Wall-crossing for framed quiver moduli

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Abstract

We investigate the wall-crossing phenomena for moduli of framed quiver representations. As main motivating examples, we present framed quivers for the type A flag manifolds, Nakajima quiver varieties of type A, framed moduli of sheaves on the projective plane, the blow-up, and the (-2)-curve, and the type A affine Laumon spaces. In particular, we give the residue formula for the type A flag manifold as an example of wall-crossing formula. We roughly explain the way developed by Mochizuki, that is, the localization technique on the enhanced master spaces, in the setting of framed quiver. Specifically, we examine the wall-crossing formulas for integrals of Euler classes over the framed quiver moduli spaces.

1 Introduction

A framed quiver representation is a quiver representation with a framing. Moduli of framed quiver representations are studied and called *framed quiver moduli* by Reineke [13]. These moduli spaces contain interesting example with applications to the representation theory and integrable systems.

Mochizuki [4] developed wall-crossing formula for moduli of parabolic Higgs bundles on algebraic surfaces. Nakajima-Yoshioka [6] interpreted this theory in terms of quivers in the study of moduli of framed sheaves on the blow up $\hat{\mathbb{P}}^2$ along a point on $\mathbb{P}^2$. These are isomorphic to moduli of representations over the framed quiver introduced in Example 2.5. In this note, we give a brief summary and materials of [10] where we study wall-crossing formulas for general framed quivers. Finally, we state the wall-crossing formulas in Theorem 5.1 for integrals of Euler classes over the framed quiver moduli.

2 Framed quiver representations

2.1 Setting

We consider $Q = (Q_0, Q_1, Q_2)$ a quiver with relations, where Q_0 is the set of vertices, Q_1 the set of arrows, and Q_2 the set of relations.

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$$\mathrm{in}(a) \bullet \xleftarrow{\ a\ } \bullet\, \mathrm{out}(a)$$

A path is a consecutive arrows: $\quad p = a_\ell a_{\ell-1} \cdots a_2 a_1$

$$\mathrm{in}(p) \bullet \xleftarrow{\ a_\ell\ } \bullet \xleftarrow{\ a_{\ell-1}\ } \bullet \longleftarrow \cdots \longleftarrow \bullet \xleftarrow{\ a_2\ } \bullet \xleftarrow{\ a_1\ } \bullet\, \mathrm{out}(p)$$

A relation is a linear combinations of paths

$$l = \sum_{\substack{p\colon \text{path} \\ \mathrm{out}(p)=i, \mathrm{in}(p)=j}} c_p^{(l)} \cdot p$$

where $c_p^{(l)} \in \mathbb{C}$ ($c_p^{(l)} = \{0, \pm 1\}$ in the following). Set $\mathrm{out}(l) = i$ and $\mathrm{in}(l) := j$

For a finite dimensional Q_0-graded vector space $V = \bigoplus_{v \in Q_0} V_v$, we consider a Q-representation $\rho = (\rho_a)_{a \in Q_1} \in \prod_{a \in Q_1} \mathrm{Hom}(V_{\mathrm{out}(a)}, V_{\mathrm{in}(a)})$ over V. For a relation $l = \sum c_p^{(l)} \cdot p$, we set

$$\rho(l) = \sum c_p^{(l)} \cdot \rho(p) \colon V_{\mathrm{out}(l)} \to V_{\mathrm{in}(l)}$$

where $\rho(p) = \rho_{a_\ell} \circ \rho_{a_{\ell-1}} \circ \cdots \circ \rho_{a_2} \circ \rho_{a_1}$ for a path $p = a_\ell a_{\ell-1} \cdots a_2 a_1$. Set

$$\mathrm{Rep}_Q(V) = \left\{ \rho \in \prod_{a \in Q_1} \mathrm{Hom}(V_{\mathrm{out}(a)}, V_{\mathrm{in}(a)}) \;\middle|\; \rho(l) = 0 \text{ for any } l \in Q_2 \right\}$$

We assume that there exists $\infty \in Q_0$ called *framing vertex* such that $\dim V_\infty = 1$. A pair (Q, ∞) is called a *framed quiver*.

For $I = Q_0 \setminus \{\infty\}$, we take a *generic* $\zeta = (\zeta_i)_{i \in I} \in \mathbb{R}^I$ and set $\zeta_\infty = -\sum_{i \in I} \zeta_i \dim V_i$. We set $\zeta(S) = \sum_{v \in Q_0} \zeta_v \dim S_v$ for any $S = \bigoplus_{v \in Q_0} S_v$.

Definition 2.1. A Q-representation ρ is said to be ζ-semisable if and only if we have $\zeta(S) \leq 0$ for any sub-representation $0 \neq S \subsetneq V$ of ρ.

We set

$$M_Q^\zeta(\alpha) = M^\zeta(\alpha) = \{\rho \in \mathrm{Rep}_Q(V) \colon \zeta\text{-semistable}\} / \prod_{i \in I} \mathrm{GL}(V_i),$$

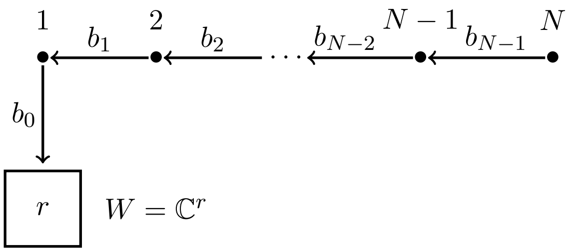
where $\alpha = (\dim V_v)_{v \in Q_0} \in (\mathbb{Z}_{\geq 0})^{Q_0}$.

2.2 Example of framed quivers

We use the following diagram:

This induces actions of diaonal torus $(\mathbb{C}^*)^r \subset \mathrm{GL}(\mathbb{C}^r)$ via the natural $\mathrm{GL}(\mathbb{C}^r)$-action on $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, V_i)$ and $\mathrm{Hom}_{\mathbb{C}}(V_i, \mathbb{C}^r)$ Some examples are presented by the following diagrams.

Example 2.2 (Flag manifold)**.**



In this case, we have

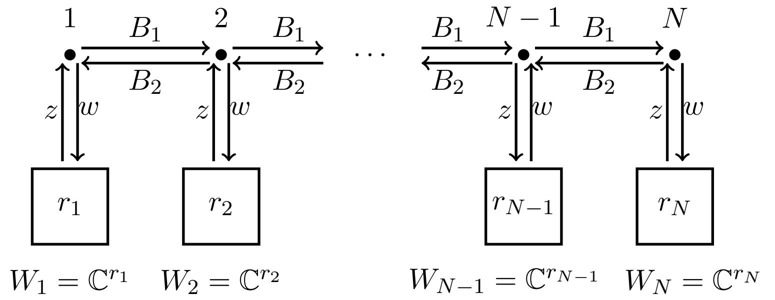
$$\mathrm{Rep}_Q(V) = \mathrm{Hom}_{\mathbb{C}}(V_1, W) \times \prod_{i=1}^{N-1} \mathrm{Hom}_{\mathbb{C}}(V_{i+1}, V_i),$$

and for a stability parameter $\zeta = (1, \ldots, 1) \in \mathbb{Z}^I$, the stable locus consists of $(b_k)_{k=0}^{N-1} \in \mathrm{Rep}_Q(V)$ such that b_k is injective for all $k = 0, 1, \ldots, N-1$. Thus we have an isomorphism

$$M^\zeta(\alpha) \cong Fl(W; d_1, \ldots, d_N), \quad (b_k)_{k=0}^{N-1} \mapsto F_\bullet = (\mathrm{im}(b_0 \cdots b_k))_{k=0}^{N-1},$$

where $Fl(W; d_1, \ldots, d_N)$ is the flag manifold parametrizing the flags $F_\bullet$ in W with $\dim F_i = \dim V_i = d_i$.

Example 2.3 (Quiver variety of type A_n)**.**

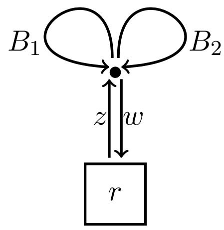


The set Q_2 of relations are

$$\begin{aligned} -B_2B_1 + zw &\in \mathrm{End}_{\mathbb{C}}(V_1) \\ B_1B_2 - B_2B_1 + zw &\in \mathrm{End}_{\mathbb{C}}(V_2) \\ &\vdots \\ B_1B_2 - B_2B_1 + zw &\in \mathrm{End}_{\mathbb{C}}(V_{N-1}) \\ B_1B_2 + zw &\in \mathrm{End}_{\mathbb{C}}(V_N). \end{aligned}$$

They are examples of Nakajima quiver varieties introduced in [5]. Here framed quivers Q are associated to the Dynkin quiver of type A. When $r_2 = \cdots = r_n = 0$ and $\zeta = (1, \ldots, 1) \in \mathbb{R}^I$, they are isomorphic to the cotangent bundle $T^*Fl(W_1; d_1, \ldots, d_N)$ of the flag manifold $Fl(W_1; d_1, \ldots, d_N)$.

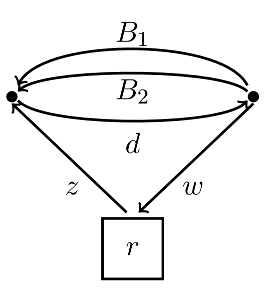
Example 2.4 (Framed moduli space on $\mathbb{P}^2$ (Jordan quiver))**.**



$Q_2\colon B_1B_2 - B_2B_1 + zw$

These framed quiver also give an example of the Nakajima quiver varieties associated to the Jordan quiver. When $\zeta < 0$, the framed moduli are isomorphic to the moduli spaces of framed sheaves on $\mathbb{P}^2$ with the rank r and the second Chern class $\dim V$. In [7], we study wall-crossing phenomena to give functional equations of Nekrasov functions associated to the fundamental matter theory.

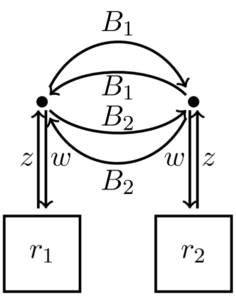
Example 2.5. [Framed moduli space on the blowup $\hat{\mathbb{P}}^2$]



$Q_2\colon B_1dB_2 - B_2dB_1 + zw$

This is the first motivating example studied in [6] to apply wall-crossing formula to the setting of framed quiver moduli. When $\zeta = (-1, -1)$, the framed quiver moduli $M^\zeta(\alpha)$ are isomorphic to the moduli spaces of framed sheaves on the projective plane $\mathbb{P}^2$ with the Chern classes determined from r and $(\dim V_i)_{i\in I}$. On the other hand, when ζ approach enough to the ray $\mathbb{R}_{>0}(-1,1)$ from the below, the framed quiver moduli $M^\zeta(\alpha)$ is isomorphic to the moduli space of framed sheaves on the blow-up $\hat{\mathbb{P}}^2$ of one point on $\mathbb{P}^2$ with the Chern classes compatible with the morphism $\hat{\mathbb{P}}^2 \to \mathbb{P}^2$. This situation is used to deduce *blow-up formula* in [6].

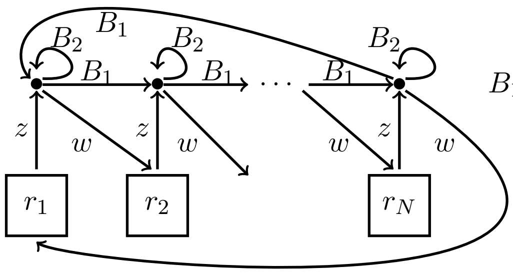
Example 2.6 (Framed moduli space on (-2)-curve)**.**



$Q_2\colon B_1B_2 - B_2B_1 + zw$

These framed quivers give the Nakajima quiver varieties associated to the Dynkin quiver of type $A_1^{(1)}$. This is analogous situation to the previous example. When $\zeta = (-1,-1)$, the framed quiver moduli $M^\zeta(\alpha)$ are isomorphic to the moduli spaces of framed sheaves on the quotient stack $[\mathbb{P}^2/\pm 1]$ with the Chern classes determined from (r_1, r_2) and $(\dim V_i)_{i\in I}$. On the other hand, when ζ approach enough to the ray $\mathbb{R}_{>0}(-1,1)$ from the below, the framed quiver moduli $M^\zeta(\alpha)$ is isomorphic to the moduli space of framed sheaves on the (-2)-curve with the Chern classes corresponding via the derived Mckay equivalence. In [8], we study wall-crossing phenomena to give functional equations among the generating functions over these moduli spaces as a refinement of the derived Mckay equivalence.

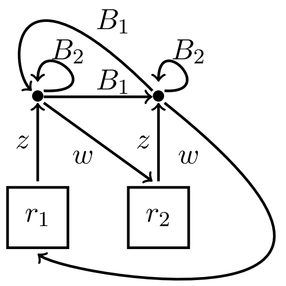
Example 2.7 (Chainsaw quiver variety (Affine Laumon space))**.**



$B_1B_2 - B_2B_1 + zw$

The framed quiver moduli spaces $M^\zeta(\alpha)$ are called the chainsaw quiver variety. In particular, when $\zeta = (-1, \ldots, -1) \in \mathbb{R}^I$, the framed quiver moduli spaces $M^\zeta(\alpha)$ are called the *affine Laumon spaces*. In [12], we study K-theoretic wall-crossing formula, and in particular, apply it to the case where $N = 2$ and $V_2 = 0$ to deduce some transformation formula for multiple hypergeometric series. One of them gives another proof to the Kajihara transformation formula [3, Theorem 1.1], which is a multiple generalization of the Euler transformation formula for the Gauss hypergeometric series. Furthermore in [12], we give a conjecture for general N and V that some generating series of K-theoretic integrals over framed quiver moduli $M^\zeta(\alpha)$ satisfies transformation formulas.

Example 2.8 (Affine Laumon space ($N = 2$, $r_1 = r_2 = 1$)).



$Q_2\colon B_1B_2 - B_2B_1 + zw$

This is the special cases of the previous example. In [1], we showed the non-stationary difference equation studied by Shakirov is gauge equivalent to the qq-Painlevé VI equation introduced by Hasegawa, and conjectrued that generating series of integrals over affine Laumon spaces gives a solution. In [2], we showed that this conjecture is true describing generating series in terms of the Jackson integrals satisfying the q-KZ equation, which is shown to be equal to the truncated Shakirov equation.

However we want to pursue possibility for another proof of the conjecture using K-theoretic wall-crossing formula developed in [12], and generalize it to arbitrary $N \geq 2$ and $(r_i)_{i \in I}$.

3 Residue formula

We give the residue formula for the type A flag manifold as an example of wall-crossing formula.

3.1 Projective space

First we explain outline of our computations for the special case. Let $W = \mathbb{C}^r$ be a r-dimensional vector space, and set

$$\mathbb{P}^{r-1} = \mathbb{P}(W) = (W \setminus \mathbf{0})/\mathbb{C}^*.$$

We compute the Euler number of $\mathbb{P}^{r-1}$ by Mochizuki method. Set $\mathcal{M} = \mathbb{P}(W \oplus \mathbb{C})$, and consider $\mathbb{C}^*_\hbar$-action defined by

$$[w_1, \ldots, w_r, x] \mapsto [w_1, \ldots, w_r, e^\hbar x]$$

for $e^\hbar \in \mathbb{C}^*_\hbar$. The fixed points set $\mathbb{P}(W \oplus \mathbb{C})^{\mathbb{C}^*_\hbar}$ is decomposed as

$$\mathbb{P}(W \oplus \mathbb{C})^{\mathbb{C}^*_\hbar} = \mathbb{P}(W) \sqcup \mathrm{pt} \xrightarrow{\iota} \mathcal{M},$$

where $\mathbb{P}(W) = \{x = 0\}$ and $\mathrm{pt} = \{[0, \ldots, 0, 1]\}$.

We introduce the equivariant Chow ring. For a variety X over $\mathbb{C}$ with $\mathbb{C}^*_\hbar$-action, We write bye $A^\bullet_{\mathbb{C}^*_\hbar}(X)$ the $\mathbb{C}^*_\hbar$-equivariant Chow ring of X.

Fact 3.1. When $X = \mathrm{pt}$, we have

$$A^\bullet_{\mathbb{C}^*_\hbar}(\mathrm{pt}) \cong \mathbb{Z}[\hbar],$$

where $\hbar = c_1(e^\hbar)$ for the weight space $e^\hbar$ with the eigenvalue $e^\hbar \in \mathbb{C}^*_\hbar$. The weight space $e^\hbar$ is regarded as a $\mathbb{C}^*_\hbar$-equivariant vector bundle over pt.

We use the localization formula for the fixed points set $X^{\mathbb{C}^*_\hbar} \xrightarrow{\iota} X$.

Fact 3.2. When X is smooth, we have the following:

(1) $X^{\mathbb{C}^*_\hbar} = \bigsqcup_{\mathfrak{J}} X_{\mathfrak{J}}$ for smooth $X_{\mathfrak{J}}$

(2) We have an isomorphism

$$\iota_* \colon A^\bullet_{\mathbb{C}^*_\hbar}(X^{\mathbb{C}^*_\hbar}) \otimes \mathbb{Q}[\hbar, \hbar^{-1}]] \cong A^\bullet_{\mathbb{C}^*_\hbar}(X) \otimes \mathbb{Q}[\hbar, \hbar^{-1}]].$$

(3) We have

$$(\iota_*)^{-1}[X] = \sum_{\mathfrak{J}} \frac{[X_{\mathfrak{J}}]}{\mathrm{Eu}(N_{\mathfrak{J}})},$$

where $\mathrm{Eu}(N_{\mathfrak{J}})$ is the Euler class of the normal bundle $N_{\mathfrak{J}}$ of $X^{\mathbb{C}^*_\hbar}$ in X.

Using this fact, we define integral over smooth $\mathbb{C}^*_\hbar$-variety X by localization. For $\Pi \colon X \to \mathrm{pt}$ and $\Pi_{\mathfrak{J}} \colon X_{\mathfrak{J}} \to \mathrm{pt}$, we have the commutative diagram:

$$\begin{tikzcd}
A^\bullet_{\mathbb{C}^*_\hbar}(X) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Q}[\hbar, \hbar^{-1}]] \arrow[r, "(\iota_*)^{-1}", "\cong"'] \arrow[d, "\Pi_*(\cdot)\cap[X]"'] & A^\bullet_{\mathbb{C}^*_\hbar}(X^{\mathbb{C}^*_\hbar}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Q}[\hbar, \hbar^{-1}]] \arrow[d, "\sum_{\mathfrak{J}} \Pi_{\mathfrak{J}*}(\cdot)\cap[X_{\mathfrak{J}}]"] \\
A_\bullet^{\mathbb{C}^*_\hbar}(\mathrm{pt}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Q}[\hbar, \hbar^{-1}]] \arrow[r, equal] & A_\bullet^{\mathbb{C}^*_\hbar}(\mathrm{pt}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Q}[\hbar, \hbar^{-1}]]
\end{tikzcd}$$

We set

$$\int_X c = \sum_{\mathfrak{J}} \pi_{\mathfrak{J}*} \frac{c|_{X_{\mathfrak{J}}}}{\mathrm{Eu}(N_{\mathfrak{J}})},$$

for $c \in A^\bullet_{\mathbb{C}^*_\hbar}(X)$. When X is proper over $\mathbb{C}$, this is equal to the usual integral $\int_X c = \Pi_* c \cap [X]$.

We perform integral over $X = \mathbb{P}(W)$. Let $N_{\mathbb{P}(W)}, N_{\mathrm{pt}}$ be normal bundles of $\mathbb{P}(W), \mathrm{pt}$ in $\mathcal{M} = \mathbb{P}(W \oplus \mathbb{C})$ respectively. Then we have

$$\begin{aligned}\frac{1}{\mathrm{Eu}(N_{\mathbb{P}(W)})} &= \frac{1}{\hbar + c_1(\mathcal{O}_{\mathbb{P}(W)}(1))} = \frac{1}{\hbar} \cdot \frac{1}{1 + c_1(\mathcal{O}_{\mathbb{P}(W)}(1))/\hbar} \\ &= \frac{1}{\hbar}\left(1 - \frac{c_1(\mathcal{O}_{\mathbb{P}(W)}(1))}{\hbar} + \cdots\right) \in A^*_{\mathbb{C}^*_\hbar}(\mathcal{M}) \otimes \mathbb{Q}[\hbar, \hbar^{-1}]\!]\end{aligned}$$

The localization formula implies

$$\int_{\mathcal{M}} \psi = \int_{\mathbb{P}(W)} \frac{\psi|_{\mathbb{P}(W)}}{\mathrm{Eu}(N_{\mathbb{P}(W)})} + \int_{\mathrm{pt}} \frac{\psi|_{\mathrm{pt}}}{\mathrm{Eu}(N_{\mathrm{pt}})}.$$

Here the left hand side is in $\mathbb{Q}[\hbar]$, while the right hand side is in $\mathbb{Q}[\hbar, \hbar^{-1}]]$.

Lemma 3.3. We have $\operatorname*{Res}_{\hbar=\infty} \int_{\mathbb{P}(W)} \frac{\psi}{\mathrm{Eu}(N_{\mathbb{P}(W)})} = \int_{\mathbb{P}(W)} \psi.$

Proof. We use the following:

$$\begin{aligned}\frac{1}{\mathrm{Eu}(N_{\mathbb{P}(W)})} &= \frac{1}{\hbar + c_1(\mathcal{O}_{\mathbb{P}(W)}(1))} \\ &= \frac{1}{\hbar} - \frac{c_1(\mathcal{O}_{\mathbb{P}(W)}(1))}{\hbar^2} + \cdots\end{aligned}$$

This implies

$$\int_{\mathbb{P}(W)} \psi = -\operatorname*{Res}_{\hbar=\infty} \int_{\mathrm{pt}} \frac{\psi|_{\mathrm{pt}}}{\mathrm{Eu}(N_{\mathrm{pt}})}. \tag{1}$$

Here $\mathrm{Res}_{\hbar=\infty}$ is an operator taking the coefficient in $\hbar^{-1}$. We compute the residue as follows. We notice

$$\begin{cases} \mathcal{V}|_{\mathrm{pt}} = e^{\hbar} \\ N_{\mathrm{pt}} = W \otimes e^{-\hbar}. \end{cases}$$

This implies that the right hand side of (1) is equal to

$$-\operatorname*{Res}_{\hbar=\infty} \frac{\psi}{(-\hbar)^r} = (-1)^{r-1} \cdot \text{ coefficients of } \hbar^{r-1} \text{ in } \psi = \psi(\hbar) \in \mathbb{Q}[\hbar, \hbar^{-1}]\!]$$

We want to apply this to $\psi = \mathrm{Eu}(T\mathbb{P}(W)) \in K(\mathbb{P}(W))$. But we need to lift ψ to the larger projective space $\mathcal{M} = \mathbb{P}(W \oplus \mathbb{C})$. From the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(W)} \to W \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \to T\mathbb{P}(W) \to 0,$$

we have the following lemma.

Lemma 3.4. We have

$$(W \otimes \mathcal{O}_{\mathcal{M}}(1) - \mathcal{O}_{\mathcal{M}})\Big|_{\mathbb{P}(W)} = T\mathbb{P}(W) \text{ in } K(\mathbb{P}(W)).$$

To define the Euler class of $W \otimes \mathcal{O}_{\mathcal{M}}(1) - \mathcal{O}_{\mathcal{M}}$ in $K(\mathcal{M})$, we introduce a new equivariant parameter $e^\theta \in \mathbb{C}^*_\theta$.

Definition 3.5. Let $\alpha = [E] - [F] \in K_{\mathbb{C}^*_\hbar}(\mathcal{M})$ with $\mathbb{C}^*_\hbar$-equivariant vector bundles E, F on $\mathcal{M}$. We define the Euler class for virtual vector bundle $\alpha = E - F$ by

$$\mathrm{Eu}^\theta(\alpha) = \frac{c_{\mathrm{rk}\,E}(E \otimes e^\theta)}{c_{\mathrm{rk}\,F}(F \otimes e^\theta)} \in A^*_{\mathbb{C}^*_\theta \times \mathbb{C}^*_\hbar}(\mathcal{M}) \otimes \mathbb{Q}(\theta)[\hbar][[\hbar^{-1}]]$$

Set

$$\psi(\theta, \hbar) = \mathrm{Eu}^\theta(W \otimes \mathcal{O}_{\mathcal{M}}(1) - \mathcal{O}_{\mathcal{M}}) \in A^*_{\mathbb{C}^*_\theta \times \mathbb{C}^*_\hbar}(\mathcal{M}) \otimes \mathbb{Q}(\theta)[\hbar][[\hbar^{-1}]].$$

Since $\psi(\theta,\hbar)|_{\mathrm{pt}} = \mathrm{Eu}^\theta(W \otimes e^{-\hbar} - 1)$, substituting $\psi = \psi(\theta, \hbar)$ into (1) we have

$$\begin{aligned}\chi(\mathbb{P}(W)) &= -\operatorname*{Res}_{\hbar=\infty} \frac{(-\hbar+\theta)^r}{\theta} \cdot \frac{1}{(-\hbar)^r} \\ &= -\operatorname*{Res}_{\hbar=\infty} \frac{1}{\theta} \cdot \frac{(\hbar-\theta)^r}{\hbar^r} = -\frac{1}{\theta} \cdot r(-\theta) = r.\end{aligned}$$

Here we used

$$\operatorname*{Res}_{\hbar=\infty} \prod_{\alpha=1}^{r} \frac{\hbar + a_\alpha}{\hbar + b_\alpha} = \sum_{\alpha=1}^{r} (a_\alpha - b_\alpha).$$

3.2 Flag manifold

Here we present the residue formula as an application of our wall-crossing formula to the framed quiver:

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[d, "b_0"'] & \overset{2}{\bullet} \arrow[l, "b_1"'] & \cdots \arrow[l, "b_2"'] & \overset{N-1}{\bullet} \arrow[l, "b_{N-2}"'] & \overset{N}{\bullet} \arrow[l, "b_{N-1}"'] \\
\boxed{r}
\end{tikzcd}$$

We set $W = \mathbb{C}^r, V_1 = \mathbb{C}^{d_1}, V_2 = \mathbb{C}^{d_2}, \ldots, V_N = \mathbb{C}^{d_N}$ with $r \geq d_1 \geq \cdots \geq d_N$, and fix stability parameters $\zeta_1 = \zeta_2 = \cdots = \zeta_N = 1$ as in Example 2.2. For the Q_0-graded vector space $V = \bigoplus_{k=1}^N V_k \oplus V_\infty$, we have

$$M_Q^\zeta(V) = \mathrm{Hom}^{\mathrm{inj}}_{\mathbb{C}}(V_1, W) \times \prod_{k=1}^{N-1} \mathrm{Hom}^{\mathrm{inj}}_{\mathbb{C}}(V_{k+1}, V_k)/G,$$

where $\mathrm{Hom}^{\mathrm{inj}}(U_1, U_2)$ denotes the set of injective $\mathbb{C}$-linear maps $U_1 \to U_2$ for vector spaces U_1 and U_2. The tautological bundle

$$\mathcal{V}_i = \mathrm{Hom}^{\mathrm{inj}}_{\mathbb{C}}(V_1, W) \times \prod_{k=1}^{m-1} \mathrm{Hom}^{\mathrm{inj}}_{\mathbb{C}}(V_{k+1}, V_i) \times V_k/G$$

corresponds to the i-th universal flag for $i = 1, \ldots, N$ via the isomorphism $M^{\zeta}_{Q}(V) \cong Fl(W; d_1, \ldots, d_N)$ in Example 2.2.

We consider a linear map $\pi \colon \mathbb{Q}[\hbar_1, \ldots, \hbar_{d_N}] \to A^{\bullet}_{\mathbb{T}}(M(W, V/V_N))$ defined by

$$\pi(\hbar_1^{\ell_1} \cdots \hbar_{d_N}^{\ell_{d_N}}) = \prod_{i=1}^{d_N} h_{\ell_i - d_{N-1}+1}(\mathcal{V}_{N-1})$$

where h_ℓ is the complete symmetric function of degree ℓ.

Theorem 3.6 (Zielenkievicz [14], O [11])**.** Let f be a symmetric polynomial. Then we have

$$\begin{aligned} &\int_{M^{\zeta}_{Q}(V)} f(\mathcal{V}_N) \\ &= (-1)^{d_N(d_{N-1}+1)} \cdot \pi \left(f(\hbar_1, \ldots, \hbar_{d_N}) \prod_{i \neq j} (\hbar_j - \hbar_i) \right). \end{aligned}$$

As an application of this theorem, we deduce the Jacobi-Trudi formula. For partition $\lambda = (\lambda_1, \lambda_2, \ldots,)$ of length $\ell(\lambda) \leq n$, set

$$S_\lambda(x_1, \ldots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{\det(x_i^{n-j})_{1 \leq i,j \leq n}}.$$

When $N = 1$ and $r = d_1 \geq \ell(\lambda)$, by Theorem 3.6 we have

$$\begin{aligned} S_\lambda(a_1, \ldots, a_r) &= \int_{M^{\zeta}_{Q}(V)} S_\lambda(\mathcal{V}_1) \\ &= \det_{1 \leq i,j \leq r} (h_{\lambda_j - j + i}). \end{aligned}$$

4 Enhancement of quiver

For a general framed quiver $Q = (Q, \infty)$, we present localization techniques developed in [4] and [6]. Fix $\beta \in (\mathbb{Z}_{\geq 0})^I$ and choose one vertex $* \in I = Q \setminus \{\infty\}$ such that $\beta_* \neq 0$, we define an enhancement $\tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2)$ of Q as follows :

$$\begin{aligned} &\tilde{Q}_0 = Q_0 \sqcup \{*(k) \mid k = 1, 2, \ldots, L\} \quad (L \geq \dim V_*) \\ &\tilde{Q}_1 = Q_1 \sqcup \{*(k) \to *(k+1)\}_{k=1}^{L}, \quad (*(L+1) = *) \\ &\tilde{Q}_2 = Q_2 \end{aligned}$$

For example, take N as $*$ in Example 2.2, then the enhanced quiver $\tilde{Q}$ is presented as follows :

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[d] & \overset{2}{\bullet} \arrow[l, "b_1"] & \cdots \arrow[l, "b_2"] & \overset{N-1}{\bullet} \arrow[l, "b_{N-2}"] & \overset{N}{\bullet} \arrow[l, "b_{N-1}"] \\
\boxed{r} & & & & \bullet \; {*(L)} \arrow[u] \\
& & & & \bullet \; {*(L-1)} \arrow[u] \\
& & & & \vdots \arrow[u] \\
& & & & \bullet \; {*(1)} \arrow[u]
\end{tikzcd}$$

We consider a $\tilde{Q}_0$-graded vector space $\tilde{V} = V \oplus \bigoplus_{k=1}^{L} \tilde{V}_{*(k)}$, where $V = \bigoplus_{v \in Q_0} V_v$ is Q_0-graded vector space with $(\dim V_v)_{v \in Q_0} = \alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$. We set $\dim \tilde{V}_{*(1)} \leq 1, \quad \dim \tilde{V}_{*(N)} = \alpha_0$,

$$\dim \tilde{V}_{*(k-1)} \leq \dim \tilde{V}_{*(k)} \leq \dim \tilde{V}_{*(k-1)} + 1 \quad (k = 2, \ldots, N-1),$$

and

$$\mathfrak{I} = \left\{ k \in [L] \,\middle|\, \dim \tilde{V}_{*(k)} - \dim \tilde{V}_{*(k-1)} = 1 \right\}.$$

We write by $Fl_X(\mathcal{E}, \mathfrak{I})$ thet full-flag bundle of $\mathcal{E}$ consisting of $F_\bullet$ of a bundle $\mathcal{E}$ on a variety X such that

$$\{k \in \mathbb{Z}_{\geq 0} \mid \dim F_k/F_{k-1} = 1\} = \mathfrak{I}.$$

We may have repetitions in $F_\bullet$, but $\dim(F_k/F_{k-1}) \leq 1$. We always put $\mathcal{F}_0 = 0$.

4.1 $(\bar{\zeta}, \ell)$-stability

We consider a pair $(\rho, F_\bullet) \in \mathrm{Rep}_Q(V) \times Fl(V_0, \mathfrak{I})$ where $Fl(V_0, \mathfrak{I})$ is the full-flag manifold of V_0. Take a dimension vector $\beta = (\beta_i)_{i \in I} \in (\mathbb{Z}_{\geq 0})^I$ with $\beta_0 \neq 0, \quad (0 \in I)$, and a *generic* stability parameter

$$\bar{\zeta} \in \beta^\perp = \{\zeta \in \mathbb{R}^I \mid \sum_{i \in I} \zeta_i \beta_i = 0\}.$$

Definition 4.1. $(\rho, F_\bullet)$ is said to be $(\bar{\zeta}, \ell)$-stable if ρ: $\bar{\zeta}$-semistable, and

(1) If $\bar{\zeta}(S) = 0$, $S_\infty = 0$ and $S \neq 0$, we have $S_0 \cap F_\ell = 0$, and

(2) If $\bar{\zeta}(S) = 0$, $S_\infty = \mathbb{C}$ and $S \neq V$, we have $F_\ell \not\subset S_0$.

We write by $M^{\bar{\zeta},\ell}(\alpha, \mathfrak{I})$ moduli of $(\bar{\zeta}, \ell)$-stable pair $(\rho, F_\bullet)$ as above. We have

$$M^{\bar{\zeta},\ell}(\alpha, \mathfrak{I}) \cong Fl_{M^{\zeta'}(\alpha)}(\mathcal{V}_0, \mathfrak{I}) \quad (\ell \gg \alpha_0, \quad \zeta' \in \mathcal{C}')$$
$$M^{\bar{\zeta},0}(\alpha, \mathfrak{I}) \cong Fl_{M^{\zeta}(\alpha)}(\mathcal{V}_0, \mathfrak{I}) \quad (\zeta \in \mathcal{C}).$$

where $\mathcal{V}_i$ is the tautological bundle over $M^\zeta(\alpha)$ for each vertex $i \in Q_0$.

4.2 Enhanced master space

Fix $\ell \in \mathfrak{I}$, and take $(\zeta^+, \eta) \in \mathcal{C}' \times (\mathbb{Q}_{>0})^N \leftrightarrow (\bar{\zeta}, \ell)$-stability, and $\zeta^- \in \mathcal{C}$ suitably.

$$\mathbb{M} = \mathbb{M}_Q(V) = \prod_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{out}(a)}, V_{\mathrm{in}(a)}), \quad \mathbb{L} = \mathbb{L}_Q(V) = \prod_{l \in Q_2} \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{out}(l)}, V_{\mathrm{in}(l)})$$

$$\widetilde{\mathbb{M}} = \mathbb{M}_{\tilde{Q}}(\tilde{V}) = \mathbb{M} \times \prod_{k=1}^{N} \mathrm{Hom}_{\mathbb{C}}(\tilde{V}_{*(k)}, \tilde{V}_{(0,k+1)}) \quad (\tilde{V}_{(0,N+1)} = V_0)$$

$$G = G_{\tilde{V}} = \prod_{i \in I} \mathrm{GL}(V_i) \times \prod_{k=1}^{N} \mathrm{GL}(\tilde{V}_{*(k)}),$$

We consider a moment map $\mu \colon \mathbb{M} \to \mathbb{L}$ corresponding to relations Q_2. Take ample G line bundle $\mathcal{L}_\pm$ corrsponding to $(\zeta^\pm, \eta)$-stability. Set $\hat{\mu} \colon \widehat{\mathbb{M}} = \mathrm{ProjSym}(\mathcal{L}_- \oplus \mathcal{L}_+) \xrightarrow{\text{projection}} \mathbb{M} \overset{\mu}{\to} \mathbb{L}$, and $\widehat{\mathbb{M}}^{ss}$ semi-stable locus with respect to $\mathcal{O}_{\widehat{\mathbb{M}}}(1)$. We set $\mathcal{M} = [\hat{\mu}^{-1}(0) \cap \widehat{\mathbb{M}}^{ss}/G]$, and consder a $\mathbb{C}^*_\hbar$-action on $\mathcal{M}$ defined by

$$\mathbb{C}^*_\hbar \curvearrowright \mathcal{M} \colon (\mathcal{A}, F_\bullet, [x_-, x_+]) \mapsto (\mathcal{A}, F_\bullet, [e^\hbar x_-, x_+]), \tag{2}$$

We write by $\mathcal{M}^{\mathbb{C}^*_\hbar}$ the fixed points set, and set $\mathcal{M}_\pm = [\hat{\mu}^{-1}(0)^{ss} \cap \mathrm{ProjSym}(\mathcal{L}_\pm)/G]$.

Proposition 4.2. We have

$$\mathcal{M}^{\mathbb{C}^*_\hbar} = \mathcal{M}_+ \sqcup \mathcal{M}_- \sqcup \bigsqcup_{\mathfrak{I}^\sharp \in \mathcal{D}^\ell(\mathfrak{I})} \mathcal{M}_{\mathfrak{I}^\sharp},$$

where $\mathcal{D}^\ell(\mathfrak{I}) = \{\mathfrak{I}^\sharp \subset \mathfrak{I} \mid |\mathfrak{I}^\sharp_0| = d^\sharp \beta_0 \text{ for } d^\sharp > 0, \text{ and } \min(\mathfrak{I}^\sharp) \leq \ell\}$.

We have $\mathcal{M}_- \cong Fl_{M^{\zeta_-}(\alpha)}(\mathcal{V}, \mathfrak{I})$, $\mathcal{M}_+ \cong M^{\bar{\zeta},\ell}(\alpha, \mathfrak{I})$, and $\mathcal{M}_{\mathfrak{I}^\sharp}$ is étale equivalent to

$$Fl_{H_{d^\sharp}}(\mathcal{V}_0^\sharp/\mathcal{V}_{\infty'}, \bar{\mathfrak{I}}^\sharp) \times M^{\bar{\zeta}, \min(\mathfrak{I}_\sharp)-1}(\alpha - d^\sharp\beta, \mathfrak{I}^\flat)$$

where $H_{d^\sharp} = M^{\zeta_-}_{Q^\sharp}(d^\sharp\beta + e_{\infty'})$ for $e_{\infty'} = \delta_{v\infty'} \in \mathbb{Z}^{Q_0^\sharp}$, and $\mathfrak{I}^\sharp \setminus \min(\mathfrak{I}^\sharp), \mathfrak{I}^\flat = \mathfrak{I} \setminus \mathfrak{I}^\sharp$. Here $Q^\sharp$ is a new quiver associated to Q and a choice of $* \in Q_0$.

When we consider Example 2.2 and choose N as $*$, then $Q^\sharp$ is presented as follows :

$$\begin{tikzcd}
\overset{1}{\bullet} & \overset{2}{\bullet} \arrow[l, "b_1"'] & \cdots \arrow[l, "b_2"'] & \overset{N-1}{\bullet} \arrow[l, "b_{N-2}"'] & \overset{N}{\bullet} \arrow[l, "b_{N-1}"'] \\
 & & & & \infty' \arrow[u]
\end{tikzcd}$$

We consider the normal bundles $N_+, N_-, N_{\mathfrak{I}^\sharp}$ of the embeddings

$$\iota_+ \colon \mathcal{M}_+ \to \mathcal{M}, \quad \iota_- \colon \mathcal{M}_- \to \mathcal{M}, \quad \iota_{\mathfrak{I}^\sharp} \colon \mathcal{M}_{\mathfrak{I}^\sharp} \to \mathcal{M}.$$

4.3 Integral

Set $M_0 = \mathrm{Spec}\Gamma(\mu^{-1}(0), \mathcal{O}_{\mu^{-1}(0)})^{\prod_{i\in I}\mathrm{GL}(V_i)}$. We assume the following two conditions.

- There exists $\mathbb{T} = (\mathbb{C}^*)^\ell$-action on all moduli spaces and natural maps to M_0 are proper $\mathbb{T}$-equivariant.
- $\mathbb{T}$-fixed poins set $M_0^{\mathbb{T}}$ consists of one point.

Then we can take integrals in $H_\bullet^{\mathbb{T}}(M_0^{\mathbb{T}}) = H_\bullet^{\mathbb{T}}(\mathrm{pt})$ after localization.

Fact 4.3. We have

$$H_\bullet^{\mathbb{T}}(\mathrm{pt}) \cong H_{\mathbb{T}}^\bullet(\mathrm{pt}) \cong \mathbb{Z}[x_1, \ldots, x_\ell] =: A$$

where $x_1 = c_1(\mathbb{C}_{t_1}), \ldots, x_\ell = c_1(\mathbb{C}_{t_\ell})$ and $\mathbb{C}_{t_1}, \ldots, \mathbb{C}_{t_\ell}$ are the weight spaces for $(t_1, \ldots, t_\ell) \in \mathbb{T}$.

$\int_M \colon H_{\mathbb{T}}^\bullet(M) \to H_\bullet^{\mathbb{T}}(M_0) \otimes_A S \cong H_\bullet^{\mathbb{T}}(M_0^{\mathbb{T}}) \otimes_A S \cong S,\ \varphi \mapsto \Pi_*(\varphi \cap [M])$ for $\mathbb{T}$-equivariant morphism $\Pi\colon M \to M_0$.

4.4 Localization

We use the following commutative diagram

$$\begin{tikzcd}
H^\bullet_{\mathbb{C}^*_\hbar\times\mathbb{T}}(\mathcal{M}) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}(\!(\hbar^{-1})\!) \arrow[r, "\cong"] \arrow[d] & H^\bullet_{\mathbb{C}^*_\hbar\times\mathbb{T}}(\mathcal{M}^{\mathbb{C}^*_\hbar}) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}(\!(\hbar^{-1})\!) \arrow[d] \\
H_\bullet^{\mathbb{C}^*_\hbar\times\mathbb{T}}(M_0) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}(\!(\hbar^{-1})\!) \arrow[r, equal] & H_\bullet^{\mathbb{C}^*_\hbar\times\mathbb{T}}(M_0) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}(\!(\hbar^{-1})\!)
\end{tikzcd}$$

where the upper horizontal arrow is given by

$$\frac{\iota_+^*}{\mathrm{Eu}(N_+)} + \frac{\iota_-^*}{\mathrm{Eu}(N_-)} + \sum_{\mathfrak{J}^\sharp\in\mathcal{D}^\ell(\mathfrak{J})} \frac{\iota_{\mathfrak{J}^\sharp}^*}{\mathrm{Eu}(N_{\mathfrak{J}^\sharp})},$$

and $\hbar$ is the first Chern class in $H^\bullet_{\mathbb{C}^*_\hbar}(\mathrm{pt})$ of the weight $e^\hbar \in \mathbb{C}^*_\hbar$. From this diagram, we have

$$\int_{\mathcal{M}} \varphi = \int_{\mathcal{M}_+} \frac{\varphi|_{\mathcal{M}_+}}{\mathrm{Eu}(N_+)} + \int_{\mathcal{M}_-} \frac{\varphi|_{\mathcal{M}_-}}{\mathrm{Eu}(N_-)} + \sum_{\mathfrak{J}^\sharp} \int_{\mathcal{M}_{\mathfrak{J}^\sharp}} \frac{\varphi|_{\mathcal{M}_{\mathfrak{J}^\sharp}}}{\mathrm{Eu}(N_{\mathfrak{J}^\sharp})}. \tag{3}$$

Substitute $\varphi = \tilde{\psi} = \frac{\psi\cdot\mathrm{Eu}^\theta(\Theta(\mathcal{F}_\bullet))}{|\mathfrak{J}|!} \in H^\bullet_{\mathbb{C}^*_\hbar\times\mathbb{T}}(\mathcal{M})$ for $\psi \in H^\bullet_{\mathbb{C}^*_\hbar\times\mathbb{T}}(\mathcal{M})$

$$\int_{M^{\bar{\zeta},\ell}(\alpha,\mathfrak{J})} \tilde{\psi} - \int_{M^{\zeta^-}(\alpha)} \psi = -\operatorname*{Res}_{\hbar=\infty} \sum_{\mathfrak{J}^\sharp\in\mathcal{D}^\ell(\mathfrak{J})} \int_{\mathcal{M}_{\mathfrak{J}^\sharp}} \frac{\tilde{\psi}|_{\mathcal{M}_{\mathfrak{J}^\sharp}}}{\mathrm{Eu}(N_{\mathfrak{J}^\sharp})},$$

where $\mathrm{Eu}^\theta(\mathcal{F}) = \mathrm{Eu}(\mathcal{F} \otimes \mathbb{C}_{e^\theta})$ for $t_\ell = e^\theta$. The last summand is equal to

$$\frac{|\mathfrak{I}^\flat|!}{|\mathfrak{I}|!} \int_{\widetilde{M}^{\min(\mathfrak{I})-1}(\tilde{V}^\flat)} \int_{Fl_{H_{d^\sharp}}(\mathcal{V}_0^\sharp/\mathcal{V}_{\infty'}, \bar{\mathfrak{I}}^\sharp)} \frac{\psi \cdot \mathrm{Eu}^\theta(\Theta(\mathcal{F}^\sharp_\bullet \oplus \mathcal{F}^\flat_\bullet))}{\mathfrak{N}(\mathcal{V}^\flat, \mathcal{V}^\sharp \otimes e^\hbar) \cdot \mathrm{Eu}(\mathfrak{H}(\mathcal{F}^\sharp_\bullet, \mathcal{F}^\flat_\bullet))}, \tag{4}$$

where

$$\Theta(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) = \sum_{i>j} \mathcal{H}om\left(\mathcal{F}_j/\mathcal{F}_{j-1}, \mathcal{F}'_i/\mathcal{F}'_{i-1}\right),$$
$$\mathfrak{H}(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) = \Theta(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) + \Theta(\mathcal{F}'_\bullet, \mathcal{F}_\bullet)$$

for two flags $\mathcal{F}_\bullet, \mathcal{F}'_\bullet$ of sheaves. When $\mathcal{F}_\bullet = \mathcal{F}'_\bullet$, we set

$$\Theta(\mathcal{F}_\bullet) = \Theta(\mathcal{F}_\bullet, \mathcal{F}_\bullet).$$

5 Application

Here we only consider integrals of equivariant Euler class as an application. See [7], [8], and [9] for another application. Set

$$\Lambda_Q(\mathcal{V}) = \sum_{a \in Q_1} \mathcal{H}om(\mathcal{V}_{\mathrm{out}(a)}, \mathcal{V}_{\mathrm{in}(a)}) - \sum_{l \in Q_2} \mathcal{H}om(\mathcal{V}_{\mathrm{out}(l)}, \mathcal{V}_{\mathrm{in}(l)}) - \sum_{i \in I} \mathcal{E}nd(\mathcal{V}_i),$$
$$\bar{\beta}_\infty = \sum_{\substack{a \in Q_1 \\ \mathrm{out}(a)=\infty}} \beta_{\mathrm{in}(a)} - \sum_{\substack{a \in Q_1 \\ \mathrm{in}(a)=\infty}} \beta_{\mathrm{out}(a)},$$
$$\tilde{\gamma}_d(\theta) = (d\beta_0 - 1)! \int_{H_{d^\sharp}} \mathrm{Eu}^\theta(\Lambda_{Q^\sharp}(\mathcal{V} \oplus \mathcal{V}_{\infty'})).$$

For $\mathfrak{I}, \mathfrak{I}' \subset [\alpha_0] = \{1, 2, \ldots, \alpha_0\}$, set

$$s(\mathfrak{I}, \mathfrak{I}') = \left|\{(l, l') \in \mathfrak{I} \times \mathfrak{I}' \mid l < l'\}\right|, - \left|\{(l, l') \in \mathfrak{I} \times \mathfrak{I}' \mid l > l'\}\right|.$$

We define a rational expression $A_{\alpha_0}(\bar{\beta}_\infty)$ by

$$A_{\alpha_0}(\bar{\beta}_\infty) = \sum_{j=1}^{\lfloor \alpha_0/\beta_0 \rfloor} \sum_{\substack{\mathfrak{I}_1 \sqcup \cdots \sqcup \mathfrak{I}_j \subset [\alpha_0] \\ \min(\mathfrak{I}_1) > \cdots > \min(\mathfrak{I}_j) \\ |\mathfrak{I}_i| \in \beta_0 \mathbb{Z}_{>0}}} \prod_{i=1}^{j} (s(\mathfrak{I}_i, \boldsymbol{\mathfrak{I}}_{>i}) - \bar{\beta}_\infty d_i) \tilde{\gamma}_{d_i}(\theta) \tag{5}$$

where $\boldsymbol{\mathfrak{I}}_{>i} = \mathfrak{I}_{i+1} \sqcup \cdots \sqcup \mathfrak{I}_j$, and $d_1 = |\mathfrak{I}_1|/\beta_0, \ldots, d_j = |\mathfrak{I}_j|/\beta_0$ for each collection $(\mathfrak{I}_1, \ldots, \mathfrak{I}_j)$ satisfying the conditions in the second summation in (5).

We set $H^\pm(\alpha) = \int_{M^{\zeta^\pm}(\alpha)} \mathrm{Eu}^\theta(\Lambda_Q(\mathcal{V}))$.

Theorem 5.1 ([10])**.** We have

$$H^+(\alpha) - H^-(\alpha) = \sum_{k=0}^{\lfloor \alpha_0/\beta_0 \rfloor} \frac{A_{k\beta_0}(\bar{\beta}_\infty)}{(k\beta_0)!} H^-(\alpha - k\beta).$$

Corollary 5.2. When Q is associated to construct quiver varieties, then wall-crossing term does not depend on $\vec{r} = (r_i)_{i \in I}$.

Question 5.3. When $\bar{\beta}_\infty = 0$, does the coefficients of wall-crossing term vanish?

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