

# Local delta invariants of weak del Pezzo surfaces with the anti-canonical degree $\geq 5$

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Let  $X$  be a log terminal Fano variety over a complex number field  $\mathbb{C}$ .

## 1 K-stability

The following problems were of importance.

When does  $X$  admit a weak Kähler Einstein (KE) metric?

Recently, the following was proved.

K-polystability  $\iff \exists$  weak KE metric

In order to interpret the criterion for K-(poly)stability of  $X$ , the delta invariant is introduced in [3],[5]. It is known that

$$\begin{aligned} \delta(X) > 1 &\iff X:\text{K-stable} \\ &\iff X:\text{K-polystable and } \#\text{Aut}(X) < \infty. \end{aligned}$$

## 2 Local delta invariant

- $Y$ : smooth variety
- $\sigma: Y \rightarrow X$ : projective birational morphism
- $E \subset Y$ : prime divisor

Let

$$\begin{aligned} A_X(E) &:= 1 + \text{ord}_E(K_Y - \sigma^*K_X), \\ S(E) &:= \frac{1}{(-K_X)^n} \int_0^\tau \text{vol}(\sigma^*(-K_X) - uE) du, \end{aligned}$$

where  $\tau$  is the pseudo effective threshold of  $E$  with respect to  $-K_X$ , that is,

$$\tau := \sup\{u \in \mathbb{Q}_{\geq 0} \mid \sigma^*(-K_X) - uE \text{ is big}\}.$$

The **local delta invariant**  $\delta_p(X)$  of  $X$  at  $p \in X$  is as follows ([2]):

$$\delta_p(X) = \inf \left\{ \frac{A_X(E)}{S(E)} \mid E:\text{divisor}/X ; p \in C_X(E) \right\}.$$

Moreover, the **delta invariant**  $\delta(X)$  of  $X$  is as follows ([3])

$$\delta(X) = \inf_{p \in X} \delta_p(X).$$

## 3 Main Results

It is known that there exist 7 (if  $K^2 = 5$ ), 6 (if  $K^2 = 6$ ), and 2 (if  $K^2 = 7$ ) types of weak del Pezzo surface in terms of the configuration of negative curves, respectively.

A weak del Pezzo surface of anti-canonical degree 8 is one of the Hirzebruch surfaces of either degree 0, 1 or 2.

**Main Result [1, Akaike]:**

We determine the whole local delta invariants for all weak del Pezzo surfaces with the anti-canonical degree  $\geq 5$ .

## 4 Application

- (1) We get the delta invariants for any du Val del Pezzo surface with the anti-canonical degree  $\geq 5$ .
- (2) The estimation of the local delta invariant of weak del Pezzo surfaces is useful for the K-stability of higher dimensional Fano varieties (cf. [4, Lemma 24, 25]).

## References

- [1] H. Akaike, Local delta invariants of weak del Pezzo surfaces with the anti-canonical degree  $\geq 5$ , arXiv:2304.09437[math.AG].
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- [4] I. Cheltsov, K. Fujita, T. Kishimoto, T. Okada, K-stable divisors in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  of degree  $(1, 1, 2)$ , arXiv:2206.08539v2 [math.AG]
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