RIMS-1979

Sign Convention for A_∞-Operations in Bott-Morse Case

By

Kaoru ONO

Januray 2024

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

SIGN CONVENTION FOR A_∞-OPERATIONS IN BOTT-MORSE CASE

KAORU ONO

We give a definition of A_∞-operations in Bott-Morse case (see Definition 2). Let L_i be a relatively spin collection of Lagrangian submanifolds, which intersects cleanly in (X,ω). (The argument presented here is also valid for immersed Lagrangian submanifolds.) Denote by R_α a connected component of L_i and L_j. (We also consider the case that $i=j$.)

We use the convention on orientation on the fiber product (in the sense of Kuranishi structure) as in Section 8.2 in [1]. In this note, the dimension of moduli spaces means their virtual dimension. Let $p : M \to N$ be a fiber bundle with oriented relative tangent bundle. Rstrict the fiber bundle to an open subset, we may assume that N is oriented. Then we give an orientation on M using the isomorphism $TM = p^*TN \oplus T_{\text{fiber}}M$, where $T_{\text{fiber}}M$ is the relative tangent bundle. Then our convention of the integration along fibers of $p : M \to N$ is

$$\int_N \alpha \wedge p_! \beta = \int_M p^*\alpha \wedge \beta,$$

where $\alpha \in \Omega^*(N)$ and $\beta \in \Omega^*(M)$, We have the following properties.

- $p_!\big((p^*\theta)\wedge\beta\big) = \theta \wedge (p_!\beta)$, where $\theta \in \Omega^*(N)$ and $\beta \in \Omega^*(M)$.
- Let $p : M \to N$ and $q : N \to B$ be fiber bundles with oriented relative tangent bundles. For $\beta \in \Omega^*(M)$, we have

$$(q\circ p)_!\beta = q_! \circ p_!(\beta).$$

Using them, we find that

$$(q\circ p)_!(p^*\theta\wedge\beta) = q_!(\theta\wedge p_!\beta). \tag{1}$$

We also have

- (base change) Let $f : S \to N$ be a smooth map (or a strongly smooth map between spaces with Kuranishi structure). Denote by $\overline{p} : f^*M \to S$ the pull-back of the fiber bundle $p : M \to N$ and $\widetilde{f} : f^*M \to M$ the bundle map covering f. Then we have

$$f^* \circ p_! = \overline{p}_! \circ \widetilde{f}^*.$$

For the definition of the integration along fibers of weakly submersive strongly smoooth map in the case of Kuranishi structure is given in Section 9.2 in [2]. We will use the Stokes type formula in Theorem 9.28 in [2], the composition formula in Theorem 10.21 in [2]. See Chapter 27 in [2] in the case with coefficients in local systems. In fact, the composition formula is a consequence of these properties.

Let $(\Sigma,\partial\Sigma)$ be a bordered Riemann surface Σ of genus 0 and with connected boundary and $\vec{z} = (z_0,\dots,z_k)$ boundary marked points respecting the cyclic order on $\partial\Sigma$. Let $u : (\Sigma,\partial\Sigma) \to (X,\cup L_i)$ be a smooth map such that $u(z_j\widehat{z_{j+1}}) \subset L_{i_j}$, $j \mod k+1$, $u(z_j) \in R_{\alpha_j}$, where R_{α_j} is a connected component of $L_{i_{j-1}} \cap L_{i_j}$. For such u and u', we introduce the equivalence relation $\sim$ so that

The author is partially supported by JSPS Grant-in-Aid for Scientific Research19H00636. He is also grateful for National Center for Theoretical Sciences, Taiwan, where a part of this work was carried out.

$u \sim u'$ when $\int_\Sigma \omega = \int_{\Sigma'} \omega$ and (2) the Maslov indices of u and u' are the same. Denote by B the equivalence class.

Consider the moduli space

$$\mathcal{M}_{k+1}(B; L_{i_0}, \dots, L_{i_k}; R_{\alpha_0}, \dots R_{\alpha_k})$$

of bordered stable maps of genus 0, with connected boundary and $(k+1)$ boundary marked points, representing the class B.

Set $\mathcal{L} = (L_{i_0}, \dots, L_{i_k})$ and $\mathcal{R} = (R_{\alpha_0}, \dots R_{\alpha_k})$ and write

$$\mathcal{M}_{k+1}(B; \mathcal{L}; \mathcal{R}) = \mathcal{M}_{k+1}(B; L_{i_0}, \dots, L_{i_k}; R_{\alpha_0}, \dots R_{\alpha_k}).$$

Denote by $ev_j^B : \mathcal{M}_{k+1}(B; \mathcal{L}; \mathcal{R}) \to R_{\alpha_j}$ the evaluation map at z_j.

For a pair of Lagrangian submanifolds L, L' which intersect cleanly, we constructed the $O(1)$-local system $\Theta^-_{R_\alpha}$ on R_α in Proposition 8.1.1 in [1]. Here R_α is a connected component of $L \cap L'$. In this note, we simply write it as Θ_{R_α}.

We recall the construction of Θ_{R_α} briefly. We assume that L, L' are equipped with spin structures. In the case of a relative spin pair, we take $TX \oplus (W \otimes \mathbb{C})$ (on the 3-skeleton of X) instead of TX and $TL \oplus W$, (resp. $TL' \oplus W$) (on the 2-skeleton of L, (resp. L') instead of TL, (resp. TL'). Then the argument goes in the same way. As written in Section 8.8 in [1], we consider the space $\mathcal{P}_{R_\alpha}(TL, TL')$ of paths of oriented Lagrangian subspaces in T_pX, $p \in R_\alpha$, of the form $\lambda(t) \oplus R_\alpha$ such that $\lambda(0) \oplus R_\alpha = T_pL$ and $\lambda(1) \oplus R_\alpha = T_pL'$. Here λ is regarded as a path of Lagrangian subspaces in $V_{R_\alpha} = (T_pL + T_pL')/(T_pL + T_pL')^{\perp_\omega} = (T_pL + T_pL')/(T_pL \cap T_pL')$, which is a symplectic vector space. Pick a compatible complex struture on it and consider the Dolbeault operator $\overline{\partial}_\lambda$ on $Z_- = (D^2 \cap \{\text{Re } z \le 0\}) \cup ([0, \infty) \times [0, 1])$.

We set $\mu(R_\alpha; \lambda) = \text{Index } \overline{\partial}_\lambda$. The parity of $\mu(R_\alpha; \lambda)$ is independent of the choice of λ above, since $\lambda \oplus T_pR_\alpha$ is a path of oriented subspaces with fixed end points, T_pL, T_pL', $p \in R_\alpha$ which are oriented. Denote by $\mu(R_\alpha) = \mu(R_\alpha; \lambda) \bmod 2$. Then we have

$$\dim \mathcal{M}_{k+1}(B; \mathcal{L}, \mathcal{R}) \equiv \dim R_{\alpha_0} + \mu(R_{\alpha_0}) - \sum_{i=1}^{k} \mu(R_{\alpha_i}) + k - 2 \mod 2.$$

We have the determinant line bundle of $\{\text{Index } \overline{\partial}_\lambda\}_{\lambda \in \mathcal{P}_{R_\alpha}(TL, TL')}$. Pick a hermitian metric on X. Denote by $P_{SO}(\lambda \oplus T_pR_\alpha)$ is the associated oriented orthogonal frame bundle of $\lambda \oplus T_pR_\alpha$. Note that $P_{SO}(\lambda \oplus T_pR_\alpha)|_{t=0}$ and $P_{SO}(\lambda \oplus T_pR_\alpha)|_{t=1}$ are canonically identified with $P_{SO}(L)|_p$ and $P_{SO}(L')|_p$, respectively. We glue the principal spin bundle $P_{Spin}(\lambda \oplus T_pR_\alpha)$ at $t = 0, 1$ with $P_{Spin}(L)|_p$ and $P_{Spin}(L')|_p$. There are two isomorphic classes of resulting spin structure on the bundle $TL \cup (\lambda \oplus T_pR_\alpha) \cup TL'$ on $L \cup [0, 1] \cup L'$, where $p \in L$ and $p \in L'$ are identified with $0, 1 \in [0, 1]$, respectively. This gives an $O(1)$-local system O_{spin} on $\mathcal{P}_{R_\alpha}(TL, TL')$. Proposition 8.1.1 in [1] states that the tensor product $\det \overline{\partial}_\lambda \otimes O_{Spin}$ descends to an $O(1)$-local system Θ_{R_α} on R_α.

Then the relative spin structure for $\{L_i\}$, namely relative spin structures for each L_i with a common oriented vector bundle $W \to X^{[3]}$, determines an isomorphism Φ^B below.

(i) Case that $k = 0$. (L is an immersed Lagrangian submanifold with clean self intersection or $R_{\alpha_0} = L$)

$$\Phi^B : ev_0^{B*}\Theta_{R_{\alpha_0}} \to ev_0^{B*}O_{R_{\alpha_0}} \otimes O_{\mathcal{M}_1(B;L)}$$

(ii) Case that $k = 1$.

$$\Phi^B : ev_0^{B*}\Theta_{R_{\alpha_0}} \to ev_0^{B*}O_{R_{\alpha_0}} \otimes O_{\mathcal{M}_2(B;\mathcal{L};\mathcal{R})} \otimes ev_1^{B*}\Theta_{R_{\alpha_1}}.$$

(iii) Case that $k \geq 2$.

$$\Phi^B : ev_0^{B*}\Theta_{R_{\alpha_0}} \to ev_0^{B*}O_{R_{\alpha_0}} \otimes O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})} \otimes \mathfrak{forget}^* O_{\mathcal{M}_{k+1}} \otimes ev_1^{B*}\Theta_{R_{\alpha_1}} \otimes \cdots \otimes ev_k^{B*}\Theta_{R_{\alpha_k}}.$$

Here $\mathcal{M}_{k+1}$ is the moduli space of bordered Riemann surfaces of genus 0, connected boundary and $(k+1)$ marked points on the boundary and $\mathfrak{forget} : \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to \mathcal{M}_{k+1}$ sends $[(\Sigma, \partial\Sigma, \vec{z}), u]$ to $[(\Sigma, \partial\Sigma, \vec{z})]$. Here $O_{R_{\alpha_0}}$, $O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}$ and $O_{\mathcal{M}_{k+1}}$ are orientation bundles of R_{α_0}, $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ and $\mathcal{M}_{k+1}$, respectively. We consider $ev_0^* O_{R_{\alpha_0}} \otimes O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}$ the orientation bundle of the relative tangent bundle of $ev_0 : \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to R_{\alpha_0}$. In the notation in [1], we write

$$\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) = R_{\alpha_0} \times {}^\circ\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$$

and

$$\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) = \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ \times \mathcal{M}_{k+1}.$$

These descriptions are considered as the splitting of tangent spaces. Using these notations, we have

$$ev_0^* O_{R_{\alpha_0}} \otimes O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})} = O_{\,^\circ\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}.$$

$$O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})} \otimes \mathfrak{forget}^* O_{\mathcal{M}_{k+1}} = O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ}.$$

We give an orientation of $\mathcal{M}_{k+1} = (\partial D^2)^{k+1}/Aut(D^2, \partial D^2)$ as the orientation of the quotient space following the convention (8.2.1.2) in [1]. Then the orientation bundle of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ is canonically isomorphic to the one of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ$. Hence, for $\mathbf{u} = [u : (\Sigma, \partial\Sigma, \vec{z}) \to (X, \cup_{L\in\mathcal{L}} L, \cup_{R_\alpha \in \mathcal{R}} R_\alpha)]$, the relative spin structure of $\mathcal{L}$, local sections σ_{α_i} of $O(1)$-local systems Θ_{α_i} around $u(z_i)$, $i = 0, 1, \ldots, k$, determines a local orientation of the relative tangent bundle of $ev_0^B : \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to R_\alpha$, at $\mathbf{u}$, i.e., the kernel of $T_{\mathbf{u}}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to T_{u(z_0)}R_{\alpha_0}$, which is denoted by $o(\sigma_{\alpha_0}; \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_k})$.

Remark 1. When $k = 0$ and $R_{\alpha_0} = L$, the orientation on $\mathcal{M}_1(B;L)$ is given in Section 8.4.1 in [1] When $k = 1$, the orientation bundle of $\mathcal{M}_2(B;\mathcal{L};\mathcal{R})$ is given in Proposition 8.8.6 in [1]. Note that $\Theta^+_{R_\alpha} \otimes O_{R_\alpha} \otimes \Theta^-_{R_\alpha}$ is canonically trivialized. We write $\Theta_{R_\alpha} = \Theta^-_{R_\alpha}$ in this note.

Hence Theorem 27.1 in [2] gives

$$(ev_0^B)_! \circ (ev_1^{B*} \times \cdots \times ev_k^{B*}) : \Omega^*(R_{\alpha_1}; \Theta_{R_{\alpha_1}}) \otimes \cdots \otimes \Omega^*(R_{\alpha_k}; \Theta_{R_{\alpha_k}}) \to \Omega^*(R_{\alpha_0}; \Theta_{R_{\alpha_0}}).$$

Namely, for $\xi_i = \zeta_i \otimes \sigma_{\alpha_i} \in \Omega^*(R_{\alpha_i}; \Theta_{\alpha_i})$, $i = 1, \ldots, k$, we define

$$\begin{aligned} & (ev_0^B)_! \circ (ev_1^{B*} \times \cdots \times ev_k^{B*})(\zeta_1 \otimes \sigma_{\alpha_1}, \ldots, \zeta_k \otimes \sigma_{\alpha_k}) \\ = \ & \big(ev_0^B; o(\sigma_{\alpha_0}; \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_k})\big)_! (ev_1^{B*}\zeta_1 \wedge \cdots \wedge ev_k^{B*}\zeta_k) \otimes \sigma_{\sigma_{\alpha_0}}. \end{aligned} \tag{2}$$

Here $\big(ev_0^B; o(\sigma_{\alpha_0}; \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_k})\big)_!$ is the integration along fibers with respect to the relative orientation $o(\sigma_{\alpha_0}; \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_k})$ of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to R_{\alpha_0}$. Note that the right hand side of (2) does not depends on σ_{α_0}, since σ_{α_0} appears twice in the right hand side of (2), and gives a differential form on R_{α_0} with coefficients in Θ_{α_0}. For general $\xi_i \in \Omega^*(R_{\alpha_i}; \Theta_{\alpha_i})$, we use partitions of unity on R_{α_i} and extend the definition of $\mathfrak{m}_k$ multi-linearly.

For $\xi \in \Omega^*(R_\alpha; \Theta_\alpha)$, we define the shifted degree

$$|\xi|' = \deg \xi + \mu(R_\alpha) - 1.$$

Definition 2. We set $\mathfrak{m}_{0,0} = 0$, $\mathfrak{m}_{(1,0)}\xi = d\xi$ on $\bigoplus \Omega^*(R_\alpha; \Theta_{R_\alpha})$, i.e., the de Rham differential on differential forms with coefficients in the local system Θ_{R_α}. For $(k, B) \neq (1, 0)$,

$$\mathfrak{m}_{k,B}(\xi_1, \dots, \xi_k) = (-1)^{\epsilon(\xi_1,\dots,\xi_k)}(ev_0^B)_! \circ (ev_1^{B*} \times \dots \times ev_k^{B*})(\xi_1 \otimes \dots, \otimes \xi_k),$$

where $\xi_i \in \Omega^*(R_{\alpha_i}; \Theta_{\alpha_i})$ and

$$\epsilon(\xi_1, \dots, \xi_k) = \left\{ \sum_{i=1}^{k} \left(i + \sum_{p=1}^{i-1} \mu\left(R_{\alpha_p}\right) \right) (\deg\, \xi_i - 1) \right\} + 1.$$

In the rest of this note, we show the filtered A_∞-relations. Denote by $\hat{\mathfrak{m}}_{k,B}$ the extension of $\mathfrak{m}_{k,B}$ as a coderivation with respect to the shifted degree $|\bullet|'$. We compute $\mathfrak{m}_{k',B'} \circ \hat{\mathfrak{m}}_{k'',B''}$. Clearly,, $\mathfrak{m}_{1,0} \circ \mathfrak{m}_{1,0} = 0$. We consider the case that $(k', B') = (1, 0)$ or $(k'', B'') = (1, 0)$. Namely, for $(k, B) \neq (1, 0)$, we have

$$\mathfrak{m}_{1,0} \circ \mathfrak{m}_{k,B}(\xi_1, \dots, \xi_k) = (-1)^{\epsilon(\xi_1,\dots,\xi_k)} d(ev_0^B)_!(ev_1^{B*}\xi_1 \wedge \dots \wedge ev_k^{B*}\xi_k) \quad (3)$$

$$\begin{aligned}
\mathfrak{m}_{k,B} \circ \hat{\mathfrak{m}}_{1,0}(\xi_1, \dots, \xi_k) &= \sum_{j=1}^{k} (-1)^{\sum_{p=1}^{j-1} |\xi_p|'} \mathfrak{m}_{k,B}(\xi_1, \dots, d\xi_j, \dots, \xi_k) \\
&= \sum_{j=1}^{k} (-1)^{\sum_{p=1}^{j-1} |\xi_p|' + \epsilon(\xi_1,\dots,d\xi_j,\dots,\xi_k)} (ev_0^B)_!(ev_1^{B*}\xi_1 \wedge \dots \wedge ev_j^{B*} d\xi_j \wedge \dots \wedge ev_k^{B*}\xi_k) \\
&= (-1)^{\epsilon(\xi_1,\dots,\xi_k)+1}(ev_0^B)_! d(ev_1^{B*}\xi_1 \wedge \dots \wedge ev_k^{B*}\xi_k) \quad (4)
\end{aligned}$$

Here we note that

$$\begin{aligned}
\sum_{p=1}^{j-1} |\xi_p|' + \epsilon(\xi_1, \dots, d\xi_j, \dots, \xi_k) &= \sum_{p=1}^{j-1} \deg\, \xi_p + \sum_{p=1}^{j-1} \big(\mu(R_{\alpha_p}) - 1\big) + \epsilon(\xi_1, \dots, \xi_k) + \big(j + \sum_{p=1}^{j-1} \mu(R_{\alpha_p})\big) \\
&\equiv \sum_{p=1}^{j-1} \deg\, \xi_p + \epsilon(\xi_1, \dots, \xi_k) + 1 \bmod 2.
\end{aligned}$$

In order to compute $\mathfrak{m}_{k',B'} \circ \hat{\mathfrak{m}}_{k'',B''}$, we discuss the relation between the orientation bundle of $\mathcal{M}_{k'+1}(B'; \mathcal{L}'; \mathcal{R}')_{ev_j^{B'}} \times_{ev_0^{B''}} \mathcal{M}_{k''+1}(B''; \mathcal{L}''; \mathcal{R}'')$ and the orientation bundle of the boundary of $\partial\mathcal{M}_{k+1}(B; \mathcal{L}; \mathcal{R})$. The codimension 1 boundary of the moduli space $\mathcal{M}_{k+1}(B; \mathcal{L}; \mathcal{R})$ is the union of the fiber products of $\mathcal{M}_{k'+1}(B'; \mathcal{L}'; \mathcal{R}')$ and $\mathcal{M}_{k''+1}(B''; \mathcal{L}''; \mathcal{R}'')$ with respect to $ev_j^{B'} : \mathcal{M}_{k'+1}(B'; \mathcal{L}'; \mathcal{R}') \to R_\alpha$ and $ev_0^{B''} : \mathcal{M}_{k''+1}(B''; \mathcal{L}''; \mathcal{R}'') \to R_\alpha$, where

$$\mathcal{L}' = (L_{i_0}, \dots, L_{i_{j-1}}, L_{i_{j+k''-1}}, \dots L_{i_k}), \mathcal{L}'' = (L_{i_{j-1}}, \dots, L_{i_{j+k''-1}})$$

$$\mathcal{R}' = (R_{\alpha_0}, \dots, R_{\alpha_{j-1}}, R_\alpha, R_{\alpha_{j+k''}}, \dots, R_{\alpha_k}), \mathcal{R}'' = (R_\alpha, R_{\alpha_j}, \dots R_{\alpha_{i_{j+k''-1}}})$$

over $j = 1, \dots, k$, k', k'' such that $k' + k'' = k + 1$, R_α a connected component of $L_{i_{j-1}} \cap L_{j+k''-1}$, all possible decomposition of B into B' and B''.

Denote by Sw the exchange of $\Theta_{R_{\alpha_1}}\otimes\cdots\otimes\Theta_{R_{\alpha_{j-1}}}$ and $O_{R_\alpha}\otimes O_{\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ}$ with the sign $(-1)^{\delta_1}$, where

$$\begin{aligned}\delta_1 &= (\sum_{p=1}^{j-1}\mu(R_{\alpha_p}))(\dim\,\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')-\dim\,R_\alpha-\dim\,\mathcal{M}_{k''+1})\\ &\equiv \big(\sum_{p=1}^{j-1}\mu(R_{\alpha_p})\big)\big(\mu(R_\alpha)-\sum_{p=j}^{j+k''-1}\mu(R_{\alpha_p})\big)\ \mathrm{mod}\ 2.\end{aligned}$$

Comparing Φ^B and $Sw\circ(id\otimes\cdots\otimes id\otimes\Phi^{B''}\otimes id\otimes\cdots\otimes id)\circ\Phi^{B'}$, we find that

$$O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ}\to O_{\mathcal{M}_{k+1}(B';\mathcal{L}';\mathcal{R}')^\circ}\otimes O_{R_\alpha}\otimes O_{\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ}$$

is $(-1)^{\delta_1}$-orientation preserving[1]. Here $\mathcal{M}_{k+1}(B';\mathcal{L}';\mathcal{R}')^\circ$ is the moduli space of bordered stable maps with a fixed domain bordered Riemann surface with fixed boundary marked points. The $O(1)$-local system $O_{\mathcal{M}_{k+1}(B';\mathcal{L}';\mathcal{R}')^\circ}\otimes O_{R_\alpha}\otimes O_{\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ}$ is the orientation bundle of the fiber product $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ_{ev_j^{B'}}\times_{ev_0^{B''}}\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ$, which is the moduli space of bordered stable maps with a fixed boundary nodal Riemann surface wth fixed boundary marked points.

Now we compare $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})=\partial\big(\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ\times\mathcal{M}_{k+1}\big)$ and $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{ev_j^{B'}}\times_{ev_0^{B''}}\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')=\big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ\times\mathcal{M}_{k'+1}\big)_{ev_j^{B'}}\times_{ev_0^{B''}}\big(\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ\times\mathcal{M}_{k''+1}\big)$. We note that $O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}=\mathbb{R}_{out}\otimes O_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}$. Here $\mathbb{R}_{out}$ is the normal bundle of the boundary oriented by the outer normal vector.

We pick local flat sections $\sigma_{\alpha_0},\dots,\sigma_{\alpha_k},\sigma_\alpha$ of $O(1)$-local systems $\Theta_{R_{\alpha_0}},\dots,\Theta_{R_{\alpha_k}},\Theta_{R_\alpha}$ and a local orientation $o_{R_{\alpha_0}}$ of R_{α_0} around $u(z_0)$. Then we can equip $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$, $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')$ and the relative tangent bundle of $ev_0^{B''}:\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')\to R_\alpha$ with local orientations induced by them. Then a local orientation of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})=R_{\alpha_0}\times^\circ\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ is given by $o_{R_{\alpha_0}}\times o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$. As the fiber product of spaces with Kuranishi structures equipped with local orieentations,

$$\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{ev_j^{B'}}\times_{ev_0^{B''}}\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')=\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')\times\ {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')$$

is locally oriented by

$$o_{R_{\alpha_0}}\times o(\sigma_{R_{\alpha_0}};\sigma_{R_{\alpha_1}},\dots,\sigma_{R_{\alpha_{j-1}}},\sigma_{R_\alpha},\sigma_{R_{\alpha_{j+k''}}},\dots,\sigma_{R_{\alpha_k}})\times o(\sigma_{R_\alpha};\sigma_{R_{\alpha_j}},\dots,\sigma_{R_{\alpha_{j+k''-1}}}).$$

We fix $z_0=+1, z_j=-1$ and consider the spaces of J-holomorphic maps $\widetilde{\mathcal{M}}_{k+1}(B;\mathcal{L},\mathcal{R})$, $\widetilde{\mathcal{M}}_{k'+1}(B';\mathcal{L}',\mathcal{R}')$, $\widetilde{\mathcal{M}}_{k''+1}(B'';\mathcal{L},''\mathcal{R}'')$ such that

$$\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})=\widetilde{\mathcal{M}}_{k'+1}(B';\mathcal{L}',\mathcal{R}')/\mathbb{R}_B,$$
$$\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')=\widetilde{\mathcal{M}}_{k'+1}(B';\mathcal{L}';\mathcal{R}')/\mathbb{R}_{B'},$$

and

$$\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')=\widetilde{\mathcal{M}}_{k''+1}(B'';\mathcal{L},''\mathcal{R}'')/\mathbb{R}_{B''}.$$

We may also write

$$\widetilde{\mathcal{M}}_{k+1}(B;\mathcal{L};\mathcal{R})=\mathcal{M}_{k+1}(B;\mathcal{L},\mathcal{R})\times\mathbb{R}_B,\text{etc.},$$

as oriented spaces.

[1](-1)-orientation preserving means orientation reversing.

The case that $z_0 = +1, z_1 = -1$ is discussed in page 699 of [1]. The case that $z_0 = +1, z_j = -1$ differs from the case that $z_0 = +1, z_1 = -1$ by an additional factor $(-1)^{j-1}$ as below.

Note that

$$\widetilde{\mathcal{M}}_{k+1}(B;\mathcal{L};\mathcal{R}) = (-1)^{j-1}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i},$$

where $z_0 = +1, z_j = -1$,

$$\widetilde{\mathcal{M}}_{k'+1}(B';\mathcal{L}';\mathcal{R}') = (-1)^{j-1}\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i},$$

where $z'_0 = +1, z'_j = -1$, and

$$\widetilde{\mathcal{M}}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') = \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i},$$

where $z''_0 = +1, z''_1 = -1$.

Remark 3. We have

$$(-1)^{j-1}\prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} = \mathcal{M}_{k+1} \times \mathbb{R}_B$$

$$(-1)^{j-1}\prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} = \mathcal{M}_{k'+1} \times \mathbb{R}_{B'}$$

and

$$\prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}. = \mathcal{M}_{k''+1} \times \mathbb{R}_{B''}.$$

Marked points of $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')$ and $\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')$ are related to marked points of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ in the following way.

$$(z'_0, \ldots, z'_{k'}) = (z_0, \ldots, z_{j-1}, z'_j, z_{j+k''}, \ldots, z_k),$$

$$(z''_0, z''_1, , \ldots, z''_{k''}) = (z''_0, z_j, \ldots, z_{j+k''-1}).$$

Here z'_j and z''_0 are identified, i.e., the boundary node of the domain curve of an element in $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$.

Then we find that

$$
\begin{aligned}
\widetilde{\mathcal{M}}_{k+1}(B;\mathcal{L};\mathcal{R}) &= (-1)^{\delta_1}\big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times_{R_\alpha} \times \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ\big) \times \\
& \quad (-1)^{j-1}\prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} \\
&= (-1)^{\delta_1+\delta_2}\big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times (-1)^{j-1}\prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i}\big) \\
& \quad \times_{R_\alpha}\big(\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}\big) \\
&= (-1)^{\delta_1+\delta_2}\big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}') \times \mathbb{R}_{B'}\big) \times_{R_\alpha} \times \big(\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') \times \mathbb{R}_{B''}\big) \\
&= (-1)^{\delta_1+\delta_2+\delta_3}\mathbb{R}_{B'-B''} \times \big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}') \times_{R_\alpha} \times \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')\big) \\
& \quad \times \mathbb{R}_{B'+B''} \\
&= (-1)^{\delta_1+\delta_2+\delta_3}\mathbb{R}_{out} \times \big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}') \times_{R_\alpha} \times \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')\big) \\
& \quad \times \mathbb{R}_B, \qquad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_2 &= (k''-1)(k'-j) + (k'-1)\big(\dim \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ - \dim R_\alpha\big), \\
\delta_3 &= \dim \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}').
\end{aligned}
$$

$\mathbb{R}_{B'-B''}$ and $\mathbb{R}_{B'+B''}$ are the oriented lines spanned by $(1,-1),(1,1) \in \mathbb{R}_{B'} \oplus \mathbb{R}_{B''}$, respectively. Note that the ordered bases $(1,0),(0,1)$ and $(1,-1),(1,1)$ give the same orientation of $\mathbb{R}_{B'} \oplus \mathbb{R}_{B''}$, $\mathbb{R}_{B'-B''}$ and $\mathbb{R}_{B'+B''}$ are identified with $\mathbb{R}_{out}$ and $\mathbb{R}_B$, respectively.

Here is an explanation of the second equality, i.e., the appearance of $(-1)^{\delta_2}$. By the convention in Section 8.2 in [1], we have

$$
\begin{aligned}
& \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times_{R_\alpha} \times \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ \\
= \; & \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^{\circ\circ} \times R_\alpha \times \; {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ, \\
= \; & \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times \; {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ
\end{aligned}
$$

where

$$\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ = \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^{\circ\circ} \times R_\alpha,$$

and

$$\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ = R_\alpha \times \; {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ.$$

Using these notations, we have

$$
\begin{aligned}
&\big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times_{R_\alpha} \times \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ\big) \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i} \\
= &\ (-1)^{\gamma_1}\big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times_{R_\alpha} \times \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ\big) \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} \\
= &\ (-1)^{\gamma_1}\big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times\ {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ\big) \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} \\
= &\ (-1)^{\gamma_1+\gamma_2}\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i} \times\ {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} \\
= &\ (-1)^{\gamma_1+\gamma_2}\big(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i}\big) \times_{R_\alpha} \big(\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}\big),
\end{aligned}
$$

where $\gamma_1 = (k''-1)(k'-j)$, i.e., $(-1)^{\gamma_1}$ is the sign of exchange of $(z_{j+k''},\dots,z_k)$ and $(z_{j+1},\dots,z_{j+k''-1})$, and $\gamma_2 = \dim\big({}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ\big)\big(\dim \mathcal{M}_{k'+1}+1\big)$. Then $\delta_2 = \gamma_1+\gamma_2$.

Now we return to the discussion on local orientations of the orientation bundle of $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{ev_j^{B'}} \times_{ev_0^{B''}} \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')$ and $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$. Recall that

$$\widetilde{\mathcal{M}}_{k+1}(B;\mathcal{L};\mathcal{R}) = \mathcal{M}_{k'+1}(B';\mathcal{L}',\mathcal{R}') \times \mathbb{R}_B. \tag{6}$$

Set $\kappa = \delta_1+\delta_2+\delta_3$. Comparing (5) and (6), we have

$$(-1)^\kappa \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{ev_j^{B'}} \times_{ev_0^{B''}} \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') \subset \partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}). \tag{7}$$

We have

$$
\begin{aligned}
\kappa \ \equiv\ &(k''-1)(k'-j) + (k'-1)\big(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1}\mu(R_{\alpha_p})\big) + \big(\sum_{p=1}^{j-1}\mu(R_{\alpha_p})\big)\big(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1}\mu(R_{\alpha_p})\big) \\
&+\dim R_{\alpha_0} + \mu(R_{\alpha_0}) - \big(\sum_{p=1}^{j-1}\mu(R_{\alpha_p}) + \mu(R_\alpha) + \sum_{p=j+k''}^{k}\mu(R_{\alpha_p})\big) + k'.
\end{aligned}
$$

From (1) in the setting of Kuranishi structures, (7), (1) and the base change formula for integration along fibers, we find that

$$
\begin{aligned}
&\big(ev_0^B|_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})};\partial o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})\big)_!\big(\prod_{i=1}^{j-1} ev_i^{B*} \times \prod_{i=j+k''}^{k} ev_i^{B*} \times \prod_{i=j}^{j+k''-1} ev_i^{B*}\big) \\
= &\ (-1)^\kappa\big(ev_0^{B'};o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_{j-1}},\sigma_\alpha,\sigma_{\alpha_{j+k''-1}},\dots,\sigma_{\alpha_k})\big)_! \\
&\circ\left(\prod_{i=1}^{j-1} ev_i^{B'*} \times \prod_{i=j+1}^{k'} ev_i^{B'*} \times \left(ev_j^{B'*}\circ\big(ev_0^{B''};o(\sigma_\alpha;\sigma_{\alpha_j},\dots,\sigma_{\alpha_{j+k''-1}})\big)_! \circ \prod_{i=1}^{k''} ev_i^{B''*}\right)\right)
\end{aligned} \tag{8}
$$

as operations applied to $(\otimes_{i=1}^{j-1}\zeta_i)\otimes(\otimes_{i=j+k''}^{k}\zeta_i)\otimes(\otimes_{i=j}^{j+k''-1}\zeta_i)$, where $\xi_i=\zeta_i\otimes\sigma_{\alpha_i}, i=1,\dots k$. Here $\partial o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$[2] is the orientation of the relative tangent bundle $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})\to R_{\alpha_0}$ induced from $o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$.

Namely, for $\mathbf{u}\in\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$, $o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$ of ${}^{\circ}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ and $\partial o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$ of the relative tangent bundle of $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})\to R_{\alpha_0}$ are related as follows: we write

$$T_{\mathbf{u}}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})=\mathbb{R}_{out}\times T_{\mathbf{u}}\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$$
$$T_{\mathbf{u}}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})=T_{u(z_0)}R_{\alpha_0}\times T_{\mathbf{u}}\ {}^{\circ}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$$

Then, under the following identification

$$\mathbb{R}_{out}\times T_{\mathbf{u}}\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})=\mathbb{R}_{out}\times T_{u(z_0)}R_{\alpha_0}\times T_{\mathbf{u}}\ {}^{\circ}\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}),$$

we define $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})\to R_{\alpha_0}$ by

$$o_{R_{\alpha_0}}\times o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})=\mathbb{R}_{out}\times o_{R_{\alpha_0}}\times\partial o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}).$$

Note that

$$ev_i^B|_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}=\begin{cases}ev_i^{B'}\circ\pi_{B'}^B, & i=1,\dots,j-1,\\ ev_{i-j+1}^{B''}\circ\pi_{B''}^B, & i=j,\dots,j+k''-1,\\ ev_{i-k''+1}^{B'}\circ\pi_{B'}^B, & i=j+k'',\dots,k,\end{cases}$$

where $\pi_{B'}^B$ and $\pi_{B''}^B$ are projections from the fiber product $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{ev_j^{B'}}\times_{ev_0^{B''}}\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')$ to $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')$ and $\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')$, respectively. Note that σ_α appears twice and the right hand side of (8) does not depends on the choice of local section σ_α of the $O(1)$-local system Θ_α.

Next, we compute $\mathfrak{m}_{k',B'}\circ\hat{\mathfrak{m}}_{k'',B''}$ with $(k',B')\neq(1,0)$, $(k'',B'')\neq(1,0)$. Armed with (8), we regard ξ_i, $i=1,\dots,k$ as differential forms on R_{α_i} in the computation below.

$$\begin{aligned}\mathfrak{m}_{k',B'}\circ\hat{\mathfrak{m}}_{k'',B''}(\xi_1,\dots,\xi_k) &= \sum_{j=1}^{k}(-1)^{\sum_{i=1}^{j-1}|\xi_i|'}\mathfrak{m}_{k',B'}(\xi_1,\dots,\mathfrak{m}_{k'',B''}(\xi_j,\dots,\xi_{j+k''-1}),\dots,\xi_k)\\ &= \sum_{j=1}^{k}(-1)^{\delta_4}(ev_0^{B'})_!\Big(ev_1^{B'*}\xi_1\wedge\cdots\wedge ev_{j-1}^{B'*}\xi_{j-1}\\ &\quad\wedge ev_j^{B'*}\big((ev_0^{B''})_!(ev_1^{B''*}\xi_j\wedge\cdots\wedge ev_{k''}^{B''*}\xi_{j+k''-1}\big)\wedge\cdots\wedge ev_{k'}^{B'*}\xi_k\Big)\\ &= (-1)^{\delta_4+\delta_5}(ev_0^{(B',B'')})_!(ev_1^{(B',B'')*}\xi_1\wedge\cdots\wedge ev_k^{(B',B'')*}\xi_k)\\ &= (-1)^{\delta_4+\delta_5+\kappa}(ev_0^B|_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})})_!(ev_1^{B*}\xi_1\wedge\dots ev_k^{B*}\xi_k)\end{aligned}\tag{9}$$

where

$$\delta_4=\sum_{i=1}^{j-1}|\xi_i|'+\epsilon(\xi_1,\dots,\mathfrak{m}_{k'',B''}(\xi_j,\dots,\xi_{j+k''-1}),\dots,\xi_k)+\epsilon(\xi_j,\dots,\xi_{j+k''-1}).$$

$$\delta_5=\big(\mu(R_\alpha)-\sum_{i=j}^{j+k''-1}\mu(R_{\alpha_i})+k''-2\big)\Big(\sum_{i=j+k''}^{k}\deg\xi_i\Big),$$

[2] $\partial o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$ is not the boundary orientation of $\partial\ {}^{\circ}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ induced from the orientation $o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$ of ${}^{\circ}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$. They differ by $(-1)^{\dim R_{\alpha_0}}$.

and $ev_j^{(B',B'')} : \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{ev_j^{B'}} \times_{ev_0^{B''}} \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') \to R_{\alpha_j}$ is the evaluation map at the j-th marked point on the fiber product $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{ev_j^{B'}} \times_{ev_0^{B''}} \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')$. Here the numbering of the marked points is the same as that on $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$. The appearance of $(-1)^{\delta_5}$ above is due to (1) and the composition formula (Theorem 10.21 in [2]). Namely, we have

$$\begin{aligned}
& (ev_0^{B'})_! \Big(ev_1^{B'*}\xi_1 \wedge \cdots \wedge ev_{j-1}^{B'*}\xi_{j-1} \wedge ev_j^{B'*}\big((ev_0^{B''})_!(ev_1^{B''*}\xi_j \wedge \cdots \wedge ev_{k''}^{B''*}\xi_{j+k''-1}\big) \\
& \wedge ev_{j+1}^{B'*}\xi_{j+k''} \wedge \cdots \wedge ev_{k'}^{B'*}\xi_k \Big) \\
= \ & (-1)^{\eta_1}(ev_0^{B'})_! \Big(\big(ev_1^{B'*}\xi_1 \wedge \cdots \wedge ev_{j-1}^{B'*}\xi_{j-1} \wedge ev_{j+1}^{B'*}\xi_{j+k''} \cdots \wedge ev_{k'}^{B'*}\xi_k\big) \\
& \wedge ev_j^{B'*} \circ (ev_0^{B''})_!(ev_1^{B''*}\xi_j \wedge \cdots \wedge ev_{k''}^{B''*}\xi_{j+k''-1}) \\
= \ & (-1)^{\eta_1}(ev_0^{B'})_! \Big(\big(ev_1^{B'*}\xi_1 \wedge \cdots \wedge ev_{j-1}^{B'*}\xi_{j-1} \wedge ev_{j+1}^{B'*}\xi_{j+k''} \cdots \wedge ev_{k'}^{B'*}\xi_k\big) \\
& \wedge (\pi_{B'})_! \circ \pi_{B''}^*(ev_1^{B''*}\xi_j \wedge \cdots \wedge ev_{k''}^{B''*}\xi_{j+k''-1}) \\
= \ & (-1)^{\eta_1}(ev_0^{B'})_! \circ (\pi_{B'})_! \big(\pi_{B'}^*(ev_1^{B'*}\xi_1 \wedge \cdots \wedge ev_{j-1}^{B'*}\xi_{j-1} \wedge ev_{j+1}^{B'*}\xi_{j+k''} \cdots \wedge ev_{k'}^{B'*}\xi_k) \\
& \wedge \pi_{B''}^*(ev_1^{B''*}\xi_j \wedge \cdots \wedge ev_{k''}^{B''*}\xi_{j+k''-1})\big) \\
= \ & (-1)^{\eta_1+\eta_2}(ev_0^{B'} \circ \pi_{B'})_!(\pi_{B'}^*(ev_1^{B'*}\xi_1 \wedge \cdots \wedge ev_{j-1}^{B'*}\xi_{j-1}) \wedge \pi_{B''}^*(ev_1^{B''*}\xi_j \wedge \cdots \wedge ev_{k''}^{B''*}\xi_{j+k''-1}) \\
& \pi_{B'}^*(ev_{j+1}^{B'*}\xi_{j+k''} \cdots \wedge ev_{k'}^{B'*}\xi_k) \\
= \ & (-1)^{\eta_1+\eta_2}(ev_0^{(B',B'')})_!(ev_1^{(B',B'')*}\xi_1 \wedge \cdots \wedge ev_k^{((B',B'')*}\xi_k),
\end{aligned}$$

where $\eta_1 = \big(((\sum_{i=j}^{j+k''-1} \deg \xi_i) + (\mu_{R_\alpha} - \sum_{i=j}^{j+k''-1} \mu(R_{\alpha_i}) + k'' - 2)\big)(\sum_{i=j+k''}^{k} \deg \xi_i)$ and $\eta_2 = (\sum_{i=j}^{j+k''-1} \deg \xi_i)(\sum_{i=j+k''}^{k} \deg \xi_i)$. Then $\delta_5 = \eta_1 + \eta_2 = \big(\mu(R_\alpha) - \sum_{i=j}^{j+k''-1} \mu(R_{\alpha_i}) + k'' - 2\big)(\sum_{i=j+k''}^{k} \deg \xi_i)$. The second equality is a consequence of the base change formula for integration along fibers, i.e., $ev_j^{B'*} \circ (ev_0^{B''})_! = (\pi_{B'})_! \circ \pi_{B''}^*$. The third equality follows from (1). Note that

$$ev_i^{(B',B'')} = \begin{cases} ev_i^{B'} \circ \pi_{B'}^B & i = 0, 1, \ldots, j-1, \\ ev_{i-j+1}^{B''} \circ \pi_{B''}^B, & i = j, \ldots, j+k''-1, \\ ev_{i-k''+1}B' \circ \pi_{B'}^B, & i = j+k'', \ldots, k. \end{cases}$$

We find that

$$\begin{aligned}
\delta_4 + \delta_5 + \kappa \ & \equiv \ \epsilon(\xi_1, \ldots, \xi_k) + 1 + k + \sum_{i=1}^{k} \deg \xi_i + \dim R_{\alpha_0} + \mu(R_{\alpha_0}) - \sum_{p=1}^{k} \mu(R_{\alpha_p}) \\
& \equiv \ \epsilon(\xi_1, \ldots, \xi_k) + 1 + \dim \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) + \sum_{i=1}^{k} \deg \xi_i \mod 2
\end{aligned}$$

Using Theorem 27.2 in [2], we have

$$\begin{aligned}
& d(ev_0^B)_!(ev_1^{B*}\xi_1 \wedge \cdots \wedge ev_k^{B*}\xi_k) \\
= \quad & (ev_0^B)_! d(ev_1^{B*}\xi_1 \wedge \cdots \wedge ev_k^{B*}\xi_k) \\
& +(-1)^{\dim\ \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})+\sum_{i=1}^k \deg\ \xi_i}(ev_0^B|_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})})_!(ev_1^{B*}\xi_1 \wedge \cdots \wedge ev_k^{B*}\xi_k). \qquad (10)
\end{aligned}$$

Combining (3), (4), (9), (10), we have

$$\begin{aligned}
& \mathfrak{m}_{1,0} \circ \mathfrak{m}_{k,B}(\xi_1, \ldots, \xi_k) + \mathfrak{m}_{k,B} \circ \hat{\mathfrak{m}}_{1,0}(\xi_1, \ldots, \xi_k) \\
& + \textstyle\sum_{(k',B'),(k'',B'')\neq(1,0)} \mathfrak{m}_{k',B'} \circ \hat{\mathfrak{m}}_{k'',B''}(\xi_1, \ldots, \xi_k) = 0.
\end{aligned}$$

References

[1] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Lagrangian Intersection Floer Theory - Anomaly and Obstruction, AMS/IP Studies in Advanced Mathematics **46**, Part II, Amer. Math. Soc. and International Press, 2009.

[2] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Kuranishi Structures and Virtual Fundamental Chains, Springer Monographs in Mathematics, Springer Nature Singapore, 2020.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan
Email address: ono@kurims.kyoto-u.ac.jp