On Dualities Related to Coupling *

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Main Sources

[M21] Mase, M., Polytope duality for families of $K3$ surfaces and coupling, *Bull. Braz. Math. Soc., New Series*, **52** (2021), 499–536.

[M22] Mase, M., Lattice duality for coupling pairs admitting polytope duality with trivial toric contribution, *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, **63** (2022), 533—559.

[M23] Mase, M., Lattice duality for families of $K3$ surfaces and coupling, submitted.

1 Introduction

Coupling is introduced by Ebeling [E06] between weight systems with $(n+1)$ integers. Focusing on $n = 3$, and we can consider weight systems that define simple $K3$ singularities. By Yonemura's classification [Y90], there are 95 such systems, let's call them $K3$ *weight systems*, exist. By an appropriate compactification, we get weight systems with 5 integers with which the weighted projective space is a toric Fano 3-fold. Parametrized by the complete anticanonical linear system, we obtain an example of $K3$ surfaces, called *weighted K3 surfaces* as a hypersurface in the Fano 3-fold. Thus, we expect to proceed to a study of $K3$ surfaces, by using toric geometry as well as a standard lattice theory.

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Strongly coupling pairs among $K3$ weight systems are all classified by [E06]. Moreover, it is investigated in [E06] that there is a relation between the coupling duality and Saito duality which concerns the reduced zeta function of some isolated hypersurface singularities, as is summarized here: denote by $\tilde{\zeta}^*_C(t)$ is the *Saito dual rational function* of $\tilde{\zeta}_C(t)$.

Theorem ([E06]). [1] *Let* $(\boldsymbol{a}, \boldsymbol{b})$ *be a coupling pair with magic square* C.
(i) *The reduced zeta function associated to* C *has a formula:*

$$\tilde{\zeta}_C(t) = \prod_{J:\text{special}} (1 - t^{d/a_J})^{(-1)^{|J|+1} \cdot a_J \cdot |\det C_{IJ}|/d}.$$

(ii) (Case $n = 3$, Corollary) *If* C *is primitive, then,* $\tilde{\zeta}_{{}^tC}(t) = \tilde{\zeta}^*_C(t)$ *holds.* □

We are motivated by a strong desire to understand a geometric structure of $K3$ surfaces, in particular, from a viewpoint of Picard lattices, an arithmetic characteristic. However, it is quite rough to investigate only the lattices. Thus, we would like to combine with some other objects. As is explained, there is an example of $K3$ surfaces associated to an IHS, for which, we can construct the Milnor lattice together with the Seifert form. In our study, we are intended to understand a relation between the Picard lattice of the families of weighted $K3$ surfaces, and the Milnor lattice of simple $K3$ singularity with the structure Seifert form.

Motivated by [E06], we are interested in giving another interpretation of coupling in terms of $K3$ surfaces. Indeed, there are many coupling pairs that are out of application of Ebeling's theorem, part (ii).

We consider the following two questions for coupling dual pairs of $K3$ weight systems formed of $(a_1, a_2, a_3; d)$ and $(b_1, b_2, b_3; h)$ together with the families $\mathcal{F}_a$ and $\mathcal{F}_b$ of weighted $K3$ surfaces.

Q.1 Are the pair of families $\mathcal{F}_a$ and $\mathcal{F}_b$ polytope-dual ?

Q.2 Does a polytope-dual pair extend to lattice-dual ?

The questions are partially affirmatively answered by the following theorems.

Main Theorem A ([M21]). *Any strongly coupling pairs extend to the polytope-dual of families except the cases where the projectivized weight systems are*

- $(1, 3, 4, 7; 15)$ *(self-dual),* $(1, 3, 4, 4; 12)$ *(self-dual), and*
- *the pair* $((1, 1, 3, 5; 10), (3, 5, 11, 19; 38))$. □

Main Theorem B ([M23', M23]). *For coupling dual pairs in [M21] except #'s 24, 27, 31 and 39* [2]*, the associated families are lattice dual.* □

We give a sketch of the proof for Main Theorems A and B in §3 following a preliminary section where we discuss coupling duality, and polytope and lattice dualities associated to weight systems. We summarize our main theorems by giving Table 1 before Appendix, where we discuss Ebeling's Theorem.

[1] See Appendix for notions used here.
[2] The numbering follows [E06].

2 Preliminary

Coupling duality

We collect necessary definition from [E06] concerning on coupling duality.

Definition 2.1. *An n-tuple $(w_1, \ldots, w_n)$ of integers is* WELL-POSED *if* $0 < w_1 \leq \cdots \leq w_n$ *and* $\gcd(w_1, \cdots, w_n) = 1$, *and for any distinct* $(n-1)$ *integers,* $\gcd(w_{i_1}, \cdots, w_{i_{n-1}}) = 1$.

We call a tuple $(w_1, \ldots, w_n; v)$ a WEIGHT SYSTEM *if $(w_1, \ldots, w_n)$ is well-posed and $v \in \mathbb{Z}_{>0}$.* ■

Take weight systems $W_{\boldsymbol{a}} := (a_1,\, a_2,\, \cdots, a_n\,;\, d)$ and $W_{\boldsymbol{b}} := (b_1,\, b_2,\, \cdots, b_n\,;\, h)$.

Definition 2.2. *A square matrix $C = (c_{ij})_{i,j=1}^n$ of size n is called a weighted magic square (associated to weight systems) if the following relations hold:*

$$C\,{}^t(a_1\, a_2\, \cdots\, a_n) = {}^t(d\, d\, \cdots\, d) \quad \text{and} \quad (b_1\, b_2\, \cdots\, b_n)C = (d\, d\, \cdots\, d). \quad ■$$

Let $C = (c_{ij})$ be the weighted magic square for $W_{\boldsymbol{a}}$ and $W_{\boldsymbol{b}}$.

Definition 2.3. (1) *The magic square C is* ALMOST PRIMITIVE *if there exist integers a_0 and b_0 such that $|\det C| = a_0 h = b_0 d$. If $a_0 = b_0 = 1$, C is* PRIMITIVE.

(2) *A* COUPLING PAIR *is $(W_{\boldsymbol{a}}, W_{\boldsymbol{b}})$ together with an almost primitive C.*

(3) *The coupled pair $(W_{\boldsymbol{a}}, W_{\boldsymbol{b}})$ is said* STRONGLY COUPLED *if*

$$\forall j,\, \exists i : c_{ij} = 0, \text{ and } \forall i,\, \exists j : c_{ij} = 0. \quad ■$$

Example 1. Let

$$W_{\boldsymbol{a}} = (1,\, 2,\, 9\,;\, 18), \quad W_{\boldsymbol{b}} = (2,\, 3,\, 11\,;\, 24), \text{ and } C = \left(\begin{smallmatrix} 9 & 0 & 1 \\ 2 & 8 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right).$$

We have

$$C\,{}^t\begin{pmatrix}1 & 2 & 9\end{pmatrix} = {}^t\begin{pmatrix}18 & 18 & 18\end{pmatrix}, \quad \begin{pmatrix}2 & 3 & 11\end{pmatrix} C = \begin{pmatrix}24 & 24 & 24\end{pmatrix}, \text{ and}$$

$$\det C = 144 = 18 \cdot 8 = 24 \cdot 6, \quad \therefore |\det C|/24 = 6, \quad |\det C|/18 = 8.$$

Moreover, C has at least one entry 0 in every row and column. Therefore, the pair $(W_{\boldsymbol{a}}, W_{\boldsymbol{b}})$ together with C is strongly coupled.

In fact [Y90], the weight systems $W_{\boldsymbol{a}}$ and $W_{\boldsymbol{b}}$ are $K3$ weight systems, that is, general quasi-homogeneous polynomials $f(x, y, z)$, and $f'(x', y', z')$ of degree 18, and 24, resp., of weights

$$wt(x, y, z) = (1, 2, 9), \quad wt(x', y', z') = (2, 3, 11)$$

determine simple $K3$ singularities. And thus, their "projectivizations", general quasi-homogeneous polynomials $F(W,X,Y,Z)$, and $F'(W',X',Y',Z')$ of degree 18, and 24, resp., of weights

$$wt(W,X,Y,Z)=(6,1,2,9),\quad wt(W',X',Y',Z')=(8,2,3,11)$$

are birational to $K3$ surfaces. ■

Let $(\boldsymbol{a},\,\boldsymbol{b})$ be a strongly coupling pair of $K3$ weight systems, and $\mathcal{F}_{\boldsymbol{a}}$, $\mathcal{F}_{\boldsymbol{b}}$ the families of weighted $K3$ surfaces parametrized respectively by the complete anticanonical linear system $|-K_{\mathbb{P}(\boldsymbol{a})}|$, and $|-K_{\mathbb{P}(\boldsymbol{b})}|$. Here, the weighted projective spaces $\mathbb{P}(\boldsymbol{a})$ of weight $\boldsymbol{a}$ and $\mathbb{P}(\boldsymbol{b})$ of weight $\boldsymbol{b}$ are toric 3-folds that are determined by polytopes $\Delta_{\boldsymbol{a}}$, and $\Delta_{\boldsymbol{b}}$. And there is a correspondence between the linear system $|-K_{\mathbb{P}(\boldsymbol{a})}|$ (resp. $|-K_{\mathbb{P}(\boldsymbol{b})}|$) and the set of lattice points in the polytope $\Delta_{\boldsymbol{a}}$ (resp. $\Delta_{\boldsymbol{b}}$).

Polytope and lattice dualities

We first discuss polytope duality.

Definition 2.4 (Polytope duality)**.** *The families $\mathcal{F}_{\boldsymbol{a}}$ and $\mathcal{F}_{\boldsymbol{b}}$ are* POLYTOPE DUAL *if there exist reflexive polytopes Δ and Δ' such that*

$$\Delta\subset\Delta_{\boldsymbol{a}},\ \Delta'\subset\Delta_{\boldsymbol{b}},\ \text{and}\ \Delta'\simeq\Delta^*$$

hold. Here, Δ^* is the polar dual of Δ. ■

Although it is "out of interest" for $K3$ surfaces, Definition 2.4 is motivated by Batyrev's theorem [B94].

Theorem 2.1 ([B94])**.** (1) *The followings are equivalent:*
(i) A polytope Δ is reflexive. (ii) The toric variety $\mathbb{P}_\Delta$ is Fano.

(2) *The families $\mathcal{F}_\Delta$ and $\mathcal{F}_{\Delta'}$ of Calabi-Yau varieties with $\Delta'\simeq\Delta^*$ are mirror pair in the sense that there is a duality between their Hodge diamonds.* □

Note that if the pair $(\Delta,\,\Delta')$ of polytopes gives a polytope duality extending a strongly coupling pair $(\boldsymbol{a},\,\boldsymbol{b})$, we have subfamilies of $K3$ surfaces $\mathcal{F}_\Delta\subset\mathcal{F}_{\boldsymbol{a}}$, and $\mathcal{F}_{\Delta'}\subset\mathcal{F}_{\boldsymbol{b}}$.

We open a discussion on lattice duality by formulating the Picard lattice of a family $\mathcal{F}_\Delta$ of $K3$ surfaces associated to a reflexive polytope Δ.

Definition 2.5 (toric contribution)**.** *The* TORIC CONTRIBUTION *is the value defined by*

$$L_0(\Delta):=\sum_{\Gamma\in\Delta^{[1]}} l^*(\Gamma)l^*(\Gamma^*),$$

where $\Delta^{[1]}$ is the set of all edges in Δ, and $l^*(\Gamma)$ is the number of all inner lattice points in an edge Γ. ■

Definition 2.6 (Picard lattice of a family)**.** *We denote by* Pic(Δ) *the Picard lattice of the family* $\mathcal{F}_\Delta$*, and* Pic$(\Delta)_{tor}$ *a sublattice of* Pic(Δ) *that is generated by the divisors which do not contribute the toric contribution.* ■

Definition 2.7 (Lattice duality)**.** *The families* $\mathcal{F}_\Delta$ *and* $\mathcal{F}_{\Delta'}$ *are* LATTICE DUAL *if the isometry*

$$\mathrm{Pic}\,(\Delta')^{\perp}_{\Lambda_{K3}} \simeq U \oplus \mathrm{Pic}\,(\Delta)_{tor}$$

holds true. Here, $\Lambda_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}$ *is the* $K3$ *lattice.* ■

Definition 2.7 is motivated by "Dolgachev-Nikulin mirror" by Dolgachev [D96]: for (primitive) sublattices M and M' of Λ_{K3}, two families of M-/M'-polarized $K3$ surfaces are mirror if the isometry $M^{\perp}_{\Lambda_{K3}} \simeq U \oplus M'$ holds. In particular, extending the study in case of rank-one lattice $M = \langle 2 \rangle$ by [D96], we are interested in more general primitive sublattices of Λ_{K3}.

3 Sketch of the Proof of Main Theorems

Main Theorem A

Step 1. If the isomorphism $\Delta_{\boldsymbol{a}} \simeq \Delta^*_{\boldsymbol{b}}$ holds, then, we may take $\Delta = \Delta_{\boldsymbol{a}}$ and $\Delta' = \Delta_{\boldsymbol{b}}$, and stop here.

Step 2. Otherwise, we try to check a subpolytope Δ of $\Delta_{\boldsymbol{a}}$ satisfies the conditions (1) Δ is reflexive, and (2) the polar dual $\Delta^* \simeq \exists \Delta' \subset \Delta_{\boldsymbol{b}}$.

Main Theorem B

Step 1. Let $X := \mathbb{P}_\Delta$. To find a lattice $\langle D_1|_{-K_X}, D_2|_{-K_X}, \ldots, D_r|_{-K_X}\rangle_{\mathbb{Z}}$ generated by the restrictions to $-K_X$ of linearly-independent toric divisors $D_1, D_2, \ldots, D_r$ with or without divisors that contribute the toric contribution. We then compute the intersection numbers $D_i|_{-K_X}.D_j|_{-K_X}$, which can be achieved by a general theory of toric geometry.

Step 2. To prove the primitivity in Λ_{K3} of the lattice obtained in Step 1.
Case 1. If the discriminant group is of prime order, then, the claim is verified.
Case 2. Otherwise, use the following criterion by Nikulin:

Corollary 3.1 ([N80, Corollary 1.12.3])**.** *Let* L *be an even unimodular lattice of signature* (l_+, l_-)*, and* K *be a sublattice with signature* (t_+, t_-)*. The lattice* K *is a primitive sublattice of* L *if*
(1) $l_+ - l_- \equiv 0 \mod 8$, (2) $l_- - t_- \geq 0$ *and* $l_+ - t_+ \geq 0$*, and*
(3) $\mathrm{rk}\, L - \mathrm{rk}\, K > \mathbf{l}(A_K)$*, where* $\mathbf{l}(A_K)$ *is the length of the discriminant group* A_K *of* K*.* □

Step 3. To prove the isometry $\mathrm{Pic}\,(\Delta')^{\perp}_{\Lambda_{K3}} \simeq U \oplus \mathrm{Pic}\,(\Delta)_{tor}$.
Case 1. If $\mathrm{Pic}\,(\Delta)_{tor}$ and $\mathrm{Pic}\,(\Delta')$ are well-known lattices such as of type ADE, or "star-shaped" $L''_{p,q,r}$, then, its orthogonal complement in Λ_{K3} is known.

Case 2. We know at least the invariants of the lattices: the signature and the discriminant number, and that the lattice is hyperbolic. Here we use the following criterion of Nikulin's:

Corollary 3.2 ([N80, Corollary 1.6.2]). *The lattices K and K' are orthogonal (in L), i.e., $K^{\perp}_{L} \simeq K'$ if and only if $-q_K \simeq q_{K'}$ for the discriminant forms.* □

4 Conclusion, Further Study, and a Table

We have seen that coupling duality for $K3$ weight systems extends to polytope duality and lattice duality in some cases. Therefore, we obtained meanings of the coupling in terms of $K3$ surfaces; interpretations in classical mirror symmetry, and in "Dolgachev-Nikulin mirror symmetry".

We prospect to see a meaning of coupling duality in terms of Seifert form on the Milnor lattice of the simple $K3$ singularity, in particular, zeta functions, in case Ebeling's theorem does not apply.

Remark 1. we obtained a numerical relation between the Seifert form for 3-dimensional IHS and the Picard lattices of weighted $K3$ surfaces in [M23'].

Here is the summarizing table of our main theorems.

#	$b; h$	Δ'	Pic (Δ')	Pic $(\Delta)_{tor}$	Δ	$a; d$
1.	1, 6, 14, 21; 42	$(-1,2,-1)$ $(-1,-1,-1)$ $(-1,-1,1)$ $(6,-1,-1)$	$U \oplus E_8$	$U \oplus E_8$	$(-1,2,-1)$ $(-1,-1,-1)$ $(-1,-1,1)$ $(6,-1,-1)$	1, 6, 14, 21; 42
2.	1, 3, 7, 10; 21					1, 6, 14, 21; 42
3.	1, 4, 9, 14; 28					1, 6, 14, 21; 42
4.	1, 5, 12, 18; 36					1, 6, 14, 21; 42
5.	1, 3, 7, 10; 21					1, 3, 7, 10; 21
6.	1, 4, 9, 14; 28					1, 3, 7, 10; 21
7.	1, 5, 12, 18; 36					1, 3, 7, 10; 21
8.	1, 4, 9, 14; 28					1, 4, 9, 14; 28
9.	1, 5, 12, 18; 36					1, 4, 9, 14; 28
10.	1, 5, 12, 18; 36					1, 5, 12, 18; 36
11. 12. 13. 14.	1, 4, 10, 15; 30	$(-1,-1,1)$ $(-1,-1,-1)$ $(\lambda,-1,-1)$ $(4,0,-1)$ $(-1,2,-1)$	$U \oplus E_7$ $(\lambda, \mu) =$ $(5,2), (6,1)$	$U \oplus A_1 \oplus E_8$	$(-1,-1,1)$ $(-1,-1,-1)$ $(4,-1,-1)$ $(0,2,-1)$ $(-1,\mu,-1)$	1, 6, 8, 15; 30
15. 16. 17. 18.	1, 3, 8, 12; 24	$(-1,-1,1)$ $(-1,-1,-1)$ $(5,-1,-1)$ $(3,-1,0)$ $(-1,2,-1)$	$U \oplus E_6$	$U \oplus A_2 \oplus E_8$	$(-1,-1,1)$ $(-1,-1,-1)$ $(3,-1,-1)$ $(0,-1,1)$ $(-1,2,-1)$	1, 6, 8, 9; 24
19.	1, 4, 6, 11; 22	$(-1,-1,1)$ $(-1,-1,-1)$ $(\lambda_1,-1,-1)$ $(3,0,-1)$ $(0,2,-1)$ $(-1,\lambda_2,-1)$	$U \oplus A_1 \oplus E_7$ $(\lambda_1, \lambda_2; \mu_1, \mu_2) =$ $(3,1;4,2), (4,1;4,1), (3,2;3,2)$	$U \oplus A_1 \oplus E_7$	$(-1,-1,1)$ $(-1,-1,-1)$ $(\mu_1,-1,-1)$ $(3,0,-1)$ $(0,2,-1)$ $(-1,\mu_2,-1)$	1, 4, 6, 11; 22
20.	1, 3, 5, 9; 18	$(-1,-1,1)$ $(-1,-1,-1)$ $(5,-1,-1)$ $(0,2,-1)$ $(-1,2,-1)$	$U \oplus A_1 \oplus E_6$	$A_2 \oplus L''_{2,4,5}$	$(0,-1,1)$ $(-1,-1,-1)$ $(2,-1,-1)$ $(2,0,-1)$ $(-1,2,-1)$	1, 4, 6, 7; 18

(continued from the previous page)

#	$b; h$	Δ'	Pic (Δ')	Pic $(\Delta)_{tor}$	Δ	$a; d$
21.	1, 3, 5, 6; 15	(−1, 2, −1) (0, −1, 1) (−1, −1, 1) (−1, −1, −1) (4, −1, −1)	$U \oplus A_2 \oplus E_6$	$A_2 \oplus L''_{3,3,4}$	(2, −1, 0) (−1, 2, −1) (0, −1, 1) (2, −1, −1) (−1, −1, −1)	1, 3, 5, 6; 15
22. 23.	1, 4, 5, 10; 20 1, 3, 4, 7; 15	(4, −1, −1) (−1, −1, 1) (−1, −1, −1) (−1, 3, −1)	$L''_{2,5,5}$	$L''_{2,5,5}$	(4, −1, −1) (−1, −1, 1) (−1, 1, 0) (−1, −1, −1)	1, 4, 5, 10; 20 1, 4, 5, 10; 20
24.	1, 3, 4, 7; 15	Not exist	....	....	Not exist	1, 3, 4, 7; 15
25.	1, 3, 4, 8; 16	(3, 0, −1) (−1, −1, 1) (4, −1, −1) (−1, 3, −1) (−1, −1, −1)	$L''_{3,4,4}$	$L''_{2,5,6}$	(3, −1, −1) (0, −1, 1) (−1, 1, 0) (−1, −1, 0) (−1, −1, −1)	1, 4, 5, 6; 16
26.	1, 3, 4, 5; 13	$(-1, -1, \lambda_1)$ (−1, 1, 0) (0, −1, 1) (−1, −1, −1) $(\lambda_2, -1, -1)$ (2, 0, −1) $(-1, \lambda_3, -1)$	$U \oplus L'_{(10,-13)}$ $(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3) =$ (1, 2, 2; 1, 2, 0), (0, 3, 2; 0, 3, 0), (1, 3, 0; 0, 2, 2), (1, 3, 2; 0, 2, 0)	$U \oplus L_{(10,-13)}$	$(-1, -1, \mu_1)$ (−1, 1, 0) (0, −1, 1) (−1, −1, −1) $(\mu_2, -1, -1)$ (2, 0, −1) $(-1, \mu_3, -1)$	1, 3, 4, 5; 13
27.	1, 3, 4, 4; 12	Not exist	....	....	Not exist	1, 3, 4, 4; 12
28. 29.	2, 3, 8, 11; 24 2, 5, 14, 21; 42	(1, 0, 0) (0, 1, 0) (0, 0, 1) (−3, −8, −12)	$U \oplus D_4 \oplus E_8$	$U \oplus D_4$	(−1, −1, 1) (−1, 2, −1) (7, −1, −1) (−1, −1, −1)	1, 2, 6, 9; 18 1, 2, 6, 9; 18
30.	2, 3, 8, 13; 26	(1, 0, 0) (0, 1, 0) (0, 0, 1) (1, 1, 1) (−2, −6, −9)	$L''_{2,4,5} \oplus E_6$	$U \oplus A_1 \oplus A_2$	(−1, −1, 1) (−1, 1, −1) (−1, −1, −1) (6, −1, −1) (0, 2, −1)	1, 2, 4, 7; 14
31.	1, 2, 4, 5; 12	Case 1. (0, −1, 1) (−1, −1, −1) $(\lambda, -1, -1)$ (−1, 2, −1) .... Case 2. (−1, 2, −1) (−1, −1, 1) (−1, −1, −1) $\mathbf{u}$, (0, −1, 1)	?? $(\lambda; \mathbf{v}) = (4; (-3, -8, -12)),$ $(5; (-2, -6, -9));$ $(\mathbf{u}; \mathbf{w}) = ((5, -1, -1); --),$ $((3, 0, -1); (-1, -2, -4)),$ $((1, 1, -1); (-2, -5, -8))$	??	Case 1. (0, 1, 0) (0, 0, 1) (2, 2, 3) $\mathbf{v}$ ...... Case 2. (0, 1, 0) (0, 0, 1) (1, 1, 2), (0, −2, −3) (−2, −6, −9), $\mathbf{w}$	2, 3, 10, 15; 30
32. 33. 34. 47.	1, 2, 2, 5; 10 2, 6, 7, 15; 30 2, 5, 6, 13; 26 5, 7, 8, 20; 40	(1, 0, 0) (0, 1, 0) (0, 0, 1) (−2, −2, −5)	$L''_{2,5,5} \oplus E_8$	$\left(\mathbb{Z}^2, \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\right)$	(−1, −1, 1) (−1, −1, −1) (4, −1, −1) (−1, 4, −1)	2, 4, 5, 9; 20 1, 2, 2, 5; 10 1, 2, 2, 5; 10 1, 1, 1, 2; 5
35. 36. 37.	1, 1, 4, 6; 12	(−1, −1, 1) (−1, −1, −1) (11, −1, −1) (−1, 2, −1)	U	$U^{\oplus 2} \oplus E_8^{\oplus 2}$	(−1, 0, 0) (0, 0, 1) (2, 4, −1) (1, −1, 0)	3, 5, 11, 14; 33
38. 40.	1, 1, 3, 5; 10	(−1, −1, 1) (−1, −1, −1) (9, −1, −1) (0, 2, −1) $(-1, \lambda, -1)$	$U \oplus E_7 \oplus E_8$ $(\lambda; \mathbf{v}, \mathbf{u}) =$ $(2; (0, -2, -3), (-1, -3, -5)), (1; (-6, -4, -1), (-5, -3, -1))$	$U \oplus A_1$	(1,0,0), (0,1,0) (0,0,1), $\mathbf{v}$, $\mathbf{u}$	3, 4, 10, 13; 30
39.	1, 1, 3, 5; 10	Not exist	....	....	Not exist	3, 5, 11, 19; 38
41. 42. 43.	1, 1, 3, 4; 9	(−1, 2, −1) (−1, −1, 1) (−1, −1, −1) (8, −1, −1) (0, −1, 1)	$U \oplus A_2$	$U \oplus E_6 \oplus E_8$	(1, 0, 0) (0, 1, 0) (0, 0, 1) (−1, −3, −4) (0, −2, −3)	3, 4, 11, 18; 36
44.	1, 1, 2, 4; 8	(−1, −1, 1) (−1, −1, −1) (7, −1, −1) (−1, 3, −1)	$U \oplus \langle -4 \rangle$	$U \oplus L_{(15,-4)}$	(1, 0, 0) (0, −1, 1) (0, 0, 1) (−1, 2, −6)	3, 4, 7, 10; 24
46.	1, 1, 1, 2; 5	(−1, 2, 0) (−1, −1, 1) (2, −1, 0) (−1, −1, −1) (4, −1, −1) (−1, 4, −1)	$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$	$U \oplus L_{16,5}$	(0, 0, 1) (2, −3, −1) (−1, 1, 0) (0, 1, 0) (1, 0, 0)	4, 5, 7, 9; 25
48. 49.	1, 1, 1, 3; 6	(−1, −1, 1) (−1, −1, −1) (5, −1, −1) (−1, 5, −1)	$U \oplus \langle -2 \rangle \oplus E_8^{\oplus 2}$	$\langle 2 \rangle$	(1, 0, 0) (0, 1, 0) (0, 0, 1) (−1, 3, −1)	5, 6, 8, 11; 30

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#	$b; h$	Δ'	$\mathrm{Pic}\,(\Delta')$	$\mathrm{Pic}\,(\Delta)_{tor}$	Δ	$a; d$
50.	1, 1, 1, 1; 4	(−1, −1, 3) (−1, −1, −1) (3, −1, −1) (−1, 3, −1)	$U \oplus \langle -4 \rangle \oplus E_8^{\oplus 2}$	$\langle 4 \rangle$	(1, 0, 0) (0, 1, 0) (0, 0, 1) (−1, −1, −1)	7, 8, 9, 12; 36
45.	3, 4, 7, 14; 28	(0, 0, 1) (0, 1, 0) (2, −2, −1) (3, −4, −2) (−2, 2, −1)	$U \oplus L_{(14,7)}$	$U \oplus \left(\mathbb{Z}^2, \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}\right)$	(−1, 1, 0) (0, −1, 1) (−1, −1, 1) (−1, −1, −1) (6, −1, −1)	1, 1, 2, 3; 7
51.	2, 2, 3, 7; 14					2, 2, 3, 7; 14

Table 1: Polytope/Lattice duality associated to strongly coupling pairs

Remark 2. In Table 1, we denote by $L_{(r,\delta)}$ and $L'_{(r,\delta)}$ the even positive-definite lattice of rank r and of discriminant δ.

Appendix

We review Ebeling's two results in subsection 1.1: a formula for the reduced zeta functions for a 3-dimensional IHS, and an interpretation of coupling in terms of Saito dual function.

For a 3-dimensional IHS defined by a quasi-homogeneous polynomial f, the Milnor lattice is the reduced homology group $\tilde{H}_*(\mathfrak{F})$ of the Milnor fibre

$$\mathfrak{F} := \{(x_1, x_2, x_3) \in \mathbb{C}^3 | f(x_1, x_2, x_3) = 1\},$$

which admits a monodromy transformation induced by the natural $\mathbb{C}^*$-action

$$\theta : \mathfrak{F} \to \mathfrak{F}; (x_1, x_2, x_3) \mapsto (e^{2\pi i a_1/d} x_1,\ e^{2\pi i a_2/d} x_2,\ e^{2\pi i a_3/d} x_3),$$

together with the induced homomorphism θ_* on the Milnor lattice $\tilde{H}_*(\mathfrak{F})$.

The reduced zeta function of θ is defined by

$$\tilde{\zeta}_C(t) := \prod_{p \geq 0} (\det (id - t\theta_*|_{\tilde{H}_p(\mathfrak{F})}))^{(-1)^p}.$$

In the following, we define a notion "special" subset, the matrix C_{IJ}, and an integer a_J [3].

A subset $J \subset \{1, 2, 3\}$ is *special* if there exists a subset $I \subset \{1, 2, 3\}$ such that $|I| = |J|$, and for all $i \in I$ and $j \in \{1, 2, 3\} \backslash J$, the (i, j)-th entry c_{ij} of C is zero. Always $\emptyset$ and $\{1, 2, 3\}$ are special. Let J be a special set. For $I \subset \{1, 2, 3\}$, define a matrix C_{IJ} by

$$C_{IJ} := (c_{ij})_{i \in I}^{j \in J}, \quad C_\emptyset := (1).$$

Define an integer a_J by $a_J := \gcd (a_j \,|\, j \in J)$. In particular, $a_\emptyset := d$.

[3] See Theorem, part (i) in subsection 1.1.

We review the definition of *Saito dual rational function* [4]. In general, for a rational function $\psi(t) = \prod_{l|h}(1-t^l)^{\alpha_l}, (\alpha_l \in \mathbb{Z})$, define the SAITO DUAL RATIONAL FUNCTION $\psi^*(t)$ by

$$\psi^*(t) = \prod_{m|h}(1-t^m)^{\alpha_{h/m}}.$$

Example 2. Consider a strongly coupled pair considered in Example 1:

$$W_{\boldsymbol{a}} = (1, 2, 9\,; 18), \quad W_{\boldsymbol{b}} = (2, 3, 11\,; 24).$$

By Ebeling's formula, the reduced zeta functions $\tilde{\zeta}_C(t)$ and $\tilde{\zeta}_{{}^tC}(t)$ are given by

$$\tilde{\zeta}_C(t) = (1-t)^{-1} \cdot (1-t^2) \cdot (1-t^{18})^7,$$

$$\tilde{\zeta}_{{}^tC}(t) = (1-t)^{-1} \cdot (1-t^8) \cdot (1-t^{24})^3.$$

The Saito dual $\tilde{\zeta}^*_{{}^tC}(t)$ of the reduced zeta function $\tilde{\zeta}_{{}^tC}(t)$ is given by

$$\tilde{\zeta}^*_{{}^tC}(t) = (1-t^{24}) \cdot (1-t^3)^{-1} \cdot (1-t)^{-3}.$$

This is an example that cannot apply Ebeling's Theorem, part (ii). ■

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[4]See Theorem, part (ii) in subsection 1.1.