The McKay correspondence for dihedral groups: The Moduli Space and the Tautological Bundles

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Abstract

A conjecture in (Ish20) states that for a finite subgroup G of $GL(2,\mathbb{C})$, a resolution Y of $\mathbb{C}^2/G$ is isomorphic to a moduli space M_θ of G-constellations for some generic stability parameter θ if and only if Y is dominated by the maximal resolution. This paper affirms the conjecture in the case of dihedral groups as a complex reflection group, and offers an extension of McKay correspondence.

Background

- (Gonzales-Springberg + Verdier) $G \subset SL(2,\mathbb{C})$ via a tautological bundle.
- (Ito + Nakamura) Explicit descriptions using the minimal resolution G-Hilbert scheme.
- (Bridgeland + King + Reid) The McKay correspondence as a derived equivalence via a Fourier-Mukai functor using the moduli space of G-clusters, for $G \subset SL(n,\mathbb{C}), n \leq 3$.
- (Craw + Ishii + Yamagishi) The McKay correspondence as a derived equivalence using the moduli space of θ-stable G-constellations, for generic parameter θ. All (projective) crepant resolutions of the quotient variety $\mathbb{C}^3/G$, for $G \subset SL(3,\mathbb{C})$.
- (Wunram) for a small $G \subset GL(2,\mathbb{C})$, a correspondence between irreducible components of the fundamental cycle of the minimal resolution f and the special non-trivial indecomposable reflexive sheaves $\mathcal{F}$ on the quotient variety X, i.e. whose $H^1(X_{min}, \tilde{\mathcal{F}}^V) = 0$, where $\tilde{F} := f^*(\mathcal{F})/(\text{torsion elements})$ whose rank corresponds to the coefficients of the said component was established
- (Ishii) Explicit descriptions using the minimal resolution G-Hilbert scheme for small subgroups of $GL(2)$. The Fourier-Mukai functor is only fully faithful.
- (Potter) Semi-orthogonal decomposition of the derived category $D^{D_{2n}}(\mathbb{C}^2)$
- (Kawamata) Gave a description of the semi-orthogonal decomposition of the derived category $D^G(\mathbb{C}^n)$, especially for $n = 3$ and $G \subset GL(3,\mathbb{C})$ in terms of maximal $\mathbb{Q}$-factorial terminalization Y in the spirit of DK-hypothesis (since the canonical divisor $K_Y \leq K_{\mathbb{C}^3/G}$).
- (Ishii) Conjectured that Y is isomorphic to M_θ for some generic stability parameter θ if and only if Y is between the minimal and maximal resolution of $\mathbb{C}^2/G$. It was shown for all small subgroups of $GL(2,\mathbb{C})$.

Definition of Terms

Moduli Space of θ-stable G-constellations

- A G-constellation on V is a G-equivariant coherent sheaf E on V such that $H^0(E)$ is isomorphic to the regular representation of G as a $\mathbb{C}[G]$-module, i.e. $H^0(E) \cong \mathbb{C}[G]$.
- Given $\theta \in \Theta$, a G-constellation E is **θ-stable** (resp. **θ-semistable**) if every proper G-equivariant coherent subsheaf $0 \subsetneq F \subsetneq E$ satisfies $\theta(H^0(F)) > 0$ (resp. $\theta(H^0(F)) \geq 0$). We regard $H^0(F)$ as an element of $R(G)$.
- A parameter $\theta \in \Theta$ is **generic** if a θ-semistable G-constellation is also θ-stable.
- The subset $\Theta^{gen} \subset \Theta$ of generic parameters is open and dense. It is the disjoint union of finitely many convex polyhedral cones C in Θ. For a generic parameter θ, by defining $C_\theta := \{\eta \in \Theta |$ every θ-stable G-constellation is η-stable$\}$, we call this convex polyhedral cone C_θ a *chamber* in Θ.

Maximal Resolution

Let (X, B) be a log terminal pair of a surface X and a $\mathbb{Q}$−divisor B. A resolution of singularities $f : Y \to X$ is a **maximal resolution** of (X, B) if $K_Y + f_*^{-1}(B) = f^*(K_X + B) + \Sigma_i a_i E_i$, where $-1 < a_i \leq 0$, and for any proper birational morphism of smooth surfaces $g : Z \to Y$ that is not an isomorphism, we have $K_Z + h_*^{-1}(B) = h^*(K_X + B) + \Sigma_j b_j F_j$, $h = fg$ and for some $b_j > 0$.

Main Results

In $G := D_{2n}$ represented by $\langle \sigma = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$:

Theorems

1. A resolution of singularities $Y \to \mathbb{C}^2/D_{2n} \cong \mathbb{C}^2$ is isomorphic to M_θ for some θ if and only if Y is dominated by the maximal resolution of the pair $(\mathbb{C}^2/D_{2n}, B)$, where B is a $\mathbb{Q}$-divisor defined by the equation $K_{\mathbb{C}^2} = \pi^*(K_{\mathbb{C}^2/D_{2n}} + B)$, and $\pi : \mathbb{C}^2 \to \mathbb{C}^2/D_{2n}$ is the projection map.
2. The maximal resolution Y_{max} of the pair $(\mathbb{C}^2/D_{2n}, B)$ is isomorphic to the quotient variety $\mathbb{Z}_n{-}Hilb(\mathbb{C}^2)/\mathbb{Z}_2$.
3. There is an obtained description of the tautological sheaves on the stack, introduced in the example, which uniquely determines the extension class.

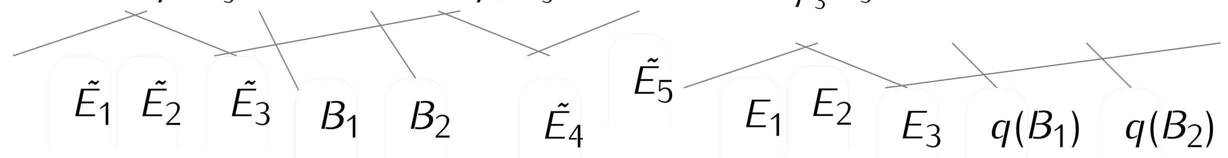
Ideas for Proof of the Main Results

- Explicitly compute the affine open covers of each of the blow-ups of $\mathbb{C}^2/D_{2n} \cong \mathbb{C}^2$ at the singular point of the boundary divisor B determined by the projection $\pi : \mathbb{C}^2 \to \mathbb{C}^2/D_{2n}$. Embed the affine covers to the crepant resolutions of $\mathbb{C}^3/D_{2n}$ via (NS17), which are realized as a moduli space of D_{2n}-constellations M_θ for some generic parameter θ.
- The maximal resolution is isomorphic to $\mathbb{Z}_n$-Hilb$(\mathbb{C}^2)/\mathbb{Z}_2$ by looking at the invariant locus $\mathbb{Z}_6$-Hilb$(\mathbb{C}^2)^{\mathbb{Z}_2}$, and can be realized via (IIN) as a moduli of θ-stable D_{2n} constellations.
- We obtain a similar description of the tautological sheaves as in (AV85).

Example for $n = 6$

The Tautological Sheaves

Let $X := \mathbb{Z}_6$-Hilb$(\mathbb{C}^2)/\mathbb{Z}_2; \mathfrak{X} := [\mathbb{Z}_6$-Hilb$(\mathbb{C}^2)/\mathbb{Z}_2]$. Consider the universal subscheme $\mathcal{Z} \subset \mathbb{Z}_6$-Hilb$(\mathbb{C}^2) \times \mathbb{C}^2$, so that we define the tautological sheaf $\mathcal{R} := p_*(O_{\mathcal{Z}}), p : \mathbb{Z}_6$-Hilb$(\mathbb{C}^2) \times \mathbb{C}^2 \to \mathbb{Z}_6$-Hilb$(\mathbb{C}^2)$ is the projection, and $q : \mathbb{Z}_6$-Hilb$(\mathbb{C}^2) \to X$ is the projection. For $\tilde{E}_i (1 \leq i \leq 5)$ the exceptional divisors in $\mathbb{Z}_6{-}Hilb(\mathbb{C}^2)$, the exceptional divisors in X are defined as $E_i := q(\tilde{E}_i)(1 \leq i \leq 3)$.
For $\epsilon \in Rep(\mathbb{Z}_6)$, ϵ_0 is the trivial representation, and ϵ_i is the scalar ϵ^i (for $1 \leq i \leq 5$). For $\rho_j \in Rep(D_{12})$, ρ_0 is the one dimensional representation of D_{12} whose σ defines the scalar 1, and τ defines the scalar 1; ρ'_0 whose σ is 1, and τ is -1; ρ_3 whose σ is -1, and τ is 1; and ρ'_3 whose σ is -1, and τ is -1. ρ_1 (resp. ρ_2) is the two dimensional representation of D_{12} defined by the matrices $\left\langle \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{bmatrix} (\epsilon^6 = 1) \right\rangle$ (resp. $\left\langle \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma = \begin{bmatrix} \epsilon^2 & 0 \\ 0 & \epsilon^{-2} \end{bmatrix} (\epsilon^6 = 1) \right\rangle$). Let $\mathfrak{B}_i := [B_i/\mathbb{Z}_2]$; and $\mathfrak{B}_{12} := [(B_1 + B_2)/\mathbb{Z}_2]$, where for $f : \mathbb{Z}_6$-Hilb$(\mathbb{C}^2) \to \mathbb{C}^2/\mathbb{Z}_6$ as the minimal resolution, and for the projection $\Pi : \mathbb{C}^2/\mathbb{Z}_6 \to \mathbb{C}^2/D_{12}$, the divisor $\tilde{B}_1$ (resp. $\tilde{B}_2$) define the equations on $\mathbb{C}^2/D_{12} \cong \mathbb{C}^2$ as $(x^3 - y^3)^2 = 0$ (resp. $(x^3 + y^3)^2 = 0$), $K_{\mathbb{C}^2/\mathbb{Z}_6} = \Pi^*(K_{\mathbb{C}^2/D_{12}} + \tilde{B}_1 + \tilde{B}_2)$, so that $B_1 := (\Pi \circ f)_*^{-1}(\tilde{B}_1)$ (resp. $B_2 := (\Pi \circ f)_*^{-1}(\tilde{B}_2)$).
The tautological bundles are defined by $\mathcal{R}_\rho := (q_*\mathcal{R} \otimes \rho^*)^{D_{12}} \in \text{Coh}(X); \tilde{\mathcal{R}}_\rho := (q_*\mathcal{R} \otimes \rho^*)^{\mathbb{Z}_6} \in \text{Coh}(\mathfrak{X})$, for $\rho \in \text{Rep}(D_{12})$.
The following non-zero degrees of tautological bundles on the irreducible exceptional curves as follows: $\deg(\mathcal{R}_{\rho'_0}|_{E_3}) = -1; \deg(\mathcal{R}_{\rho_1}|_{E_1}) = 1; \deg(\mathcal{R}_{\rho_1}|_{E_3}) = -1; \deg(\mathcal{R}_{\rho_2}|_{E_2}) = 1; \deg(\mathcal{R}_{\rho_2}|_{E_3}) = -1; \deg(\mathcal{R}_{\rho_3}|_{E_3}) = -1; \deg(\mathcal{R}_{\rho'_3}|_{E_3}) = -1$.



The Rank 1 Tautological Bundles

$\mathcal{R}_{\rho_0}$ (resp. $\mathcal{R}_{\rho'_0}$), we can realize the two sheaves as an extension via the (split) exact sequence below:

$$0 \to O_{Y_{max}} \to q_*(O_{\mathbb{Z}_6\text{-Hilb}(\mathbb{C}^2)}) \to \mathcal{L} \otimes \delta \to 0$$

where $\mathcal{R}_{\rho'_0} = \mathcal{L}$ is an invertible sheaf such that $\mathcal{L}^2 = O_{Y_{max}}(-q(B_1 + B_2))$; and δ is the nontrivial representation of $\mathbb{Z}_2$.
Also, $\mathcal{R}_{\rho_3} = O_{Y_{max}}(-q(B_1))$ and $\mathcal{R}_{\rho'_3} = O_{Y_{max}}(-q(B_2))$. These rank 1 tautological bundles are rigid, hence, the stacky description is already obtained.

The Tautological Bundle as Extension

$\mathcal{R}_{\rho_i}$ $(i = 1, 2)$, which is a rank 2 tautological bundle, we can realize $\mathcal{R}_{\rho_i}$ as an extension via the exact sequences:

$$0 \to q^*(\mathcal{R}_{\rho_i}) \to \tilde{\mathcal{R}}_{\rho_i} \to O_{B_1+B_2} \to 0 \text{ and } 0 \to O_{Y_{max}} \to q_*(O_{\mathbb{Z}_6\text{-Hilb}(\mathbb{C}^2)}) \to \mathcal{L} \to 0$$

where $\tilde{\mathcal{R}}_{\rho_i} = O(\tilde{D}_i) \oplus O(g \cdot \tilde{D}_i)$ and $q_*(\tilde{\mathcal{R}}_{\rho_i}) = O_X \oplus O_X(D_i) \otimes \mathcal{L} \otimes \delta$, where D_1 (resp. D_2) is any transversal to the exceptional divisor E_1 (resp. E_2) not intersecting E_2 (resp. E_1 and E_3) and correspondingly for $\tilde{D}_i$, which is a transversal to $\tilde{E}_i$ not intersecting on an intersection point of two exceptional divisors.
The extension class as a section of $Ext^1(O_{B_1+B_2} \otimes \delta, q^*(\mathcal{R}_{\rho_i})) \cong H^0(B_1 + B_2, \delta \otimes q^*(\mathcal{R}_{\rho_i}))) \cong \mathbb{C}[B_1] \otimes \mathbb{C}[B_2]$ corresponds to the generators of each coordinate rings, i.e. the sections are constant.

Selected References

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