The geometric exceptional set in Manin's conjecture for Batyrev and Tschinkel's example

Runxuan Gao

Nagoya University

Abstract

We give the minimal set of rational points on a Fermat cubic surface bundle X such that the image of rational points from every thin map with a- and b-value not less than X lies in this set. We conjecture that Manin's conjecture is true with this set as the exceptional set.

Exceptional set

Conjecture (Colliot-Thélène). Let X: smooth projective geometrically rational connected variety/ a number field F.

$$X(F) \neq \emptyset \Rightarrow X(F) \text{ is not a thin set.}$$

In particular, $X(F)$ is Zariski dense in X, and it is natural to study the distribution of these points with respect to some heights. It turns out that on many varieties, there is some proper closed subset possessing 100% of the rational points as the bound tends to infinity, such as exceptional curves on del Pezzo surfaces. Thus we must remove a subset of rational points to count, called the **exceptional set**, to obtain an asymptotic formula reflecting the geometry of the whole variety.

Remark. In fact, there are other types of rational points that should also be excluded to make the points equidistributed with respect to the Tamagawa measure.

Batyrev and Tschinkel's examples [BT96] are the first ones such that removing a proper closed exceptional set is not enough, which vary in each dimension ≥ 3. The first such examples in dimension 2 were found in [Gao23].

Peyre's constant

Let X be a Fano variety over a number field F with metrized line bundle $\mathcal{L} := -\mathcal{K}_X$. Denote $\Gamma := \mathrm{Gal}(\overline{F}/F)$. Normalizing $N_1(X)$ in a canonical way, we define

$$\alpha(X) := \mathrm{Vol}(\mathrm{Nef}_1(X)),$$
$$\beta(X) := \# H^1(\Gamma, \mathrm{Pic}(X_{\overline{F}})).$$

The Peyre's constant c is defined as

$$c_{\mathcal{L}}(X) := \alpha(X)\beta(X) \lim_{s\to 1}(s-1)^t L(s, \mathrm{Pic}(\overline{X})) \cdot \tau_{\mathcal{L}}(X(\mathbb{A}_F)^{\mathrm{Br}}),$$

where $L(s, \mathrm{Pic}(\overline{X}))$ is the Artin L-function associated to the Γ-module $\mathrm{Pic}(\overline{X})$,$\tau_{\mathcal{L}}$ is the Tamagawa measure on the set $X(\mathbb{A}_F)$ of adelic points, $X(\mathbb{A}_F)^{\mathrm{Br}}$ is the part that is not affected by the Brauer–Manin obstruction and $t = \mathrm{rkPic}(X)$.

The Tamagawa measure is defined as follows. For any prime p, we can define a p-adic measure ω_p on $X_{\mathbb{Q}_p}$ induced by $-\mathcal{K}_X$. Define the measure τ_p on $X(\mathbb{Q}_p)$ by

$$\tau_p := \det\left(1 - p^{-1}\mathrm{Frob}_p \mid \mathrm{Pic}(X_{\overline{\mathbb{Q}}})^{I_p}\right) \cdot \omega_p,$$

where I_p is the inertial group. We define the $\tau_{\mathcal{L}}$ to be the product of τ_ν over any $\nu \in \mathrm{Val}(\mathbb{Q})$. The Weil conjecture guarantees that the volume $\tau_{\mathcal{L}}(X(\mathbb{A}_{\mathbb{Q}}))$ is absolutely convergent.

a-invariant (Fujita invariant)

Definition. Let X be a smooth projective variety and L be a big and nef divisor on X. Then

$$a(X, L) := \min\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\}.$$

Example. When X is a Fano variety and $L = -K_X$, then $a(X, L) = 1$.

Remark. By [BCHM10], we have $a(X, L) \in \mathbb{Q} \cup \{\infty\}$ and

$$a(X, L) = \min\left\{\frac{r}{s} \in \mathbb{Q} \middle| h^0(X, sK_X + rL) > 0\right\}.$$

Definition. (X, L) is called **adjoint rigid** if the Iitaka dimension $\kappa(X, K_X + a(X, L)L) = 0$.

b-invariant

Definition. Let X be a smooth projective variety over a field F and let L be a big and nef divisor. Then

$$\mathcal{F}(X, L) := \{\alpha \in \mathrm{Nef}_1(X) \mid (K_X + a(X, L)L)\, \alpha = 0\},$$

and the b-invariant of X is defined to be

$$b(F, X, L) := \dim \mathcal{F}(X, L).$$

Example. When X is a Fano variety and $L = -K_X$, then $b(F, X, L) := \mathrm{rkPic}(X)$.

Remark. When X is not smooth, then take a smooth resolution $\beta : \widetilde{X} \to X$ and define a- and b- invariants and adjoint rigidness by $\widetilde{X}$.

Thin set

Definition. Let X be a projective variety over a number field F. Then a subset $Z \subset X(F)$ is called a **thin set** if Z is a finite union of $f(Y(F))$ where $f : Y \to X$ satisfies either

- f is not dominant, or
- f is dominant with $\deg f \geq 2$.

Conjecture. X: unirational $\Rightarrow X(F)$ is NOT a thin set. This is related to the inverse Galois problem by the weak weak approximation property.

Fundamental tools

- The finite-generatedness of the canonical ring [BCHM10]
$$R(X, K_X + a(X, L)L) = \bigoplus_{m\in\mathbb{N}} H^0(X, m(K_X + a(X, L)L)).$$
- The extension theorem of Takayama and Tsuji [Tak06], which can be seen as a generalization of the Kawamata-Viehweg vanishing.
- Geometric properties of a- and b-invariants [LT17, LST22].
- The nonexistence of adjoint rigid a-cover of Fano threefolds [BLRT22].
- Lang-Nishimura theorem on rational points.
- Hilbert's irreducibility theorem.

Main Theorem in plain language

Let $X = \left\{\sum_{i=0}^3 x_i y_i^3 = 0\right\} \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{P}^3_{\mathbf{y}}$ be the cubic surface bundle over a field F of characteristic 0 and set $L = -K_X$. Define

$$V := \{\text{the closure of deformation of lines in smooth } \pi_{\mathbf{x}}\text{-fibers}\},$$
$$T_\sigma := \{s^3 x_{\sigma(0)} x_{\sigma(1)} = t^3 x_{\sigma(2)} x_{\sigma(3)}\} \subset \mathbb{P}^1_{s,t} \times \mathbb{P}^3_{\mathbf{x}} \quad \text{for } \sigma \in \mathfrak{S}_4,$$
$$f_\sigma : X_\sigma := X \times_{\mathbb{P}^3_{\mathbf{x}}} T_\sigma \to X : \text{ the projection,}$$

Then

$$\bigcup_f f(Y(F)) = V(F) \cup \bigcup_{\sigma \in \mathfrak{S}_4} f_\sigma(X_\sigma(F)), \tag{1}$$

where $f : Y \to X$ varies over all F-thin maps such that Y is a smooth geometrically integral variety with

$$(a(Y, f^*L), b(F, Y, f^*L)) \geq (a(X, L), b(F, X, L)) \qquad \text{in lexicographical order.}$$

If we further suppose that the field F is of finite type over $\mathbb{Q}$, then any morphism f as above factors through f_σ for some $\sigma \in \mathfrak{S}_4$.

Corollary-Conjecture

(Manin's conjecture for X with an explicit exceptional set) Choosing an adelic metric $\mathcal{L} := (L, \|\cdot\|)$ on $L = -K_X$, there is an associated height function. Let $N(X(F)\backslash Z, \mathcal{L}, B)$ be the number of F-rational points on X with height $< B$, where Z is the set defined in Main Theorem. Then

$$N(X(F)\backslash Z, \mathcal{L}, B) = c_{\mathcal{L}}(X) B \log(B)(1 + o(1))$$

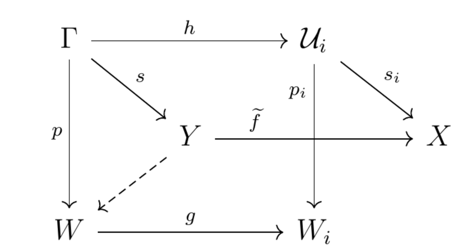
as $B \to \infty$.

Sketch of the proof I

1. Classify subvarieties of X with higher a-value and families of adjoint rigid subvarieties of X with the same a-value. The outcome is finitely many irreducible components $W_i \subset \mathrm{Hilb}(X)$.
   - Use extension theorem to reduce a subvariety of X to a singular cubic surfaces or pullback H of liner sections along $\pi_{\mathbf{x}} : X \to \mathbb{P}^3_{\mathbf{x}}$.
   - Study the adjoint rigidness and a-value of singular cubic surfaces.
   - Study singularities appeared in the classification.
2. Show that there is no adjoint rigid a-cover of X.
   - Use extension theorem to reduce to the case of a Fano 3-fold.

Sketch of the proof II

3. The proof of Main Theorem
   - Let $f : Y \to X$ be a thin map with $(a(Y, L), b(F, Y, L)) \geq (a(X, L), b(F, X, L))$. Denote the right-hand side of (1) by Z. We need to show $f(Y(F)) \subset Z$.
   - Systematically use geometric properties of a- and b-invariants proved in [LT17, LST22].
   - When Y is adjoint rigid or has higher a-value, show $f(Y(F)) \subset Z$.
   - When Y is not adjoint rigid with $a(Y, L) = a(X, L)$, the Iitaka fibration associated to $(Y, K_Y + a(Y, L)L)$ fiberizes Y birationally to a family $\Gamma \to W$ generically parametrizing adjoint rigid varieties of the same a-value.
   - By the universal property of Hilbert scheme, W factor through $W_i \subset \mathrm{Hilb}(X)$ rationally. Resolving the indeterminacy, we obtain the following commutative diagram.



   By Lang-Nishimura theorem, every rational point on Y lifts to Γ.
   - By generic flatness and nonexistence results of adjoint rigid a-covers in [LT17], $h|_{\Gamma_w}$ is F-birational for general closed point $w \in W$.
   - Systematically use of twists, Galois descents and étale fundamental groups to show every rational point $w \in \Gamma(F)$ lifts to $X_\sigma(F)$ for some $\sigma \in \mathfrak{S}_4$.
   - Use Hilbert's irreducibility theorem to prove the factoring property over Hilbertian fields.

References

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *Journal of the American Mathematical Society*, 23(2):405–468, 2010.

[BLRT22] Roya Beheshti, Brian Lehmann, Eric Riedl, and Sho Tanimoto. Moduli spaces of rational curves on Fano threefolds. *Advances in Mathematics*, 408:108557, 2022.

[BT96] V. V. Batyrev and Y. Tschinkel. Rational points on some Fano cubic bundles. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 323(1):41–46, 1996.

[Gao23] Runxuan Gao. A Zariski dense exceptional set in Manin's Conjecture: dimension 2. *Research in Number Theory*, 9(2):42, 2023.

[LST22] Brian Lehmann, Akash Kumar Sengupta, and Sho Tanimoto. Geometric consistency of Manin's conjecture. *Compositio Mathematica*, 158(6):1375–1427, June 2022.

[LT17] B. Lehmann and S. Tanimoto. On the geometry of thin exceptional sets in Manin's conjecture. *Duke Math. J.*, 166(15):2815–2869, 2017.

[Tak06] Shigeharu Takayama. Pluricanonical systems on algebraic varieties of general type. *Inventiones mathematicae*, 165(3):551–587, 2006.