THE MUKAI-TYPE CONJECTURE

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First of all, we introduce he following famous conjecture by Shigeru Mukai:

Conjecture 0.1 ([M]). *Let X be a d-dimensional smooth Fano varieties, $\rho(X)$ the Picard number, and*

$$i_X := \max\{r \in \mathbb{Z} | -K_X \sim_{\mathbb{Z}} rH \text{ for some Cartier divisor } H\}$$

be the Fano index. Then it holds that

$$d + \rho(X) - i(X) \cdot \rho(X) \geq 0.$$

Moreover the above is equal if and only if $X \simeq \mathbb{P}^{i(X)-1} \times \cdots \times \mathbb{P}^{i(X)-1}$.

If we call $d + \rho(X) - i(X) \cdot \rho(X)$ *the Mukai complexity* , the above conjectures the complexities are non-negative and smallest one is the product of projective spaces. This observation is very closed to the Shokurov conjecture charactarizing the Toric varieties.

Then we propose a new approach to the above traditional conjecture by using more modern technique from the minimal model program and Toric geometry.

First we propose the new invariant and give a new conjecture of Mukai type.

Definition 0.2. *Let X be a Fano manifold. We define the total index γ_X of X as the maximal of $\sum a_i$ for a decomposition*

$$-K_X = \sum a_i L_i,$$

where L_i are nef line bundles which is not numerically trivial and $a_i \in \mathbb{Z}_{>0}$. Note that we allow $L_i = L_j$ for $i \neq j$.

Conjecture 0.3 (Mukai type conjecture). *It holds that*

$$\dim X + \rho_X - \gamma_X \geq 0$$

and the equality holds if and only if the product of projective spaces

Note that $\dim X + \rho_X - \gamma_X$ should be called by the Shokurov complexities for generalized pairs. The above Mukai type conjecture is more accessible than the original one since that is more direct related with the Shokurov complexities than the original Mukai conjecture by using the minimal model program techniques for generalized pairs. Indeed we proposed two approach to the Mukai type conjecture. First one is to use the Kawamata–Ambro effective non-vanishing ([Am], [Ka]):

Conjecture 0.4. *Let (X, B) be a projective klt pair and L a nef line bundle on X such that $L - (K_X + B)$ is ample.*

Then $H^0(X, L) \neq 0$.

Indeed the we prove

Theorem 0.5. *Assume that Conjecture 0.4 holds. Then Conjecture 0.3 holds.*

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The second approach is to generaralize the Shokurov conjecture (which is proved by [BMSZ]) to generalized pairs. Then we need to expand the notion of Shokurov's complexieies for generalized pairs. However that is known for only surfaces in [GM].

Moreover by comparing the Shokurov and Mukai complexities, we discuss when these have relationships. In particular, we discuss when Shokurov's one is smaller than and equal to Mukai's one. For this purpose, it seems that the key point to study of the effective and nef cone of the Fano manifolds. In particular, we need to investigate about the extremal contraction under the assumption of small Mukai complexities. If it is less than one, we expect to have only fiber type. On the other hand, we have very naive question about the effective cone of Fano manifolds such that all extremal contraction are fiber type. We shall ask whether the such cone is simplicial over $\mathbb{Z}$.

In the last, we include discussion the difference of the integer total index and rational total index.

Definition 0.6. *Let X be a Fano manifold. We define the total index γ_X of X as the maximal of $\sum a_i$ for a decomposition*

$$-K_X = \sum a_i L_i,$$

where L_i are nef line bundles which is not numerically trivial and $a_i \in \mathbb{Z}_{>0}$. Note that we allow $L_i = L_j$ for $i \neq j$.

It seems to be no direct relationship with the product of the Fano index and the Picard number although we propose Conjecture 0.3 motivated with Conjecture 0.1 ;

Example 0.7. *Let X be a three point blow-up of $\mathbb{P}^2$. Then $\gamma_X = 3$ but $\rho(X) = 4$ and the Fano index $i(X) = 1$. Thus $\gamma_X < \rho(X)i(X)$.*

Example 0.8. *Let X be a one point blow-up of $\mathbb{P}^2$. Then $\gamma_X = 3$ but $\rho(X) = 2$ and the Fano index $i(X) = 1$. Thus $\gamma_X > \rho(X)i(X)$.*

We also consider the rational version of γ_X:

Definition 0.9. *Let X be a Fano manifold. We define the total index $\gamma_{X,\mathbb{Q}}$ of X as the upperlimit of $\sum a_i$ for a decomposition*

$$-K_X = \sum a_i L_i,$$

where L_i are nef line bundles which is not numerically trivial and $a_i \in \mathbb{Q}_{>0}$. Note that we allow $L_i = L_j$ for $i \neq j$.

We have a naive question about whether the integral and rational total indexes coincide:

Question 0.10. *Let X be a Fano manifold. Then $\gamma_X = \gamma_{X,\mathbb{Q}}$?*

However, the answer is no to the following example by Atsushi Ito.

Example 0.11 (Atsushi Ito)**.** *Let $\pi : S \to \mathbb{P}^2$ be a general 4-points blow-up of the projective plane. Then $\gamma_S = 2$. Indeed let E_1, E_2, E_3, E_4 be the exceptional divisors of π and L be a line on $\mathbb{P}^2$. Since $\pi^*(2L) - \sum_{i=1}^4 E_i$ is semi-ample, the decomposition*

$$-K_S \sim_{\mathbb{Z}} (\pi^*(2L) - \sum_{i=1}^{4} E_i) + \pi^* L$$

gives $\gamma_S \geq 2$. On the other hand, if $\gamma_S \geq 3$, we have a decomposition $-K_S \sim_{\mathbb{Z}} L_1 + L_2 + L_3$ such that L_i is nef and not numerical trivial Cartier divisor. By the Reimann–Roch formula, we see that $H^0(X, L_i) \neq 0$. Thus we may assume that L_i is an effective divisor. $\pi_* L_i$ is equal to a line on $\mathbb{P}^2$. Now fix i, Since $\pi^*(\pi_* L_i) = L_i + \sum_j d_j E_j$ for some non negative integers d_j. Since L_i is nef, $\sum_j d_j E_j$ is the *prime exceptional divisor. Thus we may assume that* $\pi^*(\pi_* L_i) = L_i + \epsilon_i E_i$ *(by changing the index of the exceptional divisors), where* $\epsilon_i = 0$ *or* 1*. But then* $\sum L_i$ *is not linear equivalent to* $-K_S$*. This is the contradiction. Thus we see that* $\gamma_S = 2$*. Now we see that* $\gamma_{S,\mathbb{Q}} > 2$*. Indeed, let*

$$D_i = \pi^* L - E_i, D' = 2\pi^* L - E_1 - E_2 - E_3 - E_4.$$

Then it holds that

$$1/2(D_1 + D_2 + D_3 + D_4 + D') = -K_S.$$

This decomposition gives $\gamma_{S,\mathbb{Q}} \geq 5/2 > 2$*. Thus* $\gamma_S \neq \gamma_{S,\mathbb{Q}}$

Still, we are interested in the above question for the Toric varieties.

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