Non-commutative crystalline comparison theorem

Keiho Matsumoto
lightkun0526@gmail.com
Osaka University

1 Introduction

1.1 Setting

K: finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K$: integral ring of K, k: residue field, $\mathcal{C} := \widehat{\overline{K}}$, $W := W(k)$

1.2 p-adic Hodge theory

In algebraic geometry, the concept of *period* refers to an entry of the matrix of the de Rham isomorphism:

$$H^*_{\mathrm{dR}}(M) \simeq H^*_{\mathrm{Sing}}(M;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

for a complex algebraic variety M defined over a number field K. To study the p-adic analogue of the notion of periods, Fontaine introduces the *p-adic period ring* B_{crys}. Building on Fontaine's work, Tsuji and Faltings proved the following.

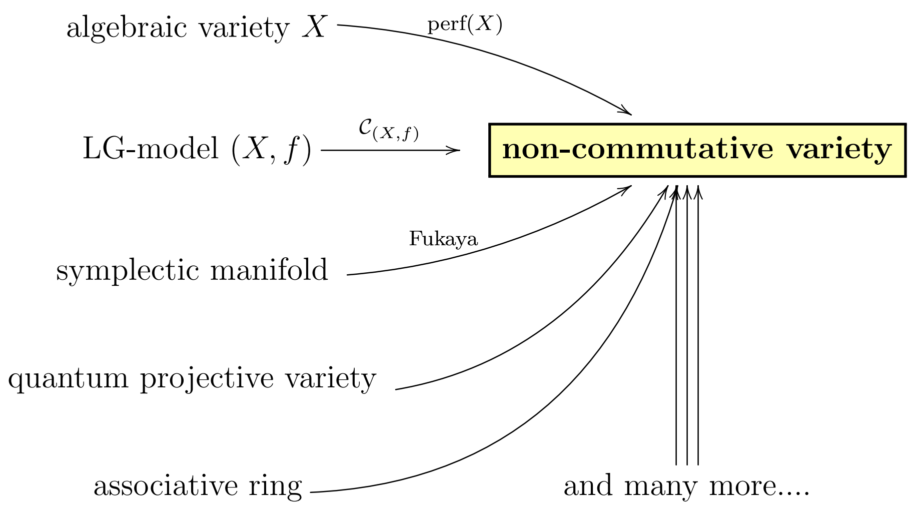
Theorem 1.1 (Tsuji, Faltings)**.** *Let X be a smooth proper variety $\mathcal{O}_K$. There is an isomorphism*

$$H^i(X_k/W) \otimes_W B_{crys} \simeq H^i(X_{\mathcal{C}};\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{crys}$$

which is compatible with the G_K-actions, the Frobenius endmorphisms and the filtration. In particular, $G_K \curvearrowright H^i(X_{\mathcal{C}};\mathbb{Q}_p)$ is a crystalline representation.

1.3 Non-commutative variety

Let R be a commutative ring. Orlov and Kontsevich introduced *non-commutative algebraic geometry* in which an R-linear, idempotent-complete, small stable ∞-category is studied as a *non-commutative variety over R*. Non-commutative algebraic geometry has been widely used in various areas of physics and mathematics.



Definition 1.2 (Orlov)**.** A non-commutative variety $\mathcal{T}$ over R *admits a geometric realization* if there are a derived scheme $\mathcal{X}$ over R s.t. $\pi_0(\mathcal{X})$ is separated scheme of finite type over R and there is a fully faithful admissible inclusion $\mathcal{T} \hookrightarrow \mathrm{perf}(\mathcal{X})$.

In many cases, a smooth proper non-commutative variety is known (or expected) to admit a geometric realization.

1.4 Non-commutative cohomology

Tsygan and Connes defined cyclic periodic homology $\mathrm{HP}(A)$ for an associative ring A which can be regarded as the *non-commutative de Rham cohomology*. Similarly, there are non-commutative versions of Betti, étale, crystalline and Hodge cohomology groups for non-commutative variety $\mathcal{T}$ over R:

variety X over R	NC variety $\mathcal{T}$ over R	
$H^*_{\mathrm{dR}}(X/R)$	$\pi_*\mathrm{HP}(\mathcal{T}/R)$	
$H^*_{\mathrm{Sing}}(X,\mathbb{Z})$	$\pi_*K^{\mathrm{top}}(\mathcal{T})$	$R=\mathbb{C}$
$H^*_{\mathrm{Zar}}(X,\Omega^*_{X/R})$	$\pi_*\mathrm{HH}(\mathcal{T}/k)$	
$H^*_{\acute{e}t}(X,\mathbb{Z}_p)$	$\pi_*L_{K(1)}K(\mathcal{T})$	$(\mathrm{ch}(R),p)=1$
$H^*_{\mathrm{crys}}(X/W(R))$	$\pi_*\mathrm{TP}(\mathcal{T};\mathbb{Z}_p)$	$R=\mathbb{F}_{p^n}$

1.5 Comparison theorems

Some of comparison theorems between non-commutative cohomologies are previously known or expected. For a smooth proper NC variety $\mathcal{T}$ over $\mathbb{C}$, Kontsevich expected that there is the *non-commutative Hodge decomposition*:

$$\pi_i\mathrm{HP}(\mathcal{T}/\mathbb{C}) \simeq \bigoplus\nolimits_{n\in\mathbb{Z}} \pi_{i+2n}\mathrm{HH}(\mathcal{T}/\mathbb{C}),$$

and Kaledin proved it. For a smooth proper NC variety $\mathcal{T}$ over $W(\mathbb{F}_{p^n})$, Scholze announced that there is the *non-commutative Berthelot-Ogus isomorphism*:

$$\pi_i\mathrm{TP}(\mathcal{T}_{\mathbb{F}_{p^n}};\mathbb{Z}_p) \simeq \pi_i\mathrm{HP}(\mathcal{T}/W(\mathbb{F}_{p^n})).$$

2 Non-commutative p-adic Hodge

Inspired by Kaledin and Scholze's results, we expected the following comparison theorem, which can be regarded as a non-commutative analogue of Theorem 1.1.

Conjecture 2.1 (M, 2022)**.** *Let $\mathcal{T}$ be a smooth proper NC variety over $\mathcal{O}_K$. Then there is an isomorphism*

$$\pi_i\,\mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p) \otimes_W B_{crys} \simeq \pi_i L_{K(1)}K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} B_{crys}$$

which is compatible with the G_K-actions, the Frobenius endmorphisms. In particular, $G_K \curvearrowright \pi_i L_{K(1)}K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a crystalline representation.

Example 2.2. If $\mathcal{T} = \mathrm{perf}(X)$ for a smooth proper variety X over $\mathcal{O}_K$, then the conjecture holds for $\mathcal{T}$ by Theorem 1.1.
$\because$ There are the following canonical isomorphisms due to Thomason and Bhatt-Morrow-Scholze:

$$\pi_i L_{K(1)}K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \bigoplus_{n\in\mathbb{Z}} H^{i+2n}_{\acute{e}t}(X_{\mathcal{C}},\mathbb{Q}_p(n))$$

$$\pi_i\mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)[\frac{1}{p}] \simeq \bigoplus_{n\in\mathbb{Z}} H^{i+2n}_{\mathrm{cry}}(X_k/W[\frac{1}{p}])(n)$$

3 Main result

We can prove a non-commutative version of Bhatt-Morrow-Scholze's comparison theorem. By using it and Du-Liu's prismatic $(\phi,\hat{G})$-mod theory, we have the following result.

Theorem 3.1 (M, 2023)**.** *Let $\mathcal{T}$ be a smooth proper NC variety over $\mathcal{O}_K$. We assume that $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization, then Conjecture 2.1 holds for $\mathcal{T}$.*