Ascending Chain Condition for Minimal Log Discrepancies for Generalized Fano Pairs

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Introduction

Generalized pairs, as a generalization of log pairs, play an important role in Birkar's proof of the Borisov-Alexeev-Borisov conjecture (the BBAB theorem) [Birkar19, Birkar21a]. On the other hand, the ascending chain condition (ACC) for minimal log discrepancies (mld) is conjectured to hold for pairs with the coefficients of their boundaries in a fixed DCC set. The ACC conjecture and the lower semi-continuity conjecture for mlds imply the termination of flips [Shokurov].

Using the lower bound for the log canonical thresholds (lct) and the finiteness of mlds for bounded generalized pairs, we show the ACC conjecture holds for bounded generalized pairs. As a consequence, applying the BBAB theorem, the ACC for mlds for generalized Fano pairs holds.

Notation and Conventions

- We work over $\mathbb{C}$. Every variety is assumed to be normal, projective and $\mathbb{Q}$-factorial.
- A generalized pair $(X, B+M)$ consists of a variety X, an effective divisor B on X and a b-nef b-divisor M over X such that K_X+B+M is $\mathbb{R}$-Cartier. Equivalently, M can be viewed as the image of a nef divisor M' on a birational model X' over X. We say the coefficients of M are in some set $I \subseteq \mathbb{R}_{\geq 0}$ if $M' = \sum m_i M'_i$ for some $m_i \in I$ and some Cartier divisors M'_i on X'.

BBAB Theorem (Birkar)

Let d be a natural number and let ϵ be a positive real number. Then the varieties X such that, with some generalized boundaries $B+M$,

- $(X, B+M)$ is generalized ϵ-lc of dimension d, and
- $-(K_X+B+M)$ is nef and big,

form a bounded family.

Lower bound of lct [Birkar21b]

Let d and n be natural numbers and let ϵ be a positive real number. Then there exist a positive real number t such that if $(X, B+M)$ is an ϵ-lc generalized pair of dimension d with

- $\dim X = d$,
- there is a very ample divisor A on X with $A^d \leq n$,
- $B+D+M+N$ is a generalized boundary for some effective divisor D on X and some b-nef b-divisor N over X, and
- $A-(B+D+M+N) \geq 0$,

then $(X, B+tD+M+tN)$ is generalized lc.

Finiteness of mlds [CGN]

Let d and k be natural numbers and let $I \subseteq \mathbb{R}_{\geq 0}$ be a finite set. Then the set of mlds $\{\mathrm{mld}(X, B+M)\}$ with

- $\dim X = d$,
- the coefficients of B and M are in I, and
- kK_X is Cartier,

is finite.

Acc for mlds for generalized Fano pairs

Let d be a natural number and let $0 \in I \subseteq \mathbb{R}_{\geq 0}$ be a DCC set. Then the set of mlds $\{\mathrm{mld}(X, B+M)\}$ with

- $\dim X = d$,
- the coefficients of B and M in I, and
- $-(K_X+B+M)$ is nef and big,

satisfies ACC.

Sketch of Proof.

1. It is enough to show this for a sequence $\{(X_i, B_i+M_i)\}_{i=1}^{\infty}$ of such generalized pairs.
2. We may assume that $\{a_i := \mathrm{mld}(X_i, B_i+M_i)\}_{i=1}^{\infty}$ is strictly increasing. We may assume that (X_i, B_i+M_i) is generalized ϵ-lc for every i for some fixed positive real number ϵ.
3. Applying the BBAB theorem, this then implies that the generalized pairs (X_i, B_i+M_i) form a bounded family.
4. There exists a positive real number t such that $(X_i, (1+t)B_i+(1+t)M_i)$ is generalized lc for each i.
5. The numbers of terms in both B_i and M_i are bounded. So we may assume that $B_i = \sum_{j=1}^{k} b_{ij}B_{ij}$ and $M'_i = \sum_{j=1}^{k} m_{ij}M'_{ij}$ for some real numbers b_{ij}, $m_{ij} \in I$, some integral divisors B_{ij} on X_i and some Cartier divisors M'_{ij} on a model X' over X.
6. We may assume that $\{b_{ij}\}_{i=1}^{\infty}$ and $\{m_{ij}\}_{i=1}^{\infty}$ are non-decreasing for all j. Let $b_{0j} = \lim_{i\to\infty} b_{ij}$ and $m_{0j} = \lim_{i\to\infty} m_{ij}$ for each j.
7. Let $\hat{B}_i = \sum_{j=1}^{k} b_{0j}B_{ij}$ and $\hat{M}'_i = \sum_{j=1}^{k} m_{0j}M'_{ij}$ for each i. Then $B_i+M_i \leq \hat{B}_i+\hat{M}_i \leq (1+c_it)(B_i+M_i)$ for some $c_i \geq 0$ with $= \lim_{i\to\infty} c_i = 0$.
8. This implies that $\mathrm{mld}(\mathrm{X_i}, \hat{\mathrm{B}}_\mathrm{i}+\hat{\mathrm{M}}_\mathrm{i}) \in [(1-c_i)a_i, a_i]$.
9. We may assume that $\{\mathrm{mld}(\mathrm{X_i}, \hat{\mathrm{B}}_\mathrm{i}+\hat{\mathrm{M}}_\mathrm{i})\}_{i=1}^{\infty}$ is strictly increasing, which contradicts the finiteness of mlds.

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