepsilon_0))\partial_x\phi_{i,1}(x,0) + (\phi_{i',0}+\psi_{i',0})^{p_i}(x,0) \\ &\geq -c_1(2+\varepsilon_0)|\partial_x f_i(x)| + (\gamma_\phi+\gamma_\psi)^{p_i} \geq 0 \quad \text{in} \quad B_{R^*+c_1T^*} \quad (i=1,2). \end{aligned}$$

Moreover, we see that

$$D_{i,-}J_{i,1}^{\pm} = (\partial_t \pm \lambda\partial_x)D_{i,-}\phi_{i,1} = (\partial_t \pm \lambda\partial_x)(\phi_{i',0}+\psi_{i',0})^{p_i} = 0 \quad \text{in} \quad K_{R^*,T^*,1}$$

for $i=1,2$. Therefore, we have

$$J_{i,1}^{\pm} \geq 0 \quad \text{in} \quad K_{R^*,T^*,1} \quad (i=1,2). \tag{3.2}$$

Similarly we obtain

$$L_{i,1}^{\pm} \geq 0 \quad \text{in} \quad K_{R^*,T^*,1} \quad (i=1,2). \tag{3.3}$$

Next, we assume that (3.1) holds in the case of $n=k$. It follows from (A4) that

$$\begin{aligned} J_{i,k+1}^{\pm}(x,0) &= (\partial_t \pm \lambda\partial_x)\phi_{i,k+1}(x,0) \\ &= (c_i \pm c_1(1+\varepsilon_0))\partial_x\phi_{i,k+1}(x,0) + (\phi_{i',k}+\psi_{i',k})^{p_i}(x,0) \\ &\geq -c_1(2+\varepsilon_0)|\partial_x f_i(x)| + (\gamma_\phi+\gamma_\psi)^{p_i} \geq 0 \quad \text{in} \quad B_{R^*+c_1T^*} \quad (i=1,2). \end{aligned}$$

Moreover, we have

$$\begin{aligned} D_{i,-}J_{i,k+1}^{\pm} &= (\partial_t \pm \lambda\partial_x)D_{i,-}\phi_{i,k+1} = (\partial_t \pm \lambda\partial_x)(\phi_{i',k}+\psi_{i',k})^{p_i} \\ &= p_i(\phi_{i',k}+\psi_{i',k})^{p_i-1}(J_{i',k}^{\pm}+L_{i',k}^{\pm}) \geq 0 \quad \text{in} \quad K_{R^*,T^*,1} \quad (i=1,2). \end{aligned}$$

Hence we obtain

$$J_{i,k+1}^{\pm} \geq 0 \quad \text{in} \quad K_{R^*,T^*,1} \quad (i=1,2). \tag{3.4}$$

By the same arguments, we also obtain that

$$L_{i,k+1}^{\pm} \geq 0 \quad \text{in} \quad K_{R^*,T^*,1} \quad (i=1,2). \tag{3.5}$$

It follows from (3.2), (3.3), (3.4) and (3.5) that (3.1) for all $n \in \mathbb{N}$.

□

We shall construct a classical solution of (2.1). Fix $(x,t) \in K_{R^*,T^*,1}$. Thanks to Lemma 3.1, we can define $\phi$ and $\psi$ as

$$\phi_i(x,t) = \lim_{n\to\infty} \phi_{i,n}(x,t) = \left(\sup_{n\in\mathbb{N}} \phi_{i,n}(x,t)\right) \quad \psi_i(x,t) = \lim_{n\to\infty} \psi_{i,n}(x,t) = \left(\sup_{n\in\mathbb{N}} \psi_{i,n}(x,t)\right)$$

for $i = 1,2$. Moreover, thanks to Lemma 3.2, we can define the following function:

$$T(x) = \sup\{t \in (0,T^*) \mid \sum_{i=1,2} (\phi_i + \psi_i)(x,t) < \infty\} \quad \text{for} \quad x \in B_{R^*}.$$

**Lemma 3.3.** *Assume* (A2)–(A4). *Then,* $(\phi_1, \psi_1, \phi_2, \psi_2)$ *is a unique* $(C^{3,1}\Omega)^4$ *solution of* (2.1). *Here,* $\Omega = \left\{(x,t) \in \mathbb{R}^2 \mid x \in B_{R^*},\ 0 < t < T(x)\right\}$.

*Proof.* Through this proof, we fix $(x_0,t_0) \in \Omega$ and define $B_i(t)$ as

$$B_i(t) = \{x \in \mathbb{R} \mid |x_0 - x| < c_i(t_0 - t)\} \text{ for } (x_0,t_0) \in \Omega \quad (i = 1,2).$$

First, we show that there exits a positive constant $C_0$ such that

$$\sum_{i=1,2} \|(\phi_i + \psi_i)(\cdot,t)\|_{L^\infty(B_1(t))} \le C_0 \quad \text{for} \quad 0 \le t \le t_0 \tag{3.6}$$

by showing a contradiction. We assume that there exists $\tilde{t}$ such that

$$0 \le \tilde{t} \le t_0 \quad \text{and} \quad \sum_{i=1,2} \|(\phi_{i'} + \psi_{i'})(\cdot,\tilde{t})\|_{L^\infty(B_1(t))} = \infty. \tag{3.7}$$

Then, it follows from Lemma 3.2 and (3.7) that

$$\infty = \sum_{i=1,2} (\phi_i + \psi_i)(x_0,t_0) < \infty.$$

This is a contradiction. Hence, we obtain (3.6). Next, we shall show that $(\phi_1, \psi_2, \phi_2, \psi_2) \in \left(C^{3,1}(K_{-,1}(x_0,t_0))\right)^4$. For $i = 1,2$, we notice that

$$\begin{aligned}
&|(\phi_{i,n+1} - \phi_{i,n})(x,t)| \\
&= \Big| f(x + c_i t) + \int_0^t (\phi_{i',n} + \psi_{i',n})^{p_i}(x + c_i(t-s), s)ds \\
&\qquad - f(x + c_i t) - \int_0^t (\phi_{i',n-1} + \psi_{i',n-1})^{p_i}(x + c_i(t-s), s)ds \Big| \\
&\le \int_0^t \left|(\phi_{i',n} + \psi_{i',n})^{p_i} - (\phi_{i',n-1} + \psi_{i',n-1})^{p_i}\right| (x + c_i(t-s), s)ds \\
&\le p_i \int_0^t ((\phi_{i',n} + \psi_{i',n} + \theta(\phi_{i',n-1} + \psi_{i',n-1}))^{p_i - 1} \\
&\qquad \times \left|(\phi_{i',n} + \psi_{i',n}) - (\phi_{i',n-1} + \psi_{i',n-1})\right| (x + c_i(t-s), s)ds
\end{aligned}$$

for some $\theta \in (0,1)$. Hence, we have

$$\begin{aligned}
&|(\phi_{i,n+1} - \phi_{i,n})(x,t)| \\
&\le 2^{p_i-1} p_i \int_0^t \left( (\phi_{i',n} + \psi_{i',n})^{p_i-1} + (\phi_{i',n-1} + \psi_{i',n-1})^{p_i-1} \right) \\
&\qquad\qquad \times \left| (\phi_{i',n} + \psi_{i',n}) - (\phi_{i',n-1} + \psi_{i',n-1}) \right| (x + c_i(t-s), s) ds.
\end{aligned}$$

for $i = 1, 2$ In the same way, we have

$$\begin{aligned}
&|(\psi_{i,n+1} - \psi_{i,n})(x,t)| \\
&\le 2^{p_i-1} p_i \int_0^t \left( (\phi_{i',n} + \psi_{i',n})^{p_i-1} + (\phi_{i',n-1} + \psi_{i',n-1})^{p_i-1} \right) \\
&\qquad\qquad \times \left| (\phi_{i',n} + \psi_{i',n}) - (\phi_{i',n-1} + \psi_{i',n-1}) \right| (x - c_i(t-s), s) ds
\end{aligned}$$

for $i = 1, 2$. Since (3.6) yields that

$$\begin{aligned}
&\sum_{i=1,2} \Big( \|(\phi_{i,n+1} - \phi_{i,n})(\cdot, t)\|_{L^\infty(B_1(t))} + \|(\psi_{i,n+1} - \psi_{i,n})(\cdot, t)\|_{L^\infty(B_1(t))} \Big) \\
&\le \sum_{i=1,2} 2^{p_i} p_i \int_0^t \Big( \|(\phi_{i',n} + \psi_{i',n})(\cdot, s_1)\|_{L^\infty(B_1(s_1))}^{p_i-1} + \|(\phi_{i',n-1} + \psi_{i',n-1})(\cdot, s_1)\|_{L^\infty(B_1(s_1))}^{p_i-1} \Big) \\
&\quad \times \Big( \|(\phi_{i',n} - \phi_{i',n-1})(\cdot, s_1)\|_{L^\infty(B_1(s_1))} + \|(\psi_{i',n} - \psi_{i',n-1})(\cdot, s_1)\|_{L^\infty(B_1(s_1))} \Big) ds_1 \\
&\le \sum_{i=1,2} 2^{p_i+1} p_i C_0^{p_i-1} \int_0^t \Big( \|(\phi_{i',n} - \phi_{i',n-1})(\cdot, s_1)\|_{L^\infty(B_1(s_1))} \\
&\qquad\qquad + \|(\psi_{i',n} - \psi_{i',n-1})(\cdot, s_1)\|_{L^\infty(B_1(s_1))} \Big) ds_1 \\
&\vdots \\
&\le \sum_{i=1,2} \left( 2^{p_i+1} p_i C_0^{p_i-1} \right)^n \int_0^t \cdots \int_0^{s_{n-1}} \Big( \|(\phi_{i',1} - \phi_{i',0})(\cdot, s_n)\|_{L^\infty(B_1(s_n))} \\
&\qquad\qquad + \|(\psi_{i',1} - \psi_{i',0})(\cdot, s_n)\|_{L^\infty(B_1(s_n))} \Big) ds_n \cdots ds_1 \\
&\le \sum_{i=1,2} 4C_0 \frac{(2^{p_i+1} p_i C_0^{p_i-1} T^*)^n}{n!} \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

for $t \in [0, t_0]$. Moreover, we see that

$$D_{i,-} D_\theta \phi_{i,n+1} = D_{i,+} D_\theta \psi_{i,n+1} = p_{i'} (\phi_{i',n} + \psi_{i',n})^{p_{i'}-1} (D_\theta \phi_{i',n} + D_\theta \psi_{i',n}),$$

where $D_\theta = \cos\theta \partial_x + \sin\theta \partial_t$. Let $W(t)$ define by

$$W(t) = 2\left(\frac{1}{c_1} + 1\right) C_0^{p_1+p_2} \exp\Big(2(p_1+p_2) C_0^{p_1+p_2-1} t\Big).$$

Then $W(t)$ satisfies

$$W(t) = 2\left(\frac{1}{c_2}+1\right)C_0^{p_1+p_2} + \int_0^t 2(p_1+p_2)C_0^{p_1+p_2-1}W(s)ds.$$

We shall show

$$\|D_\theta\phi_{i,n}(\cdot,t)\|_{L^\infty(B_1(t))},\ \|D_\theta\psi_{i,n}(\cdot,t)\|_{L^\infty(B_1(t))} \le W(t) \quad (i=1,2) \tag{3.8}$$

for $n \in \mathbb{N}\cup\{0\}$ and $t \in [0,t_0]$. In the case of $n=0$, we see that

$$\|D_\theta\phi_{i,0}(\cdot,t)\|_{L^\infty(B_1(t))},\ \|D_\theta\psi_{i,0}(\cdot,t)\|_{L^\infty(B_1(t))} = 0 \quad (i=1,2)$$

for $t\in[0,t_0]$. It follows from (A4) that $|\partial_x f_i(x)| \le \frac{1}{2c_i}(f_{i'}+g_{i'})^{p_i}$. Moreover, we notice

$$\begin{aligned} D_{i,-}D_\theta\phi_{i,n+1} &= D_\theta D_{i,-}\phi_{i,n+1} = D_\theta(\phi_{i',n}+\psi_{i',n})^{p_i} \\ &= p_i(\phi_{i',n}+\psi_{i',n})^{p_i-1}D_\theta(\phi_{i',n}+\psi_{i',n}) \quad (i=1,2). \end{aligned}$$

Assuming that (3.8) holds for $n=k$, one can estimate $W(t)$ as follows.

$$\begin{aligned} &\|D_\theta\phi_{i,k+1}(\cdot,t)\|_{L^\infty(B_1(t))} \\ &\le \|(\cos\theta + c_i\sin\theta)\partial_x f_i(\cdot) + \sin\theta|(f_{i'}+g_{i'})|^{p_i}(\cdot)\|_{L^\infty(B_1(0))} \\ &+ \int_0^t \left\|p_i(\phi_{i',k}+\psi_{i',k})^{p_i-1}(D_\theta\phi_{i',k}+D_\theta\psi_{i',k})(\cdot,s)\right\|_{L^\infty(B_1(t))}ds \\ &\le 2\left(\frac{1}{c_i}+1\right)\|(f_{i'}+g_{i'})(\cdot)\|^{p_i}_{L^\infty(B_1(0))} + \int_0^t 2p_iC_0^{p_i-1}W(s)ds \\ &\le 2\left(\frac{1}{c_i}+1\right)C_0^{p_i} + \int_0^t 2p_iC_0^{p_i-1}W(s)ds \\ &\le 2\left(\frac{1}{c_2}+1\right)C_0^{p_1+p_2} + \int_0^t 2(p_1+p_2)C_0^{p_1+p_2-1}W(s)ds = W(t) \quad (i=1,2) \end{aligned}$$

for $t\in[0,t_0]$. Similarly, we can show $\|D_\theta\psi_{i,k+1}\|_{L^\infty(B_1(t))} \le W(t) \quad (i=1,2)$ for $t\in[0,t_0]$. Therefore we have (3.8) for $n\in\mathbb{N}$. Let $C_1 = W(T^*)$ satisfy

$$\|D_\theta\phi_{i,n}\|_{L^\infty(B_1(t))} \le C_1, \quad \|D_\theta\psi_{i,n}\|_{L^\infty(B_1(t))} \le C_1 \quad (i=1,2)$$

for $t \in (0, t_0)$ and $n \in \mathbb{N} \cup \{0\}$. We notice that

$$
\begin{aligned}
&|D_\theta \phi_{i,n+1}(x,t) - D_\theta \phi_{i,n}(x,t)| \\
&= |(\cos\theta + c_i \sin\theta)\partial_x f_i(x + c_i t) + \sin\theta (f_{i'} + g_{i'})^{p_i}(x + c_i t) \\
&\qquad\qquad + \int_0^t p_i (\phi_{i',n} + \psi_{i',n})^{p_i - 1}(D_\theta \phi_{i',n} + D_\theta \psi_{i',n})(x + c_i(t-s), s) ds \\
&\quad -(\cos\theta + c_i \sin\theta)\partial_x f_i(x + c_i t) - \sin\theta (f_{i'} + g_{i'})^{p_i}(x + c_i t) \\
&\qquad\qquad - \int_0^t p_i (\phi_{i',n-1} + \psi_{i',n-1})^{p_i - 1}(D_\theta \phi_{i',n-1} + D_\theta \psi_{i',n-1})(x + c_i(t-s), s) ds \Big| \\
&\le \int_0^t p_i \left|\phi_{i',n} + \psi_{i',n}\right|^{p_i - 1} \left|D_\theta \phi_{i',n} + D_\theta \psi_{i',n} - D_\theta \phi_{i',n-1} - D_\theta \psi_{i',n-1}\right| (x + c_i(t-s), s) ds \\
&\quad + \int_0^t p_i \left|D_\theta \phi_{i',n-1} + D_\theta \psi_{i',n-1}\right| \\
&\qquad\qquad \times |(\phi_{i',n} + \psi_{i',n})^{p_i - 1} - (\phi_{i',n-1} + \psi_{i',n-1})^{p_i - 1}|(x + c_i(t-s), s) ds \\
&\le \int_0^t p_i \left|\phi_{i',n} + \psi_{i',n}\right|^{p_i - 1} \\
&\qquad\qquad \times \left(|D_\theta \phi_{i',n} - D_\theta \phi_{i',n-1}| + |D_\theta \psi_{i',n} - D_\theta \psi_{i',n-1}|\right)(x + c_i(t-s), s) ds \\
&\quad + \int_0^t 2^{p_i - 2} p_i (p_i - 1) \left|D_\theta \phi_{i',n-1} + D_\theta \psi_{i',n-1}\right| \left((\phi_{i',n} + \psi_{i',n})^{p_i - 2} + (\phi_{i',n-1} + \psi_{i',n-1})^{p_i - 2}\right) \\
&\qquad \times \left(|\phi_{i',n} - \phi_{i',n-1}| + |\psi_{i',n} - \psi_{i',n-1}|\right)(x + c_i(t-s), s) ds.
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
&|D_\theta \psi_{i,n+1}(x,t) - D_\theta \psi_{i,n}(x,t)| \\
&\le \int_0^t p_i \left|\phi_{i',n} + \psi_{i',n}\right|^{p_i - 1} \\
&\qquad\qquad \times \left(|D_\theta \phi_{i',n} - D_\theta \phi_{i',n-1}| + |D_\theta \psi_{i',n} - D_\theta \psi_{i',n-1}|\right)(x - c_i(t-s), s) ds \\
&+ \int_0^t 2^{p_i - 2} p_i (p_i - 1) \left|D_\theta \phi_{i',n-1} + D_\theta \psi_{i',n-1}\right| \left((\phi_{i',n} + \psi_{i',n})^{p_i - 2} + (\phi_{i',n-1} + \psi_{i',n-1})^{p_i - 2}\right) \\
&\qquad \times \left(|\phi_{i',n} - \phi_{i',n-1}| + |\psi_{i',n} - \psi_{i',n-1}|\right)(x - c_i(t-s), s) ds.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\sum_{i=1,2} \Big( \|(D_\theta \phi_{i,n+1} - D_\theta \phi_{i,n})(\cdot, t)\|_{L^\infty(B_1(t))} + \|(D_\theta \psi_{i,n+1} - D_\theta \psi_{i,n})(\cdot, t)\|_{L^\infty(B_1(t))} \Big) \\
&\le \sum_{i=1,2} \tilde{C}_1 \int_0^t \Big( \|(D_\theta \phi_{i',n} - D_\theta \phi_{i',n-1})(\cdot, s)\|_{L^\infty(B_1(s))} + \|(D_\theta \psi_{i',n} - D_\theta \psi_{i',n-1})(\cdot, s)\|_{L^\infty(B_1(s))} \Big) ds \\
&\quad + \sum_{i=1,2} \tilde{C}_2 \int_0^t \Big( \|(\phi_{i',n} - \phi_{i',n-1})(\cdot, s)\|_{L^\infty(B_1(s))} + \|(\psi_{i',n} - \psi_{i',n-1})(\cdot, s)\|_{L^\infty(B_1(s))} \Big) ds
\end{aligned}
$$

Hence, we have

$$\begin{aligned}
&\sum_{i=1,2}\Big(\|(D_\theta\phi_{i,n+1}-D_\theta\phi_{i,n})(\cdot,t)\|_{L^\infty(B_1(t))}+\|(D_\theta\psi_{i,n+1}-D_\theta\psi_{i,n})(\cdot,t)\|_{L^\infty(B_1(t))}\Big)\\
&\le \sum_{i=1,2}(\tilde C_1)^n\int_0^t\cdots\int_0^{s_{n-1}}\big(\|D_\theta\phi_{i,1}(\cdot,s_n)-D_\theta\phi_{i,0}(\cdot,s_n)\|_{L^\infty(B_1(s_n))}\\
&\qquad\qquad +\|D_\theta\psi_{i,1}(\cdot,s_n)-D_\theta\psi_{i,0}(\cdot,s_n)\|_{L^\infty(B_1(s_n))}\big)\,ds_n\cdots ds_1\\
&\qquad\qquad +\sum_{i=1,2}\sum_{k=0}^{n-1}\tilde C_1^k\tilde C_2\int_0^t\cdots\int_0^{s_{k-1}}4C_0\frac{(2^{p_i+1}p_iC_0^{p_i-1}T^*)^{n-k}}{(n-k)!}ds_k\cdots ds_1\\
&\le 8C_0\frac{(\tilde C_1T)^n}{n!}+2^{(p_i+1)n+3}C_0\tilde C_1^n\tilde C_2(T^*)^n\sum_{k=0}^{n-1}\frac{1}{k!(n-k)!}\to 0 \quad \text{as} \quad n\to\infty
\end{aligned}$$

for $t\in[0,t_0]$. Here,

$$\tilde C_1=2(p_1+p_2)C_0^{p_1+p_2-1},\quad \tilde C_2=2^{p_1+p_2+1}(p_1+p_2)(p_1+p_2-1)C_1C_0^{p_1+p_2-2}.$$

Thus, there exists $(\phi_1^{(1)},\psi_1^{(1)},\phi_2^{(1)},\psi_1^{(1)})\in(L^\infty(K_{-,1}(x_0,t_0)))^4$ such that

$$\|D_\theta\phi_{i,n}-\phi^{(1)}\|_{L^\infty(B_1(t))}+\|D_\theta\psi_{i,n}-\psi^{(1)}\|_{L^\infty(B_1(t))}\to 0 \quad \text{as} \quad n\to\infty \quad (i=1,2)$$

for $t\in[0,t_0]$. Therefore we obtain that $(\phi_1,\psi_1,\phi_2,\psi_2)\in(W^{1,\infty}(K_{-,1}(x_0,t_0)))^4$. By repeating this process, we obtain $(\phi_1,\psi_1,\phi_2,\psi_2)\in(W^{4,\infty}(K_{-,1}(x_0,t_0)))^4$. This means that

$$(\phi_1,\psi_1,\phi_2,\psi_2)\in(C^{3,1}(K_-(x_0,t_0)))^4.$$

Finally, we shall show the uniqueness of solution of (2.1). We assume that

$$(\phi_1^*,\psi_1^*,\phi_2^*,\psi_2^*),(\phi_1^{**},\psi_1^{**},\phi_2^{**},\psi_2^{**})\in(W^{4,\infty}(\tilde\Omega))^4$$

satisfy (2.1) where we set

$$\tilde\Omega=\big\{(x,t)\in\mathbb{R}^2\mid x\in B_{R^*},\ 0<t<\min\{T^*(x),T^{**}(x)\}\big\}$$

and $T^*$ and $T^{**}$ be the blow-up curves of $(\phi_1^*,\psi_1^*,\phi_2^*,\psi_2^*)$ and $(\phi_1^{**},\psi_1^{**},\phi_2^{**},\psi_2^{**})$ respectively.

Let us assume that $(x_0,t_0)\in\tilde\Omega$. Then, there exits a positive constant $C_0$ such that

$$\begin{cases}\|\phi_i^*(\cdot,t)\|_{L^\infty(B_1(t))},\ \|\psi_i^*(\cdot,t)\|_{L^\infty(B_1(t))}\le C_0\\ \|\phi_i^{**}(\cdot,t)\|_{L^\infty(B_1(t))},\ \|\psi_i^{**}(\cdot,t)\|_{L^\infty(B_1(t))}\le C_0\end{cases}\qquad (i=1,2).$$

for $t \in [0, t_0]$. It follows from Lemma 3.2 that

$$\sup_{t\in[0,t_0]} \sum_{i=1,2} \left( \|(\phi_i^* - \phi_i^{**})(\cdot,t)\|_{L^\infty(B_1(t))} + \|(\psi_i^* - \psi_i^{**})(\cdot,t)\|_{L^\infty(B_1(t))} \right)$$
$$\leq \sup_{t\in[0,t_0]} \sum_{i=1,2} \int_0^t 2 \left\| \left( (\phi_{i'}^* + \psi_{i'}^*)^{p_i} - (\phi_{i'}^{**} + \psi_{i'}^{**})^{p_i} \right) (\cdot,s) \right\|_{L^\infty(B_1(s))} ds$$
$$\leq \sup_{t\in[0,t_0]} \sum_{i=1,2} \int_0^t 2^{p_i+1} p_i C_0^{p_i-1} \left( \|(\phi_{i'}^* - \phi_{i'}^{**})(\cdot,s)\|_{L^\infty(B_1(s))} + \|(\psi_{i'}^* - \psi_{i'}^{**})(\cdot,s)\|_{L^\infty(B_1(s))} \right) ds$$
$$\leq 2^{p_i+1} p_i C_0^{p_i-1} t \sup_{t\in[0,t_0]} \sum_{i=1,2} \left( \|(\phi_{i'}^* - \phi_{i'}^{**})(\cdot,t)\|_{L^\infty(B_1(t))} + \|(\psi_{i'}^* - \psi_{i'}^{**})(\cdot,t)\|_{L^\infty(B_1(t))} \right)$$
$$= 2^{p_i+1} p_i C_0^{p_i-1} t \sup_{t\in[0,t_0]} \sum_{i=1,2} \left( \|(\phi_i^* - \phi_i^{**})(\cdot,t)\|_{L^\infty(B_1(t))} + \|(\psi_i^* - \psi_i^{**})(\cdot,t)\|_{L^\infty(B_1(t))} \right)$$

for $t \in [0, t_0]$. Hence we obtain

$$\sum_{i=1,2} \left( \|(\phi_i^* - \phi_i^{**})(\cdot,t)\|_{L^\infty(B_1(t))} + \|(\psi_i^* - \psi_i^{**})(\cdot,t)\|_{L^\infty(B_1(t))} \right) = 0 \tag{3.9}$$

for $t \in \left[0, \dfrac{1}{2^{p_1+p_2+1}(p_1+p_2)C_0^{p_1+p_2-1}}\right]$. Since the constant $\dfrac{1}{2^{p_1+p_2+1}(p_1+p_2)C_0^{p_1+p_2-1}}$ does not depend on $t$, we obtain that (3.9) for $t \in [0, t_0]$. Therefore, we have that

$$(\phi_1^*, \psi_1^*, \phi_2^*, \psi_2^*) = (\phi_1^{**}, \psi_1^{**}, \phi_2^{**}, \psi_2^{**}) \quad \text{in} \quad \tilde{\Omega}.$$

Moreover, this means that $T^*(x) = T^{**}(x)$ for $x \in B_{R^*}$.

□

The following lemma guarantees that the solution of (2.1) blows-up in $K_{R^*,T^*,1}$.

**Lemma 3.4.** *Assume* (A1)–(A4). *Then, we have*

$$T(x) \leq T^* \quad \text{for} \quad x \in B_{R^*}. \tag{3.10}$$

*Proof.* We consider $\{(\tilde{\phi}_n, \tilde{\psi}_n)\}_{n\in\mathbb{N}\cup\{0\}}$ such that $\tilde{\phi}_0 \equiv \gamma_\phi$, $\tilde{\psi}_0 \equiv \gamma_\psi$ and

$$\begin{cases} \dfrac{d\tilde{\phi}_{n+1}}{dt} = (\tilde{\phi}_n + \tilde{\psi}_n)^p, & t > 0, \\ \dfrac{d\tilde{\psi}_{n+1}}{dt} = (\tilde{\phi}_n + \tilde{\psi}_n)^p, & t > 0, \\ \tilde{\phi}_{n+1}(0) = \gamma_\phi, \quad \tilde{\psi}_{n+1}(0) = \gamma_\psi, \end{cases}$$

for $n \in \mathbb{N} \cup \{0\}$. Thanks to (A1), it is enough to show that

$$\phi_{i,n}(x,t) \geq \tilde{\phi}_n(t) (\geq \gamma_\phi > 1) \quad \psi_{i,n}(x,t) \geq \tilde{\psi}_n(t) (\geq \gamma_\phi > 1) \quad (i = 1,2) \tag{3.11}$$

for $n \in \mathbb{N} \cup \{0\}$ . First, in the case of $n = 0$, we can easily confirm (3.11) holds. Assuming that (3.11) holds for $n = k$, we obtain that

$$\begin{aligned}\phi_{i,k+1}(x,t) - \tilde{\phi}_{k+1}(t) &= f_i(x + c_i t) - \gamma_\phi \\ &\quad + \int_0^t \left\{ (\phi_{i',k} + \psi_{i',k})^{p_i}(x + c_i(t-s), s) - (\tilde{\phi}_{i',k} + \tilde{\psi}_{i',k})^p(s) \right\} ds \\ &\geq 0 \quad \text{for} \quad (x,t) \in K_{R^*,T^*,1} \quad (i = 1,2)\end{aligned}$$

because of $p = \min\{p_1, p_2\}$. By the same arguments, we have that

$$\psi_{i,k+1}(x,t) - \tilde{\psi}_{k+1}(t) \geq 0 \quad \text{for} \quad (x,t) \in K_{R^*,T^*,1} \quad (i = 1,2).$$

Therefore we have (3.11) for $n \in \mathbb{N}$.

□

## § 4. Lipschitz continuity and the blow-up rates

This section is devoted to prove show that the blow-up rates of $\phi_i, \psi_i$ $(i = 1,2)$ and the blow-up curve $T$ is Lipschitz continuous in $B_{R^*}$.

**Proposition 4.1.** *Assume* (A1)–(A4). *Then, there exist positive constants $C_1$ and $C_2$ which depend only on $c_i, p_i, f_i, g_i$ $(i = 1,2)$ and $\varepsilon_0$ such that*

$$C_1(T(x) - t)^{-r_i} \leq (\phi_i + \psi_i)(x,t) \leq C_2(T(x) - t)^{-r_i} \quad \textit{for} \quad (x,t) \in \Omega \quad (i = 1,2) \tag{4.1}$$

*where $r_i = (p_i + 1)/(p_i p_{i'} - 1)$.*

*Proof.* First, we show that there exist positive constants $C_1$ and $C_2$ such that

$$C_1(\phi_{i',n} + \psi_{i',n})^{p_i} \leq \partial_t \phi_{i,n}, \partial_t \psi_{i,n} \leq C_2(\phi_{i',n} + \psi_{i',n})^{p_i} \text{ in } K_{R^*,T^*,1} \quad (i = 1,2). \tag{4.2}$$

For $n \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned}D_{i,-}\partial_t \phi_{i,n+1} &= \partial_t D_{i,-}\phi_{i,n+1} = \partial_t(\phi_{i',n} + \psi_{i',n})^{p_i} \\ &= p_i(\phi_{i',n} + \psi_{i',n})^{p_i - 1}(\partial_t \phi_{i',n} + \partial_t \psi_{i',n}) \quad \text{in} \quad K_{R^*,T^*,1} \quad (i = 1,2).\end{aligned} \tag{4.3}$$

By Lemma 3.2, we obtain

$$\begin{aligned}&D_{i,-}(\phi_{i',n} + \psi_{i',n})^{p_i} \\ &= p_i(\phi_{i',n} + \psi_{i',n})^{p_i - 1}(\partial_t \phi_{i',n} - c_i \partial_x \phi_{i',n} + \partial_t \psi_{i',n} - c_i \partial_x \psi_{i',n}) \\ &\leq 2p_i(\phi_{i',n} + \psi_{i',n})^{p_i - 1}(\partial_t \phi_{i',n} + \partial_t \psi_{i',n}) \quad \text{in} \quad K_{R^*,T^*,1} \quad (i = 1,2)\end{aligned} \tag{4.4}$$

for $n \in \mathbb{N} \cup \{0\}$. We set

$$J_{\phi_i,n+1} = 2\partial_t \phi_{i,n+1} - (\phi_{i',n} + \psi_{i',n})^{p_i}$$

for $n \in \mathbb{N} \cup \{0\}$. By (4.3) and (4.4), we have

$$D_{i,-} J_{\phi_i,n+1} \geq 0 \quad \text{in} \quad K_{R^*,T^*,1} \quad (i = 1, 2) \tag{4.5}$$

for $n \in \mathbb{N} \cup \{0\}$. It follows from (A4) that

$$\begin{aligned} J_{\phi_i,n+1}(x, 0) &= 2\partial_t \phi_{i,n+1}(x, 0) - (\phi_{i',n} + \psi_{i',n})^{p_i}(x, 0) \\ &= 2c_i \partial_x \phi_{i,n+1}(x, 0) + (\phi_{i',n} + \psi_{i',n})^{p_i}(x, 0) \\ &\geq \Big(-2c_1 |\partial_x f_i(x)| + (\gamma_\phi + \gamma_\psi)^{p_i}\Big) \geq 0 \quad \text{in} \quad B_{R^*+c_1T^*} \quad (i = 1, 2) \end{aligned} \tag{4.6}$$

for $n \in \mathbb{N} \cup \{0\}$. Since (4.5) and (4.6) yields that

$$J_{\phi_i,n} \geq 0 \quad \text{in} \quad K_{R^*,T^*,1} \quad (i = 1, 2) \tag{4.7}$$

for $n \in \mathbb{N}$. It follows from Lemma 3.2 that

$$\begin{aligned} \partial_t \phi_{i,n+1} &= c_i \partial_x \phi_{i,n+1} + (\phi_{i',n} + \psi_{i',n})^{p_i} \\ &\leq \frac{1}{1+\varepsilon_0} \partial_t \phi_{i,n+1} + (\phi_{i',n} + \psi_{i',n})^{p_i} \quad \text{in} \quad K_{R^*,T^*,1} \quad (i = 1, 2). \end{aligned}$$

for $n \in \mathbb{N}$. Hence we have that

$$\partial_t \phi_{i,n+1} \leq \frac{1+\varepsilon_0}{\varepsilon_0} (\phi_{i',n} + \psi_{i',n})^{p_i} \quad \text{in} \quad K_{R^*,T^*,1} \quad (i = 1, 2). \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\frac{1}{2} (\phi_{i',n} + \psi_{i',n})^{p_i} \leq \partial_t \phi_{i,n+1} \leq \frac{1+\varepsilon_0}{\varepsilon_0} (\phi_{i',n} + \psi_{i',n})^{p_i} \quad \text{in} \quad K_{R^*,T^*,1} \quad (i = 1, 2)$$

for $n \in \mathbb{N} \cup \{0\}$. Similarly, we can prove

$$\frac{1}{2} (\phi_{i',n} + \psi_{i',n})^{p_i} \leq \partial_t \psi_{i,n} \leq \frac{1+\varepsilon_0}{\varepsilon_0} (\phi_{i',n} + \psi_{i',n})^{p_i} \quad \text{in} \quad K_{R^*,T^*,1} \quad (i = 1, 2). \tag{4.9}$$

Therefore we obtain

$$C_1 (\phi_{i'} + \psi_{i'})^{p_i} \leq \partial_t \phi_i, \partial_t \psi_i \leq C_2 (\phi_{i'} + \psi_{i'})^{p_i} \quad \text{in} \quad \Omega \quad (i = 1, 2). \tag{4.10}$$

Next, we show that there exist positive constants $C_3$ and $C_4$ such that

$$C_3 (\phi_i + \psi_i)^{1+r_i^{-1}} \leq \partial_t \phi_i, \partial_t \psi_i \leq C_4 (\phi_i + \psi_i)^{1+r_i^{-1}} \quad \text{in} \quad \Omega \quad (i = 1, 2), \tag{4.11}$$

where $r_i = (p_i+1)/(p_ip_{i'}-1)$. By (4.10), there exist positive constants $C, C'$ such that

$$C(\phi_{i'}+\psi_{i'})^{p_i} \le \frac{\partial(\phi_i+\psi_i)}{\partial t} \le C'(\phi_{i'}+\psi_{i'})^{p_i} \quad \text{in} \quad \Omega \quad (i=1,2).$$

Hence, there exist positive constants $C''$ such that

$$\int_{(\phi_i+\psi_i)(x,0)}^{(\phi_i+\psi_i)(x,t)} s^{p_{i'}}\,ds \le C'' \int_{(\phi_{i'}+\psi_{i'})(x,0)}^{(\phi_{i'}+\psi_{i'})(x,t)} s^{p_i}\,ds \quad \text{for} \quad (x,t)\in\Omega \quad (i=1,2).$$

Thus, there exists a positive constant $C'''$ such that

$$\begin{aligned}&(\phi_i+\psi_i)^{p_{i'}}(x,t)\\ &\le C'''(\phi_{i'}+\psi_{i'})^{(p_i+1)p_{i'}/(p_{i'}+1)}(x,t)\\ &\times\left(1+\frac{(\phi_i+\psi_i)^{p_{i'}+1}(x,0)-(\phi_{i'}+\psi_{i'})^{p_i+1}(x,0)}{(\phi_{i'}+\psi_{i'})^{p_i+1}(x,t)}\right)^{p_{i'}/(p_{i'}+1)} \quad \text{for} \quad (x,t)\in\Omega \quad (i=1,2).\end{aligned}$$

It follows from Lemma 3.2 that there exists a positive constant $\bar{C}$ such that

$$0 \le 1+\frac{(\phi_i+\psi_i)^{p_{i'}+1}(x,0)-(\phi_{i'}+\psi_{i'})^{p_i+1}(x,0)}{(\phi_{i'}+\psi_{i'})^{p_i+1}(x,t)} \le \bar{C} \quad \text{for} \quad (x,t)\in\Omega \quad (i=1,2).$$

Therefore we have

$$(\phi_i+\psi_i)(x,t) \le (C''')^{1/p_{i'}}\bar{C}(\phi_{i'}+\psi_{i'})^{(p_i+1)/(p_{i'}+1)}(x,t) \quad \text{for} \quad (x,t)\in\Omega \quad (i=1,2).$$

Hence it follows from (4.10) that we obtain (4.11).

Finally, we shall show (4.1). We notice that it follows from Lemma 3.4 that

$$\sum_{i=1,2}(\phi_i+\psi_i)(x,t)\to\infty \quad \text{as} \quad t\to T(x) \quad \text{for} \quad (x\in B_{R^*}).$$

Let $x_0 \in B_{R^*}$ and assume

$$(\phi_1+\psi_1)(x_0,t)\to\infty \quad \text{as} \quad t\to T(x_0).$$

Then,

$$\partial_t\phi_1(x_0,t), \partial_t\psi_1(x_0,t)\to\infty \quad \text{as} \quad t\to T(x_0).$$

It follows from (4.10) that

$$(\phi_2+\psi_2)(x_0,t)\to\infty \quad \text{as} \quad t\to T(x_0).$$

Similarly, we can prove

$$(\phi_1+\psi_1)(x_0,t)\to\infty \quad \text{as} \quad t\to T(x_0)$$

when we assume

$$(\phi_2+\psi_2)(x_0,t)\to\infty \quad \text{as} \quad t\to T(x_0).$$

Hence, we see that

$$\begin{cases}(\phi_1+\psi_1)(x_0,t)\to\infty \\ (\phi_2+\psi_2)(x_0,t)\to\infty\end{cases} \quad \text{as} \quad t\to T(x) \quad \text{for} \quad x\in B_{R^*}.$$

It follows from (4.11) that there exist positive constants $C, C'$ such that

$$C\int_{(\phi_i+\psi_i)(x_0,t)}^{(\phi_i+\psi_i)(x_0,T(x_0)-\varepsilon)} z^{-1-r_i^{-1}}dz \le T(x_0)-\varepsilon-t \le C'\int_{(\phi_i+\psi_i)(x_0,t)}^{(\phi_i+\psi_i)(x_0,T(x_0)-\varepsilon)} z^{-1-r_i^{-1}}dz.$$

Hence we obtain (4.1) by letting $\varepsilon\to 0$. □

Lemma 3.2 yields that the blow-up curve $T$ is Lipschitz continuous.

**Lemma 4.2.** *Assume* (A1)–(A4). *Then, we have*

$$|T(x)-T(x')| \le \frac{1}{c_1(1+\varepsilon_0)}|x-x'| \quad \textit{for} \quad x,x'\in B_{R^*}. \tag{4.12}$$

*Proof.* This proof is based on the Implicit Function Theorem. Let $\varepsilon$ be a sufficiently small positive constant such that $T(x)-\varepsilon>0$. For $M_\varepsilon=\sup_{x\in B_{R^*}}(\phi_1+\psi_1)(x,T(x)-\varepsilon)$, there exists an implicit function $E_\varepsilon\in C^1(B_{R^*})$ such that

$$(\phi_1+\psi_1)(x,E_\varepsilon(x))=M_\varepsilon \quad \text{for} \quad x\in B_{R^*}.$$

For $x_1,x_2\in B_{R^*}$, let $k=E_\varepsilon(x_1)-E_\varepsilon(x_2)$ and consider $H\in C^1(0,1)$ such that

$$H(\xi)=(\phi_1+\psi_1)(x_1+\xi(x_2-x_1),t+\xi k).$$

We notice that $H$ satisfies that

$$\begin{aligned}&H(0)=(\phi_1+\psi_1)(x_1,t),\\ &H(1)=(\phi_1+\psi_1)(x_2,t+k)=(\phi_1+\psi_1)(x_2,t+E_\varepsilon(x_2)-E_\varepsilon(x_1)).\end{aligned}$$

Then, we have $H(0)=H(1)=M_\varepsilon$ where we set $t=E_\varepsilon(x_1)$. By using Rolle's Theorem, there exists $\xi'\in(0,1)$ such that

$$\begin{aligned}H'(\xi')=&(x_2-x_1)\partial_x(\phi_1+\psi_1)(x_1+\xi'(x_2-x_1),E_\varepsilon(x_1)+\xi'k)\\ &+k\partial_t(\phi_1+\psi_1)(x_1+\xi'(x_2-x_1),E_\varepsilon(x_1)+\xi'k)=0.\end{aligned} \tag{4.13}$$

Since Lemma 3.2 and (4.13) yield that

$$|E_\varepsilon(x_1) - E_\varepsilon(x_2)| = |k| = \left| \frac{-\partial_x(\phi_1 + \psi_1)(x_1 + \xi'(x_2 - x_1), E_\varepsilon(x_1) + \xi' k)}{\partial_t(\phi_1 + \psi_1)(x_1 + \xi'(x_2 - x_1), E_\varepsilon(x_1) + \xi' k)} \right| |x_1 - x_2|$$
$$\leq \frac{1}{c_1(1+\varepsilon_0)} |x_1 - x_2|.$$

Thus, $E_\varepsilon$ is Lipschitz continuous. Finally, we will prove Lipschitz continuity of $T$ in $B_{R^*}$. By using the definition of $M_\varepsilon$, we have

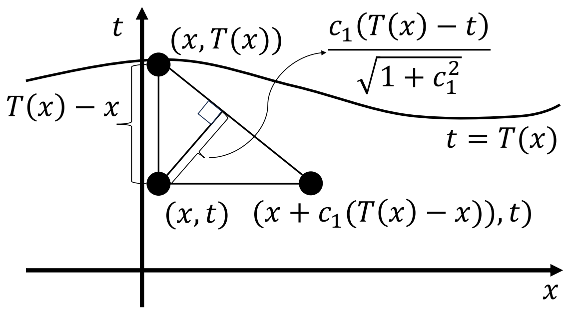
$$|T(x_1) - T(x_2)| \leq |T(x_1) - E_\varepsilon(x_1)| + |E_\varepsilon(x_1) - E_\varepsilon(x_2)| + |E_\varepsilon(x_2) - T(x_2)|$$
$$\leq 2\varepsilon + \frac{1}{c_1(1+\varepsilon_0)} |x_1 - x_2| \quad \text{for} \quad x_1, x_2 \in B_{R^*}.$$

Since we let $\varepsilon > 0$ take an arbitrary value, this completes the proof.

□

By $d(x,t)$, we denote the distance from a point $(x,t)$ in $\Omega$ to $\Gamma = \{(x, T(x)) \mid x \in B_{R^*}\}$. It follows from Lemma 4.2 that we obtain the following results.

$$\frac{c_1(T(x) - t)}{\sqrt{1 + c_1^2}} \leq d(x,t) \leq T(x) - t.$$




By replacing $T(x) - t$ by $d(x,t)$ in Proposition 4.1, we obtain the following Corollary.

**Corollary 4.3.** *Assume* (A1)–(A4). *Then, there exist positive constants* $C_1$ *and* $C_2$ *which depend only on* $c_i, p_i, f_i, g_i$ $(i = 1, 2)$ *and* $\varepsilon_0$ *such that*

$$\text{(4.14)} \quad \begin{cases} C_1(T(x) - t)^{-r_i} \leq \phi_i(x,t) \leq C_2(T(x) - t)^{-r_i} \\ C_1(T(x) - t)^{-r_i} \leq \psi_i(x,t) \leq C_2(T(x) - t)^{-r_i} \end{cases} \quad \textit{for} \quad (x,t) \in \Omega \quad (i = 1, 2),$$

*where* $r_i = (p_i + 1)/(p_i p_{i'} - 1)$.

*Proof.* It follows from Proposition 4.1 that there exist positive constants $C, C'$ such that

$$C d^{-r_i}(x,t) \leq (\phi_i + \psi_i)(x,t) \leq C' d^{-r_i}(x,t) \quad \text{for} \quad (x,t) \in \Omega \quad (i = 1, 2).$$

Therefore, there exists a positive constant $C'', C'''$ we have

$$
\begin{aligned}
\phi_i(x, T(x)-\varepsilon) &= f_i(x + c_1(T(x)-\varepsilon)) \\
&\qquad + \int_0^{T(x)-\varepsilon} (\phi_{i'} + \psi_{i'})^{p_i}(x + c_i(T(x)-\varepsilon - s), s)ds \\
&\geq \int_{T(x)-2\varepsilon}^{T(x)-\varepsilon} (\phi_{i'} + \psi_{i'})^{p_i}(x + c_i(T(x)-\varepsilon - s), s)ds \\
&\geq C''\varepsilon \inf_{T(x)-2\varepsilon \leq s \leq T(x)-\varepsilon} d(x + c_i(T(x)-\varepsilon - s), s)^{-r_{i'}p_i} \\
&\geq C'''\varepsilon \cdot \varepsilon^{-r_{i'}p_i} = C'''\varepsilon^{-r_i} \quad \text{for} \quad (x,t) \in \Omega \quad (i = 1, 2).
\end{aligned}
$$

By the same arguments, we can show that there exist positive constants $C_1, C_2$ such that

$$
\begin{cases}
\phi_i(x, T(x)-\varepsilon) \leq C_2\varepsilon^{-r_i} \\
C_1\varepsilon^{-r_i} \leq \psi_i(x, T(x)-\varepsilon) \leq C_2\varepsilon^{-r_i}
\end{cases}
\quad \text{for} \quad (x,t) \in \Omega \quad (i = 1, 2).
$$

□

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