FEIGIN AND ODESSKII'S ELLIPTIC ALGEBRAS

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Abstract. This article summarizes some of the results in joint papers [CKS19, CKS21, CKS23] with Alex Chirvasitu and S. Paul Smith. We studied elliptic algebras introduced by Feigin and Odesskii in 1989, which are noncommutative graded algebras $Q_{n,k}(E,\eta)$ parametrized by an elliptic curve E, a point $\eta \in E$, and coprime positive integers $n > k$. These algebras are a generalization of Sklyanin algebras, recognized as important examples of Artin-Schelter regular algebras. One of our main results is that $Q_{n,k}(E,\eta)$ has the same Hilbert series as the polynomial ring in n variables when η is not a torsion point.

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1. Introduction

This article summarizes some of the results in joint papers [CKS19, CKS21, CKS23] with Alex Chirvasitu and S. Paul Smith.

In 1989, Feigin and Odesskii [FO89, OF89] defined a family of graded algebras $Q_{n,k}(E,\eta)$ called *elliptic algebras*. These algebras have played an important role in noncommutative algebraic geometry, especially in the study of Artin-Schelter regular algebras.

Let $\tau \in \mathbb{H}$ be a complex number that is in the upper half plane, and define the lattice $\Lambda := \mathbb{Z} + \mathbb{Z}\tau$. Consider the elliptic curve $E := \mathbb{C}/\Lambda$ and fix a closed point $\eta \in E$. Let $n > k \geq 1$ be coprime integers. The algebra $Q_{n,k}(E,\eta)$ is defined to be the graded $\mathbb{C}$-algebra generated by n variables x_i in degree one, indexed by $i \in \mathbb{Z}/n\mathbb{Z}$, with n^2 quadratic relations

$$\sum_{r\in\mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)}\, x_{j-r}x_{i+r} = 0 \quad (i,j \in \mathbb{Z}/n\mathbb{Z}), \tag{1.1}$$

where θ_α ($\alpha \in \mathbb{Z}/n\mathbb{Z}$) are certain theta functions of order n that are quasi-periodic with respect to the lattice Λ (see [CKS21, §2.2.5]). More precisely, they are holomorphic functions on $\mathbb{C}$ characterized by the following properties (up to common scalar multiple):

$$\theta_\alpha(z+\tfrac{1}{n}) = e^{\frac{2\pi\sqrt{-1}\alpha}{n}}\theta_\alpha(z),$$
$$\theta_\alpha(z+\tfrac{1}{n}\tau) = e^{2\pi\sqrt{-1}(-z-\frac{1}{2n}-\frac{n-1}{2n}\tau)}\theta_{\alpha+1}(z).$$

The zeros of θ_α are the points in $-\frac{\alpha}{n}+\frac{1}{n}\mathbb{Z}+\mathbb{Z}\tau$ (all have multiplicity one), so the denominator in the relations (1.1) can be zero when η is in $\frac{1}{n}\Lambda$ (that is, η is an n-torsion point). Nonetheless, there is a way to extend the definition to all $\eta \in E$, and then, $Q_{n,k}(E,0)$ is the polynomial ring in n variables ([CKS21, Proposition 5.1]). So $\{Q_{n,k}(E,\eta)\}_{\eta\in E}$ can be thought as a family of noncommutative deformations of the polynomial ring.

These algebras have been studied from various perspectives. One significant aspect is that they are a rich source of Artin-Schelter regular algebras.

A graded $\mathbb{C}$-algebra A generated by finitely many degree-one elements $x_1,\dots,x_r$ over $A_0=\mathbb{C}$ with relations $f_1,\dots,f_s$ is called *Artin-Schelter regular* (or *AS-regular* for short) of dimension n if

(1) the global dimension of A is n;
(2) $\operatorname{Ext}^i_A(\mathbb{C},A) \cong \begin{cases}\mathbb{C} & \text{if } i=n,\\ 0 & \text{if } i\neq n;\end{cases}$
(3) the sequence $\{\dim_{\mathbb{C}} A_i\}_{i=0}^{\infty}$ has polynomial growth.

Artin-Schelter [AS87] gave a partial classification of 3-dimensional AS-regular algebras. Their first observation was that the 3-dimensional AS-regular algebras are either of *quadratic type* or of *cubic type*, the former has 3 generators (in degree 1) and 3 quadratic relations (as the polynomial ring in 3 variables does), and the latter has 2 generators and 2 cubic relations.

Artin-Tate-Van den Bergh [ATVdB90] showed that one can associate a triple $(X,\sigma,\mathcal{L})$ to a 3-dimensional AS-regular algebra, where X is a scheme (a closed subscheme of $\mathbb{P}^2$ or $\mathbb{P}\times\mathbb{P}$), σ is an automorphism of X, and $\mathcal{L}$ is an invertible sheaf on X, and that the algebra can be recovered from the associated triple. In this way, the classification of 3-dimensional AS-regular algebras can be reduced to the classification of such triples. In the recent works by Itaba-Matsuno [IM21] and Matsuno [Mat21], the classification of quadratic 3-dimensional algebras has been completed, in the sense that they wrote down the relations for each isomorphism classes explicitly. We may associate a triple to a higher-dimensinal AS-regular algebra, but there is no method to recover the algebra from the triple in general. Thus the classification for AS-regular algebras of dimension ≥ 4 is not in sight.

When $k=1$, $Q_{n,1}(E,\eta)$ are known as *Sklyanin algebras*, and have been considered as important examples of quadratic Artin-Schelter regular algebras. The name comes from the appearance of $Q_{4,1}(E,\eta)$ in Sklyanin's papers on the quantum scattering inverse method [Skl82]. The list of 3-dimensional quadratic Artin-Schelter regular algebras of Artin-Schelter [AS87] or Artin-Tate-Van den Bergh [ATVdB90] suggests that $Q_{3,1}(E,\eta)$ are the most "standard" family among those algebras.

In 1992, Smith-Stafford [SS92] showed that $Q_{4,1}(E,\eta)$ is a 4-dimensional AS-regular algebra, and that it is a noetherian domain that has the same Hilbert series as the polynomial ring in 4 variables. This result was generalized by Tate and Van den Bergh [TVdB96] to $Q_{n,1}(E,\eta)$ for arbitrary n; they showed that $Q_{n,1}(E,\eta)$ is n-dimensional AS-regular and has many other good homological properties shared with a polynomial ring in n variables.

When $k>1$, not much was known about $Q_{n,k}(E,\eta)$. Thus, Alex Chirvasitu, S. Paul Smith, and the author initiated a project to investigate these algebras. This leads to the results that we will review in later sections.

2. Hilbert series and AS-regularity

Feigin-Odesskii ([FO98, Ode02], for example) claimed that the algebras $Q_{n,k}(E,\eta)$ have the same Hilbert series as the polynomial ring in n variables (namely $(1-t)^{-n}$) for generic η, and provided some ideas for the proof. We made the statement more precise, and gave a complete proof for that.

Theorem 2.1 ([CKS23, Theorem 1.1]). *Assume that $\eta\in E$ is not a torsion point.*

(1) *The Hilbert series of $Q_{n,k}(E,\eta)$ is the polynomial ring in n variables.*
(2) *$Q_{n,k}(E,\eta)$ is a Koszul algebra whose global dimension is n.*

We will briefly sketch the proof for the first statement. The key fact is that the defining relations for $Q_{n,k}(E,\eta)$ come from a certain solution to the quantum Yang-Baxter equation. Let V be the degree-one part of the algebra $Q_{n,k}(E,\eta)$, namely the vector space with basis $\{x_i\}_{i\in\mathbb{Z}/n\mathbb{Z}}$. Define the linear operator $R_\eta(z)\colon V\otimes_{\mathbb{C}}V\to V\otimes_{\mathbb{C}}V$ by

$$R_\eta(z)(x_i\otimes x_j):=\frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)}\sum_{r\in\mathbb{Z}/n\mathbb{Z}}\frac{\theta_{j-i+r(k-1)}(-z+\eta)}{\theta_{j-i-r}(-z)\theta_{kr}(\eta)}\,x_{j-r}\otimes x_{i+r}.$$

The family of operators $\{R_\eta(z)\}_{z\in\mathbb{C}}$ satisfies the *quantum Yang-Baxter equation with one spectral parameter*:

$$R_\eta(u)_{12}R_\eta(u+v)_{23}R_\eta(v)_{12}=R_\eta(v)_{23}R_\eta(u+v)_{12}R_\eta(u)_{23}$$

on $V\otimes_{\mathbb{C}}V\otimes_{\mathbb{C}}V$, where $A_{12}:=A\otimes\mathrm{id}_V$ and $A_{23}:=\mathrm{id}_V\otimes A$ for an operator A on $V\otimes_{\mathbb{C}}V$. This solution is essentially Belavin's elliptic solution ([Bel80]; see [CKS23, §1.3]).

It is easy to see that the space for quadratic relations for $Q_{n,k}(E,\eta)$ is the image of $R_\eta(\eta)$, that is, $Q_{n,k}(E,\eta)$ is the tensor algebra $T_{\mathbb{C}}V$ modulo the ideal generated by $\operatorname{Im}R_\eta(\eta)$.

We construct an operator $F_d(-\eta)$ on $V^{\otimes d}$ by composing some operators of the form $\mathrm{id}_V^{\otimes s}\otimes R_\eta(z)\otimes\mathrm{id}_V^{\otimes t}$ for various z and $s+t+2=d$, and show that

$$\sum_{s+t+2=d}V^{\otimes s}\otimes\operatorname{Im}R_\eta(\eta)\otimes V^{\otimes t}\subseteq\operatorname{Ker}F_d(-\eta),\tag{2.1}$$

where the left-hand side is the space of degree-d relations for $Q_{n,k}(E,\eta)$. We do not present the construction of $F_d(-\eta)$ here but $F_2(-\eta)=R_\eta(-\eta)$ in the case of $d=2$. (2.1) can be deduced using the fact that $R_\eta(z)$ satisfies the quantum Yang-Baxter equation, with some additional observation on $R_\eta(z)$ combined.

We will consider the limit $\eta \to 0$. It is easy to see that the map $\eta \mapsto R_\eta(\eta)$ is holomorphic on $\mathbb{C}$ so taking the limit is essentially substitution $\eta = 0$, but we should note that the limit of $R_\eta(\eta)$ may be, and actually is, different from the limit of $R_\eta(-\eta)$. This is because the map $(\eta, z) \mapsto R_\eta(z)$ is not holomorphic on $\mathbb{C}^2$. Taking the limit $\eta \to 0$, the image of the operator $R_\eta(\eta)$ becomes the space of quadratic relations for $Q_{n,k}(E, 0)$, which is the polynomial ring in n variables, so the dimension of the left-hand side of (2.1) becomes the dimension of the space of degree-d relations for the polynomial ring, which is $n^d - \binom{n+d-1}{d}$. On the other hand, it can be observed by a direct computation that

$$\lim_{\eta \to 0} F_d(-\eta) = \prod_{m=1}^{d-1} m! \cdot \sum_{\sigma \in S_d} \sigma,$$

where the symmetric group S_d acts on $V^{\otimes d}$ by permuting tensorands. So the dimension of the right-hand side of (2.1) also becomes $n^d - \binom{n+d-1}{d}$.

For a continuous family of operators, the dimension of the image attains the largest value on a Zariski-open subset, so the dimension of the left-hand side of (2.1) has the same property, and hence it is $\geq n^d - \binom{n+d-1}{d}$ for a general point η. Similarly, the dimension of the kernel attains the smallest value on a Zariski-open subset, so the dimension of the right-hand side of (2.1) is $\leq n^d - \binom{n+d-1}{d}$ for general η. Since we have the inclusion (2.1) for all η, it follows that (2.1) is an equality for general η and, in that case, they have dimension $n^d - \binom{n+d-1}{d}$. Moreover,

$$\dim \sum_{s+t+2=d} V^{\otimes s} \otimes \operatorname{Im} R_\eta(\eta) \otimes V^{\otimes t} \leq n^d - \binom{n+d-1}{d} \leq \dim \operatorname{Ker} F_d(-\eta) \tag{2.2}$$

for all η. We have to show that the inequalities in (2.2) are equalities whenever η is not a torsion point. Since the treatment for general d is complicated [CKS23, §6.4], we will focus on the case $d = 2$ here. In this case, (2.1) is

$$\operatorname{Im} R_\eta(\eta) \subseteq \operatorname{Ker} R_\eta(-\eta)$$

and we already know from (2.2) that

$$\dim \operatorname{Im} R_\eta(\eta) \leq \binom{n}{2} \leq \dim \operatorname{Ker} R_\eta(-\eta). \tag{2.3}$$

Thus it suffices to compute the dimension of $\operatorname{Ker} R_\eta(z)$ for all $z \in \mathbb{C}$ (by the rank-nullity theorem). The key observation for this is that, for all $p \in \mathbb{C}$,

$$\dim \operatorname{Ker} R_\eta(p) \leq \operatorname{mult}_p \det R_\eta(z), \tag{2.4}$$

where the right-hand side is the multiplicity of the function $z \mapsto \det R_\eta(z)$ at $z = p$. (2.4) is not specific to $R_\eta(z)$ and can be shown elementarily ([CKS23, Lemma 4.1]). By looking at the definition of $R_\eta(z)$, we notice that the function $z \mapsto \det R_\eta(z)$ is a theta function of order n^4 with respect to Λ, and hence has exactly n^4 zeros in each fundamental parallelogram for Λ. So we obtain

$$\sum_p \dim \operatorname{Ker} R_\eta(p) \leq \sum_p \operatorname{mult}_p \det R_\eta(z) = n^4, \tag{2.5}$$

where p runs over a fundamental parallelogram. The final step is to show that this is an equality when τ is not a torsion point. By (2.3), we already know that $\dim \operatorname{Ker} R_\eta(-\eta) \geq \binom{n}{2}$, and a similar argument shows that $\dim \operatorname{Ker} R_\eta(\eta) \geq \binom{n+1}{2}$. We observe that $R_\eta(z+\zeta)$ has the same rank as that of $R_\eta(z)$ for all $\zeta \in \frac{1}{n}\Lambda$ ([CKS23, Proposition 2.6]), so $\dim \operatorname{Ker} R_\eta(-\eta+\zeta) \geq \binom{n}{2}$ and $\dim \operatorname{Ker} R_\eta(\eta+\zeta) \geq \binom{n+1}{2}$ for all $\zeta \in \frac{1}{n}\Lambda$. There are n^2 such ζ in the fundamental parallelogram for Λ. If $\eta \notin \frac{1}{2n}\Lambda$, then $\{\, \eta+\zeta \mid \zeta \in \frac{1}{n}\Lambda \,\}$ and $\{\, -\eta+\zeta \mid \zeta \in \frac{1}{n}\Lambda \,\}$ have no intersection, so the dimensions of the kernels add up to (at least)

$$\binom{n}{2} \cdot n^2 + \binom{n+1}{2} \cdot n^2 = n^4.$$

Therefore (2.5) is an equality and so is (2.4).

We also have a result on the AS-regularity:

Theorem 2.2 ([CKS23, Theorem 1.2]). *$Q_{n,k}(E,\eta)$ is n-dimensional AS-regular for all but countably many $\eta \in E$.*

This result is of weaker form than Theorem 2.1 in the sense that we cannot detect the points where $Q_{n,k}(E,\eta)$ is (possibly) not AS-regular. Nevertheless, this fact tells us that elliptic algebras provide a large family of AS-regular algebras.

3. Point modules

The statements of Theorem 2.1 and Theorem 2.2 do not involve the number k at all, so the reader may wonder how different $Q_{n,k}(E,\eta)$ are from $Q_{n,1}(E,\eta)$. In fact, they are quite different, and looking at the structure of point modules is one way to observe this.

For a graded $\mathbb{C}$-algebra A generated by finitely many degree-one elements $x_1, \ldots, x_r$ over $A_0 = \mathbb{C}$, a *point module* is a graded (right) A-module M such that

(1) $\dim_{\mathbb{C}} M_i = \begin{cases} 1 & \text{if } i \geq 0, \\ 0 & \text{if } i < 0; \end{cases}$

(2) M is generated in degree zero, that is, $M = M_0 A$.

Considering point modules for a given algebra was essential in the work of Artin-Tate-Van den Bergh [ATVdB90]. Indeed, in the triple $(X, \sigma, \mathcal{L})$ constructed from a 3-dimensional AS-regular algebra A, the closed points of the scheme X parametrizes the isomorphism classes of point modules over A. Such X is called the *point scheme* of A. The point scheme can be defined for A that is not necessarily 3-dimensional AS-regular. It is defined to be an inverse limit of schemes (more precisely, an inverse limit of sets, each of which has a natural scheme structure), but the inverse system often stabilizes, which makes the inverse limit an actual scheme.

The point scheme of the Sklyanin algebras $Q_{n,1}(E,\eta)$ are understood when η is a general point. For all $n \geq 3$, except for $n = 4$, the point scheme is the elliptic curve E ([Smi94]). When $n = 4$, the point scheme is the union of E and 4 additional points. If we embed E into $\mathbb{P}^3$ as an elliptic normal curve of degree 4, those four points are realized as the vertices (singular points) of the four singular quadrics that contain E ([SS92]).

The point scheme for $Q_{n,k}(E,\eta)$ for $k>1$ is not well-understood. Feigin-Odesskii [OF89, FO98] claimed that the point scheme of $Q_{n,k}(E,\eta)$ is a finite product E^g modulo an action of a finite group, where the number g is the length of the negative continued fraction for n/k:

$$\frac{n}{k} \;=\; n_1 - \cfrac{1}{n_2 - \frac{1}{\cdots - \frac{1}{n_g}}} \quad (n_1,\dots,n_g \text{ are integers } \geq 2). \tag{3.1}$$

However, if $k=1$, then $g=1$, so according to this claim the point scheme of $Q_{n,1}(E,\eta)$ should be a quotient of E, but this is not the case when $n=4$, because the point scheme has 4 additional points.

In [CKS19], we defined the action of a finite group explicitly and concluded that the point scheme always *contains* the scheme described by Feigin-Odesskii. To describe the action, let $\varepsilon\colon E^g \to E^{g+1}$ be the injection defined by

$$\varepsilon(z_1,\dots,z_g) := (z_1, z_2-z_1,\dots,z_g-z_{g-1},-z_g).$$

and identify E^g with $\operatorname{Im}\varepsilon$, which consists of those points in E^{g+1} whose sum of coordinates is zero. The symmetric group S_{g+1} acts on E^{g+1} by permutation, and it induces the action on $\operatorname{Im}\varepsilon$, hence on E^g. Define Σ to be the subgroup of S_{g+1} generated by transpositions $(i,i+1)$ for all $1\leq i\leq g$ satisfying $n_i=2$.

Theorem 3.1 ([CKS19, Theorem 1.2, Proposition 5.12])**.** *For all $\eta\in E$, the point scheme of* $Q_{n,k}(E,\eta)$ contains $X_{n/k} := E^g/\Sigma$.

This reveals a significant difference between $Q_{n,1}(E,\eta)$ and $Q_{n,k}(E,\eta)$ for $k>1$. For example, if $n=5$ and $k=2$, then the negative continued fraction is

$$\frac{5}{2} = 3 - \frac{1}{2},$$

so $g=2$ and $(n_1,n_2)=(3,2)$. A direct computation shows that $X_{5/2}=E^2/\Sigma$ is the symmetric product S^2E. So the point scheme of $Q_{5,2}(E,\eta)$ contains $X_{5/2}$. In contrast, when $\eta\in E$ is a general point, the point scheme of $Q_{5,1}(E,\eta)$ is E, which cannot contain S^2E.

References

[AS87] M. Artin and W. F. Schelter, *Graded algebras of global dimension* 3, Adv. in Math. **66** (1987), no. 2, 171–216. MR 917738 (88k:16003)

[ATVdB90] M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 33–85. MR 1086882 (92e:14002)

[Bel80] A. A. Belavin, *Discrete groups and integrability of quantum systems*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 4, 18–26, 95. MR 595725

[CKS19] A. Chirvasitu, R. Kanda, and S. P. Smith, *The characteristic variety for Feigin and Odesskii's elliptic algebras*, arXiv:1903.11798v4.

[CKS21] A. Chirvasitu, R. Kanda, and S. P. Smith, *Feigin and Odesskii's elliptic algebras*, J. Algebra **581** (2021), 173–225. MR 4249759

[CKS23] A. Chirvasitu, R. Kanda, and S. P. Smith, *Elliptic R-matrices and Feigin and Odesskii's elliptic algebras*, Selecta Math. (N.S.) **29** (2023), no. 2, Paper No. 31, 81 pages. MR 4563694

[FO89] B. L. Feigin and A. V. Odesskii, *Sklyanin algebras associated with an elliptic curve*, Preprint deposited with Institute of Theoretical Physics of the Academy of Sciences of the Ukrainian SSR (1989), 33 pages.

[FO98] B. L. Feigin and A. V. Odesskii, *Vector bundles on an elliptic curve and Sklyanin algebras*, Topics in quantum groups and finite-type invariants, Amer. Math. Soc. Transl. Ser. 2, vol. 185, Amer. Math. Soc., Providence, RI, 1998, pp. 65–84. MR 1736164 (2001f:14063)

[IM21] A. Itaba and M. Matsuno, *Defining relations of 3-dimensional quadratic AS-regular algebras*, Math. J. Okayama Univ. **63** (2021), 61–86. MR 4174859

[Mat21] M. Matsuno, *A complete classification of 3-dimensional quadratic AS-regular algebras of type EC*, Canad. Math. Bull. **64** (2021), no. 1, 123–141. MR 4242997

[Ode02] A. V. Odesskii, *Elliptic algebras*, Uspekhi Mat. Nauk **57** (2002), no. 6(348), 87–122. MR 1991863

[OF89] A. V. Odesskii and B. L. Feigin, *Sklyanin elliptic algebras*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 3, 45–54, 96. MR 1026987 (91e:16037)

[Skl82] E. K. Sklyanin, *Some algebraic structures connected with the Yang-Baxter equation*, Funktsional. Anal. i Prilozhen. **16** (1982), no. 4, 27–34, 96. MR 684124 (84c:82004)

[Smi94] S. P. Smith, *Point modules over Sklyanin algebras*, Math. Z. **215** (1994), no. 2, 169–177. MR 1259453

[SS92] S. P. Smith and J. T. Stafford, *Regularity of the four-dimensional Sklyanin algebra*, Compositio Math. **83** (1992), no. 3, 259–289. MR 1175941 (93h:16037)

[TVdB96] J. T. Tate and M. Van den Bergh, *Homological properties of Sklyanin algebras*, Invent. Math. **124** (1996), no. 1-3, 619–647. MR 1369430 (98c:16057)

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